Momentum relaxation of a relativistic Brownian particle

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The momentum relaxation of a relativistic Brownian particle immersed in a fluid is studied on the basis of the Fokker-Planck equation for the relativistic Ornstein-Uhlenbeck process. An analytical expression is derived for the short-time relaxation rate. The relaxation spectrum has both discrete and continuum components. It is shown that the Fokker-Planck equation under consideration is closely related to the Schrödinger equation for the hydrogen atom. Hence it follows that there is an infinite number of discrete states. The momentum autocorrelation function is calculated numerically for a strongly relativistic particle.

I. INTRODUCTION

It was shown by Uhlenbeck and Ornstein [1] that the Langevin description of the motion of a nonrelativistic free Brownian particle is equivalent to a linear Fokker-Planck equation for the conditional probability distribution of velocity [2]. The fundamental solution of the linear Fokker-Planck equation is Gaussian. From the Gaussian distribution of velocity Uhlenbeck and Ornstein derived the corresponding Gaussian conditional distribution of position. At long times this tends to the Gaussian solution of the diffusion equation, as found by Einstein.

In some situations in high-energy physics, plasma physics, and astrophysics it is necessary to consider relativistic velocities. The relativistic generalization of the Ornstein-Uhlenbeck theory was formulated by Debbasch *et al.* [3]. It is based on a nonlinear Fokker-Planck equation for the conditional distribution of momentum. At long times the distribution tends to the thermal equilibrium distribution derived by Jüttner [4] as a generalization of the nonrelativistic Maxwell-Boltzmann distribution.

The theory of relativistic Brownian motion on the basis of the Langevin description was discussed by Dunkel and Hänggi [5,6]. They showed that only a particular treatment of the multiplicative noise leads to a Fokker-Planck equation consistent with the Jüttner equilibrium distribution. However, their Fokker-Planck equation differs from that derived by Debbasch *et al.* [3]. The difference corresponds to a different dependence of the generalized friction tensor on momentum. The theory has been reviewed by Dunkel and Hänggi [7,8].

Recently Debbasch *et al.* [9] have studied thermal relaxation of a relativistic particle diffusing in a fluid at equilibrium via a numerical solution of an Ornstein-Uhlenbeck-type Fokker-Planck equation in one dimension. From a Fourier analysis of the numerical solution of the equation with imaginary time variable, they concluded that the relaxation spectrum has both discrete and continuum components.

In the following we study momentum relaxation in the three-dimensional relativistic Ornstein-Uhlenbeck process on the basis of the thermal momentum autocorrelation function. An exact expression is derived for the short-time rate of relaxation. We show that the Fokker-Planck equation is

closely related to the Schrödinger equation for the hydrogen atom. The relation shows that there is an infinite number of discrete modes, corresponding to the bound states of the hydrogen atom, and a continuous spectrum corresponding to the scattering states. For a strongly relativistic particle the relaxation at intermediate and long times is dominated by the discrete states with the smallest relaxation rates. The relaxation function is studied on the basis of an approximation involving a numerical solution of the dominating discrete states, hydrogen atom bound-state wave functions for the discrete states with larger rates, and a Padé-type approximation to the contribution from the continuous spectrum.

II. FOKKER-PLANCK EQUATION

We consider a Brownian particle of rest mass m_0 immersed in a viscous fluid consisting of much lighter particles. Due to collisions between the Brownian particle and the particles of the fluid the momentum of the Brownian particle varies stochastically in time. The probability distribution of momentum $P(\boldsymbol{p},t)$ is assumed to satisfy the Fokker-Planck equation [3]

$$\frac{\partial P}{\partial t} = D \frac{\partial}{\partial \boldsymbol{p}} \cdot \left(\frac{\partial P}{\partial \boldsymbol{p}} + \frac{\partial U}{\partial \boldsymbol{p}} P \right), \qquad (2.1)$$

with momentum diffusion coefficient D, and dimensionless energy

$$U(p) = \beta c \sqrt{p^2 + q^2}, \qquad q = m_0 c, \qquad (2.2)$$

where c is the velocity of light, and β is a measure of inverse temperature. It follows from the H theorem [10] that in the course of time the distribution function $P(\boldsymbol{p},t)$ tends to the equilibrium distribution

$$P_{eq}(p) = \exp[-U(p)]/\mathcal{Z}(\beta), \quad \mathcal{Z}(\beta) = \int \exp[-U(p)]d\boldsymbol{p}, \qquad (2.3)$$

where $\mathcal{Z}(\beta)$ is the normalization factor [4]. The width of the Jüttner distribution $P_{eq}(p)$ is characterized by the dimensionless parameter $\chi = \beta c q$. For $\chi \gg 1$ the distribution tends to the Maxwell-Boltzmann distribution. In this nonrelativistic limit Eq. (2.1) describes an Ornstein-Uhlenbeck process [1] with $D = \zeta/\beta = k_B T \zeta$, where k_B is Boltzmann's constant, T is the absolute temperature, and ζ is the friction coefficient.

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We consider in particular the initial distribution function $P(\boldsymbol{p},0) = P_{eq}(p) + f(\boldsymbol{p},0)P_{eq}(p)$ with a chosen factor $f(\boldsymbol{p},0)$. We write the time-dependent distribution as

$$P(\boldsymbol{p},t) = P_{eq}(p) + f(\boldsymbol{p},t)P_{eq}(p). \tag{2.4}$$

The function $f(\boldsymbol{p},t)$ satisfies the equation

$$\frac{\partial f}{\partial t} = \mathcal{L}f, \tag{2.5}$$

with adjoint Fokker-Planck operator

$$\mathcal{L} = D\left(\frac{\partial}{\partial \boldsymbol{p}} - \frac{\partial U}{\partial \boldsymbol{p}}\right) \cdot \frac{\partial}{\partial \boldsymbol{p}}. \tag{2.6}$$

Consider in particular the initial value $f(\boldsymbol{p},0) = p_z$. The relaxation of momentum may be characterized by the mean value

$$C(t) = \int p_z f(\boldsymbol{p},t)P_{eq}(p)\,d\boldsymbol{p}. \tag{2.7}$$

This may be expressed as

$$C(t) = \langle p_z^2 \rangle G(t), \quad \langle p_z^2 \rangle = \int p_z^2 P_{eq}(p)\,d\boldsymbol{p}, \tag{2.8}$$

with dimensionless autocorrelation function $G(t)$ with initial value $G(0) = 1$. The mean relaxation time

$$\tau_M = \int_0^\infty G(t)dt \tag{2.9}$$

may be expressed as

$$\tau_M = -\langle p_z \mathcal{L}^{-1} p_z \rangle / \langle p_z^2 \rangle. \tag{2.10}$$

We introduce spherical coordinates (p,θ,φ) in momentum space. Then $p_z = p\cos\theta$, and it follows from isotropy that the factor $f(\boldsymbol{p},t)$ takes the form $f(\boldsymbol{p},t) = g(p,t)\cos\theta$ with a spherically symmetric function $g(p,t)$. We write this in the form

$$g(p,t) = \frac{\psi(p,t)}{p} \exp[U(p)/2]. \tag{2.11}$$

By substitution into Eq. (2.5) we find that $\psi(p,t)$ satisfies the equation

$$\frac{\partial \psi}{\partial t} = D\left[\frac{\partial^2 \psi}{\partial p^2} - V(p)\psi\right], \tag{2.12}$$

with

$$V(p) = \frac{2}{p^2} - \frac{U'}{p} + \frac{1}{4}U'^2 - \frac{1}{2}U'', \tag{2.13}$$

where the prime denotes differentiation with respect to p. The equation is similar to a radial Schrödinger equation for a particle in a central potential. The first term is the centrifugal part due to the angular dependence of the distribution. It is easy to show that $V(p)$ is positive, diverges as $2/p^2$ as $p \to 0$, and tends to

$$V_\infty = \frac{\chi^2}{4q^2} = \frac{1}{4}\beta^2 c^2 \tag{2.14}$$

as $p \to \infty$. In Fig. 1 we plot the reduced potential $V(p)/V_\infty$ for $\chi = 1$, $q = 1$ as a function of p/q. For intermediate values
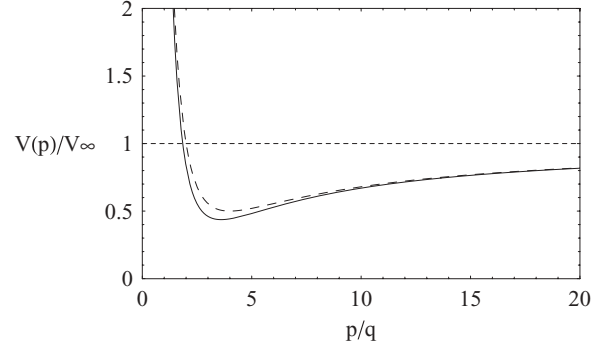

FIG. 1. Reduced potential $V(p)/V_\infty$ for $q = 1$, $\chi = 1$ as a function of p/q (solid curve). We also plot the function $8q^2/p^2 - 4q/p + 1$ corresponding to the potential of the hydrogen atom (dashed curve).

of p/q the function shows a potential well with reduced values less than unity, but larger than zero.

III. TIME DEPENDENCE

The time dependence of the autocorrelation function $G(t)$ can be expressed in terms of the solution $\psi(p,t)$ of Eq. (2.12). The equation can be interpreted as a diffusion-reaction equation describing diffusion on the half line $p > 0$ and absorption in proportion to the potential $V(p)$. We consider the initial value

$$\psi(p,0) = p^2 \exp[-U(p)/2]. \tag{3.1}$$

Then the autocorrelation function can be expressed as

$$G(t) = \int_0^\infty \psi(p,0)\psi(p,t)\,dp \bigg/ \int_0^\infty \psi(p,0)^2\,dp. \tag{3.2}$$

We write Eq. (2.12) as

$$\frac{\partial \psi}{\partial t} = -\mathcal{M}\psi, \tag{3.3}$$

with linear operator $\mathcal{M}$. The operator can be decomposed as

$$\mathcal{M} = \mathcal{S} + DV_\infty. \tag{3.4}$$

We call

$$\mathcal{S} = D\left[-\frac{\partial^2}{\partial p^2} + V(p) - V_\infty\right] \tag{3.5}$$

the Schrödinger operator. The potential $V(p) - V_\infty$ tends to zero at infinity, and has a negative part. Hence the operator $\mathcal{S}$ has bound states with negative eigenvalues $\{\lambda_{bj}\}$, and scattering states with a continuous spectrum of positive eigenvalues λ_f. From Eq. (2.13) we find

$$V(p) - V_\infty = -\frac{\beta c}{p} + O(1/p^2) \quad \text{as } p \to \infty, \tag{3.6}$$

so that the potential decays slowly for large p, in analogy to the hydrogen atom. Therefore there is an infinite number of bound states.

We can write the correlation function as

$$G(t) = \sum_{j=0}^\infty w_{bj} e^{-\mu_{bj}t} + \int_{\gamma/4}^\infty w_f(\mu_f) e^{-\mu_f t}\,d\mu_f, \tag{3.7}$$

with the abbreviations

$$\mu_b = \lambda_b + \tfrac{1}{4}\gamma, \quad \mu_f = \lambda_f + \tfrac{1}{4}\gamma, \quad \gamma = D\beta^2 c^2, \quad (3.8)$$

where $\{\mu\}$ are the eigenvalues of $\mathcal{M}$ and $\{\lambda\}$ are the eigenvalues of $\mathcal{S}$. The weights w follow from the expansion of the initial value $\psi(p,0)$ in terms of the eigenfunctions of the Schrödinger operator $\mathcal{S}$,

$$\frac{\psi(p,0)}{\sqrt{N_0}} = \sum_{j=0}^{\infty} c_{bj}\phi_{bj}(p) + \int_{\gamma/4}^{\infty} c_f(\mu)\phi_f(\mu,p)\,d\mu, \quad (3.9)$$

with norm

$$N_0 = \int_0^{\infty} \psi(p,0)^2\,dp, \quad (3.10)$$

and normalized eigenfunctions $\{\phi\}$. In particular, for the bound states

$$w_{bj} = c_{bj}^2, \quad c_{bj} = \int_0^{\infty} \phi_{bj}(p)\psi(p,0)\,dp/\sqrt{N_0}. \quad (3.11)$$

Here we have used the orthonormality relations

$$\int_0^{\infty} \phi_{bj}(p)\phi_{bk}(p)dp = \delta_{jk}, \quad \int_0^{\infty} \phi_{bj}(p)\phi_f(\mu,p)dp = 0. \quad (3.12)$$

The norm N_0 can be evaluated explicitly. It is given by

$$N_0 = \int_0^{\infty} p^4 e^{-U(p)}dp = q^5 \int_{\chi}^{\infty} (U^2 - \chi^2)^{3/2} e^{-U} U\,dU$$
$$= 3q^5 K_3(\chi)/\chi^2, \quad (3.13)$$

with the modified Bessel function $K_3(\chi)$.

It is convenient to define the scalar product

$$(\phi_1,\phi_2) = \int_0^{\infty} \phi_1(p)\phi_2(p)dp. \quad (3.14)$$

Then the correlation function can be expressed as

$$G(t) = (\psi_0, e^{-\mathcal{M}t}\psi_0)/N_0, \quad N_0 = (\psi_0,\psi_0), \quad (3.15)$$

with the abbreviation $\psi_0 = \psi(p,0)$. Hence the short-time relaxation rate is given by

$$\gamma_S = -\frac{dG}{dt}\bigg|_{t=0} = (\psi_0, \mathcal{M}\psi_0)/N_0. \quad (3.16)$$

The numerator can be evaluated as in Eq. (3.13). It takes the value

$$(\psi_0, \mathcal{M}\psi_0) = \frac{D}{\beta^3 c^3} \int_{\chi}^{\infty} (U^2 - \chi^2)^{3/2} e^{-U}\,dU$$
$$= 3Dq^3 K_2(\chi)/\chi. \quad (3.17)$$

Hence the short-time relaxation rate is given by

$$\gamma_S = D\frac{\chi K_2(\chi)}{q^2 K_3(\chi)}. \quad (3.18)$$

For large χ this behaves as

$$\gamma_S = \gamma_{nr}\left[1 - \frac{5}{2\chi} + O(\chi^{-2})\right], \quad (3.19)$$

where γ_{nr} is the nonrelativistic rate coefficient [1]

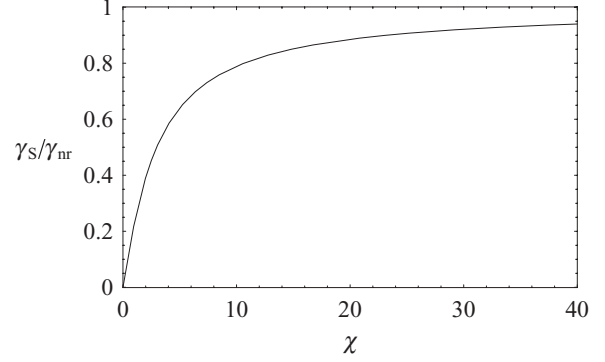$$\gamma_{nr} = \frac{D\chi}{q^2} = \frac{D\beta}{m_0}. \quad (3.20)$$

FIG. 2. Ratio $\gamma_S/\gamma_{nr} = K_2(\chi)/K_3(\chi)$ as a function of χ.

For small χ the short-time relaxation rate behaves as

$$\gamma_S = \gamma_{nr}\left[\frac{1}{4}\chi - \frac{1}{32}\chi^3 + O(\chi^5 \log \chi)\right]. \quad (3.21)$$

The first term equals DV_{∞}. In Fig. 2 we plot the ratio $\gamma_S/\gamma_{nr} = K_2(\chi)/K_3(\chi)$ as a function of χ. This shows that the short-time rate coefficient is always less than its nonrelativistic limit value.

The long- and intermediate-time behavior of the correlation function $G(t)$ is dominated by the first few terms of the sum in Eq. (3.7). The first few bound states $\{\phi_{bj}(p)\}$ are easily found numerically by the shooting method, so that the corresponding weights w_{bj} can be evaluated. We estimate the contribution from the remaining bound states by approximating the eigenfunctions by the P states of the hydrogen atom. The contribution from the scattering states can then be estimated from the Laplace transform of $\psi(p,t)$ at a few selected values of the Laplace variable.

The bound-state eigenfunctions of the P-state hydrogen Schrödinger operator

$$\mathcal{S}_{\mathrm{H}} = D\left[-\frac{\partial^2}{\partial p^2} + \frac{2}{p^2} - \frac{Z}{p}\right] \quad (3.22)$$

are given by

$$\phi_{\mathrm{H}j}(p) = \frac{1}{\sqrt{N_{\mathrm{H}j}}} p^2 e^{-\kappa p} L_j^3(2\kappa p), \quad \kappa = \frac{Z}{2(j+2)}, \quad (3.23)$$

with associated Laguerre polynomial $L_j^3(\rho)$ and normalization factor

$$N_{\mathrm{H}j} = \int_0^{\infty} \left[p^2 e^{-\kappa p} L_j^3(2\kappa p)\right]^2 dp$$
$$= \frac{2}{Z^5}(j+1)(j+2)^7(j+3). \quad (3.24)$$

The corresponding eigenvalues are

$$\lambda_{\mathrm{H}j} = -D\frac{Z^2}{4(j+2)^2}, \quad j = 0,1,2,\ldots. \quad (3.25)$$

In addition the operator $\mathcal{S}_{\mathrm{H}}$ has a continuous spectrum of positive eigenvalues with corresponding scattering eigenstates.

IV. NUMERICAL RESULTS

To illustrate the calculation of the autocorrelation function $G(t)$ we consider in particular the value $\chi = 1$. We can choose
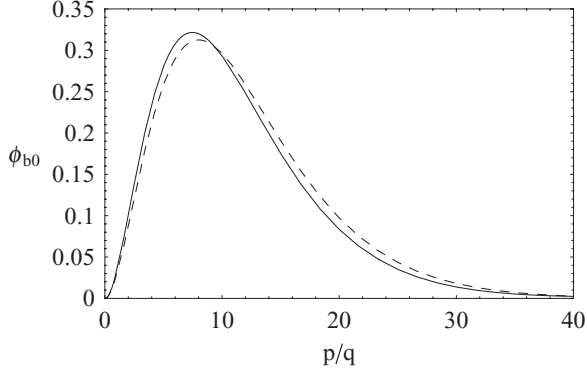
FIG. 3. Bound-state eigenfunction $\phi_{b0}(p)$ for $q = 1$, $\chi = 1$ as a function of p/q (solid curve). We compare with the $2P$-state wave function of the hydrogen atom (dashed curve).

units such that $m_0 = 1$, $c = 1$. Then also $q = 1$. In Fig. 1 we also plot the function $8q^2/p^2 - 4q/p + 1$, corresponding to the potential of the hydrogen atom. In Figs. 3–5 we plot the bound-state eigenfunctions corresponding to the lowest three eigenvalues, and compare with the corresponding eigenfunctions of $\mathcal{S}_H$ for $Z = 1$ and $j = 0,1,2$. In view of Fig. 1 it is not surprising that the eigenfunctions are remarkably similar to those of the hydrogen atom. The comparison of eigenvalues yields

$$
\begin{aligned}
\lambda_0 &= -0.067\,41\gamma, \quad \lambda_{H0} = -0.062\,50DZ^2, \\
\lambda_1 &= -0.029\,19\gamma, \quad \lambda_{H1} = -0.027\,78DZ^2, \\
\lambda_2 &= -0.016\,21\gamma, \quad \lambda_{H2} = -0.015\,63DZ^2.
\end{aligned}
\tag{4.1}
$$

We note that the values of μ_{bj} in Eq. (3.8) are positive. The corresponding weights are calculated from Eq. (3.11) as

$$
w_{b0} = 0.6327, \quad w_{b1} = 0.0858, \quad w_{b2} = 0.0284. \tag{4.2}
$$

The remaining bound-state eigenfunctions and eigenvalues are approximated by those of the operator $\mathcal{S}_H$. This yields for the remaining weight

$$
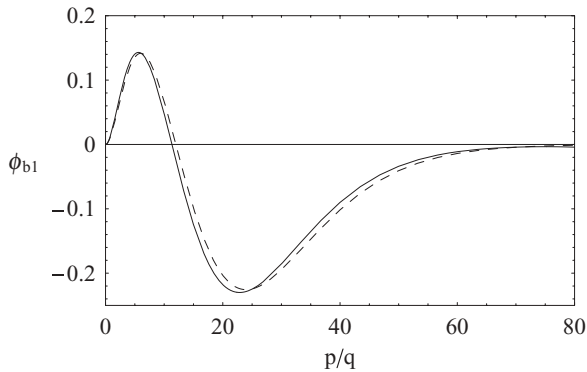R_b = \sum_{j=3}^{\infty} w_{bj} \approx 0.042, \tag{4.3}
$$

FIG. 4. Bound-state eigenfunction $\phi_{b1}(p)$ for $q = 1$, $\chi = 1$ as a function of p/q (solid curve). We compare with the $3P$-state wave function of the hydrogen atom (dashed curve).
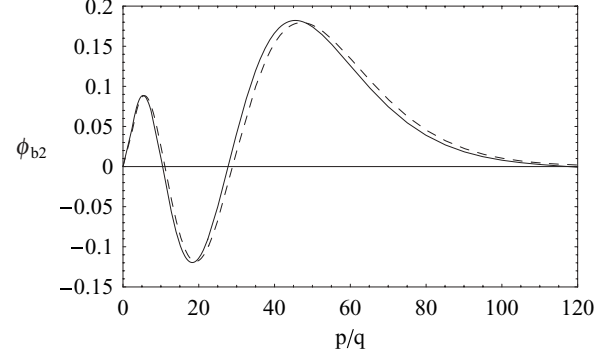

FIG. 5. Bound-state eigenfunction $\phi_{b2}(p)$ for $q = 1$, $\chi = 1$ as a function of p/q (solid curve). We compare with the $4P$-state wave function of the hydrogen atom (dashed curve).

so that the total weight of the bound states is

$$
W_b = w_{b0} + w_{b1} + w_{b2} + R_b \approx 0.789. \tag{4.4}
$$

The remaining weight $W_f = 1 - W_b = 0.211$ is contributed by the continuous spectrum of scattering states.

The Laplace transform of the correlation function is defined as

$$
\hat{G}(s) = \int_0^{\infty} e^{-st} G(t) dt. \tag{4.5}
$$

From Eq. (3.7) we find the expression

$$
\hat{G}(s) = \sum_{j=0}^{\infty} \frac{w_{bj}}{s + \mu_{bj}} + \int_{\gamma/4}^{\infty} \frac{w_f(\mu)}{s + \mu} d\mu. \tag{4.6}
$$

The nature of the potential $V(p)$ suggests that a two-pole approximation in the complex $\sqrt{s + \gamma/4}$ plane will be a good approximation to the contribution from the continuous spectrum [11]. Thus we approximate

$$
\int_{\gamma/4}^{\infty} \frac{w_f(\mu)}{s + \mu} d\mu \approx \frac{W_f \tau_f}{1 + \sigma_f \sqrt{(s + \gamma/4)\tau_f} + (s + \gamma/4)\tau_f}, \tag{4.7}
$$

with two parameters σ_f and τ_f. The expression has been constructed such that the contribution $G_f(t)$ from the scattering states has the initial value $G_f(0) = W_f$. The two parameters σ_f and τ_f are determined from the requirement that the value of $\hat{G}(s)$ at two selected values of s is correct. We choose in
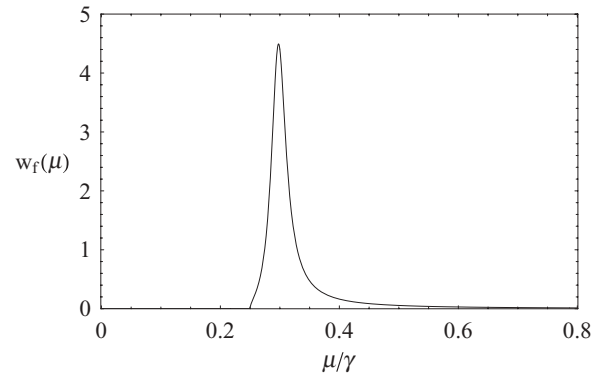

FIG. 6. Weight function $w_f(\mu)$ for $q = 1$, $\chi = 1$ as a function of μ/γ, as given by Eq. (4.10).
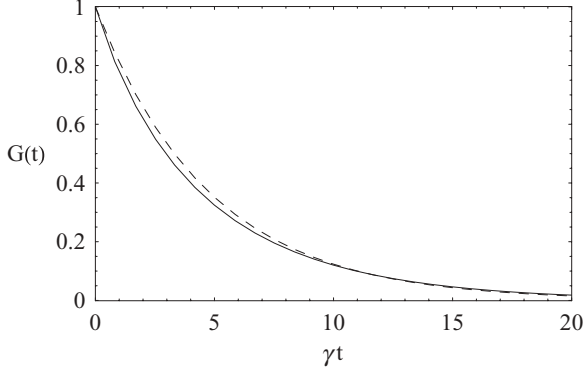
FIG. 7. Autocorrelation function $G(t)$ for $q = 1$, $\chi = 1$ as a function of γt (solid curve), compared with the exponential $\exp(-t/\tau_M)$ (dashed curve).
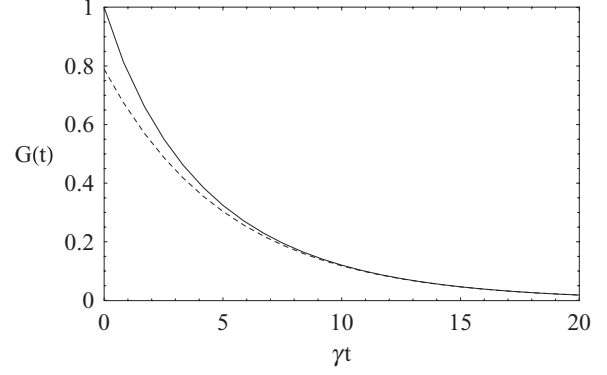

FIG. 8. Autocorrelation function $G(t)$ for $q = 1$, $\chi = 1$ as a function of γt (solid curve), compared with the bound-state contribution $G_b(t)$ (dashed curve).

particular the values $s = 0$ and $s = -0.1\gamma$. The second value lies on the negative axis between $-\mu_{b0}$ and 0. The value of $\hat{G}(s)$ at these two points is evaluated from the numerical solution of the Laplace transform of Eq. (2.12),

$$D\left[\frac{\partial^2 \hat{\psi}(p,s)}{\partial p^2} - V(p)\hat{\psi}(p,s)\right] = s\hat{\psi}(p,s) - \psi(p,0),$$
(4.8)

with initial value $\psi(p,0)$ given by Eq. (3.1). The numerical fit yields for the two parameters

$$\sigma_f = 0.308, \qquad \tau_f = 20.43/\gamma.$$
(4.9)

In Fig. 6 we plot the corresponding approximation to the weight function $w_f(\mu)$, given by

$$w_f(\mu) = \frac{1}{\pi}W_f\sigma_f\tau_f\sqrt{(\mu - \gamma/4)\tau_f}\big[1 + (\sigma_f^2 - 2)$$
$$\times (\mu - \gamma/4)\tau_f + (\mu - \gamma/4)^2\tau_f^2\big],$$
(4.10)

with integral

$$W_f = \int_{\gamma/4}^{\infty} w_f(\mu)d\mu.$$
(4.11)

From the solution at $s = 0$ we find for the mean relaxation time τ_M, defined by Eq. (2.9), the value $\tau_M = 4.7812/\gamma$. The approximation to $\hat{G}_f(s)$, given by Eq. (4.7), corresponds to the behavior

$$G_f(t) = W_f e^{-\gamma t/4}[A_+ w(-iy_+\sqrt{t/\tau_f}) + A_- w(-iy_-\sqrt{t/\tau_f})]$$
(4.12)

with values $A_\pm$ and $y_\pm$ given by

$$y_\pm = -\frac{1}{2}\sigma_f \pm \frac{1}{2}\sqrt{\sigma_f^2 - 4}, \quad A_\pm = \frac{\pm y_\pm}{y_+ - y_-}.$$
(4.13)

The function $w(z) = \exp(-z^2)\text{erfc}(-iz)$ is related to the error function of complex argument [12].

The correlation function $G(t)$ is the sum $G_b(t) + G_f(t)$, where the bound-state contribution $G_b(t)$ is given by the sum in Eq. (3.7), and $G_f(t)$ by the integral. We approximate the sum by the first three terms plus an infinite sum determined from hydrogen P eigenfunctions and eigenvalues, as given by Eqs. (3.14)–(3.16) with $DZ^2 = \gamma$. In Fig. 7 we plot the approximate result for the correlation function $G(t)$ and compare with the exponential $\exp(-t/\tau_M)$. In Fig. 8 we compare the correlation function $G(t)$ with the bound-state contribution $G_b(t)$. The scattering states contribute significantly only for short times.

V. DISCUSSION

We have studied momentum relaxation for a particular form of the Fokker-Planck equation for the distribution function of momentum of a relativistic Brownian particle, as given by Eq. (2.1). The transformation to a Schrödinger-type equation, and a comparison with the eigenfunctions and eigenvalues of the hydrogen atom, allowed interesting conclusions on the nature of the relaxation spectrum.

Clearly the same method can be used for other moments with different angular dependence of the momentum distribution function. Also it may be applied to different forms of the Fokker-Planck equation [6] and in one dimension [3,5]. A quick comparison of different equations can be made on the basis of the short-time relaxation rate. In the present case this is given by the analytic expression Eq. (3.18). The method may also be useful in the mathematical study of the spectral properties of relativistic diffusion [13].

Both discrete rates, corresponding to bound states, and the continuum of scattering states contribute to the relaxation of momentum. The two contributions are characterized by the weights W_b and $W_f = 1 - W_b$. Mathematically there is an interesting connection with the theory of dissociation in electrolyte solutions [14–16].

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