Determination of the critical coupling of explosive synchronization transitions in scale-free networks by mean-field approximations

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An explosive synchronization can be observed in scale-free networks when Kuramoto oscillators have natural frequencies equal to their number of connections. The present paper reports on mean-field approximations to determine the critical coupling of such explosive synchronization. It has been verified that the equation obtained for the critical coupling has an inverse dependence on the network average degree. This expression differs from those whose frequency distributions are unimodal and even. In this case, the critical coupling depends on the ratio between the first and second statistical moments of the degree distribution. Numerical simulations were also conducted to verify our analytical results.

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I. INTRODUCTION

The synchronization of coupled oscillators has been intensively investigated because of its ubiquity in the real world [1,2]. When a collection of oscillators is coupled as a network, a synchronous state emerges [2,3]. Such an onset of coherent collective behavior has been verified between neurons in the central nervous system, communication networks, power grids, social interactions, animal behavior, ecosystems, and circadian rhythm [2].

The level of synchronization of a system is the consequence of a combination of the type of oscillators, the connectivity organization, the time-delay, and the interaction function [1,2]. Particularly, the network topology has a strong influence on the value of the critical coupling [4–8] and on the stability of the fully synchronized state [2,9–11]. For instance, Watts and Strogatz [12] verified that the decrease in the average shortest path length in small-world networks allows a more efficient coupling, enhancing the synchronization level. In addition, Nishikawa *et al.* [11] suggested that networks with a homogeneous degree distribution are more synchronizable than heterogeneous ones.

The network structure is important not only to enhance the level of synchronization, but also to permit the occurrence of phase transitions. Many works have verified second-order phase transitions in networks of Kuramoto oscillators [2]. Recently, Gardeñes *et al.* [13] showed that a first-order nonequilibrium synchronization transition can be observed in scale-free networks. They suggested that this event is a consequence of a positive correlation between the heterogeneity of the connections and the natural frequencies of the oscillators [13]. First-order phase transitions were also obtained experimentally and numerically by considering a network of Rössler units [14]. Indeed, such phenomenon has been attracting the interest of many researchers of complex networks (e.g., Refs. [14–16]).

Although the explosive synchronization has been observed in scale-free networks, the analytical expression that describes the critical coupling has not been determined yet. The present paper addresses this problem by considering mean-field approximations.

II. EXPLOSIVE SYNCHRONIZATION

The Kuramoto model considers a set of N oscillators coupled by the sine of their phase differences and phase oscillators at arbitrary frequencies [17]. Each oscillator is characterized by its phase $\theta_i(t)$, $i = 1,\dots,N$. In complex networks, each oscillator i obeys an equation of motion defined as

$$\frac{d\theta_i}{dt} = \omega_i + \lambda \sum_{i=1}^{N} A_{ij} \sin(\theta_j - \theta_i), \quad i = 1,\dots,N, \quad (1)$$

where λ is the coupling strength, ω_i is the natural frequency of oscillator i, and A_{ij} are the elements of the adjacency matrix A, so that $A_{ij} = 1$ when nodes i and j are connected and $A_{ij} = 0$ otherwise. The general Kuramoto model considers a random distribution of the natural frequencies and phases according to a specific distribution $g(\omega)$ [1,2]. In most of the cases, the frequency distributions are unimodal and symmetric around a mean value ω_0 [2].

Here, we considered a modified version of the Kuramoto model as proposed by Gardeñes *et al.* [13]. More specifically, the natural frequency ω_i of node i was assigned to be equal to its node degree k_i, i.e., $\omega_i = k_i$. Therefore, we have $g(\omega) = P(k)$, in which $P(k)$ is the degree distribution. This choice for the frequency distribution leads to the explosive synchronization in scale-free networks [13]. When the positive correlation between the network structure and dynamics is broken, a first-order transition is no longer observed, whereas a second-order transition occurs [13].

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III. MEAN FIELD APPROACH

In order to analyze the interplay between structure and dynamics in the Gardeñes *et al.* model, we considered the mean-field approach proposed by Ichinomiya [18]. First, we characterized the network according to its degree distribution $P(k)$ and introduced the density of the nodes with phase θ at time t for a given degree k, denoted by $\rho(k;\theta,t)$, which is normalized according to

$$\int_0^{2\pi} \rho(k;\theta,t)d\theta = 1. \tag{2}$$

The continuum limit of Eq. (1) is taken by considering the absence of degree correlation between the nodes in the network. Observe that this is a typical assumption in mean-field approximation [1]. In this regime, the probability that a random edge is attached to a node with degree k and phase θ at time t is given as

$$\frac{kP(k)\rho(k;\theta,t)}{\langle k\rangle}, \tag{3}$$

where $\langle k\rangle$ is the network average degree. Replacing $\omega_i = k_i$ in Eq. (1) and taking the continuum limit in the mean-field approach using Eq. (3), we obtained

$$\frac{d\theta(t)}{dt} = k + \lambda k \int dk' \int d\theta' \frac{k'P(k')}{\langle k\rangle}\rho(k';\theta',t)\sin(\theta'-\theta). \tag{4}$$

The order parameter, which quantifies the synchronization level of the network, is defined as [18,19]

$$re^{i\psi(t)} = \frac{1}{\langle k\rangle}\int dk \int d\theta kP(k)\rho(k;\theta,t)e^{i\theta}, \tag{5}$$

where $0 \leqslant r \leqslant 1$ and $\psi(t)$ is the average phase of the oscillators.

Multiplying Eq. (5) by $e^{-i\theta'}$, taking the imaginary part and including it in Eq. (4), we obtained

$$\frac{d\theta}{dt} = k + \lambda kr\sin(\psi-\theta), \tag{6}$$

which is Eq. (4) written in terms of the order parameter.

In order to define the equations of motion in function of known parameters of the network, we set a reference rotating frame $\psi(t) = \Omega t$, where Ω is the average frequency of the network. In the present model, i.e., $g(\omega) = P(k)$, the average frequency is equal to the network average degree ($\Omega = \langle k\rangle$) [13]. Defining a new variable as $\phi(t) \equiv \theta(t) - \psi(t)$ and replacing it in Eq. (6), we obtained

$$\frac{d\phi}{dt} = (1-\lambda r\sin\phi)k - \langle k\rangle. \tag{7}$$

The density of oscillators ρ can be redefined in terms of the new variable ϕ, i.e., $\rho = \rho(k;\phi,t)$. This density of oscillators must satisfy the continuity equation [18]

$$\frac{\partial\rho(k;\phi,t)}{\partial t} + \frac{\partial}{\partial\phi}[v_\phi\rho(k;\phi,t)] = 0, \tag{8}$$

where $v_\phi = \frac{d\phi}{dt}$. Since we were interested in the analysis of the steady state of the system, we obtained the time-independent solutions of Eq. (8), i.e.,

$$\rho(k;\phi) = \begin{cases} \delta\left(\phi - \arcsin\left[\frac{1}{\lambda r}\left(\frac{k-\langle k\rangle}{k}\right)\right]\right) & \text{if } \frac{|k-\langle k\rangle|}{k} \leqslant \lambda r, \\ \frac{A(k)}{|(k-\langle k\rangle)-\lambda kr\sin\phi|} & \text{otherwise,} \end{cases} \tag{9}$$

where $\delta(\cdot)$ is the Dirac delta function and $A(k)$ is the normalization factor. The first solution refers to the synchronous state, i.e., $\frac{d\phi}{dt} = 0$, corresponding to the oscillators entrained by the mean field. On the other hand, the second solution is the density of the nonentrained oscillators, i.e., $\rho(k;\phi) \sim \frac{1}{|v_\phi|}$ [3,18]. Thus, to compute the integrals in Eq. (5), we have redefined it in terms of variable ϕ and separated the contributions of entrained and nonentrained oscillators

$$\langle k\rangle r = \int\left(\int_{\frac{|k-\langle k\rangle|}{k}\leqslant\lambda r} dk + \int_{\frac{|k-\langle k\rangle|}{k}>\lambda r} dk\right) \times P(k)k\rho(k;\phi)e^{i\phi}d\phi. \tag{10}$$

Rewriting the second integral in Eq. (10) and noting that $\rho(k;\phi)$ is π periodic in ϕ, we obtained

$$\int_0^{2\pi}\int_{\langle k\rangle/(1-\lambda r)}^{\infty} P(k)k\frac{\sqrt{(k-\langle k\rangle)^2 - k^2\lambda^2r^2}}{2\pi\,(k-\langle k\rangle - k\lambda r\sin\phi)}e^{i\phi}dkd\phi$$

$$+\int_0^{2\pi}\int_{k_{\min}}^{\langle k\rangle/(1+\lambda r)} P(k)k\frac{\sqrt{(k-\langle k\rangle)^2 - k^2\lambda^2r^2}}{2\pi\,(\langle k\rangle - k + k\lambda r\sin\phi)}e^{i\phi}dkd\phi = 0$$

where $k_{\min}$ is the minimum degree in the network. Thus, only the contribution of the oscillators entrained in the mean-field is accounted in the summation of Eq. (10):

$$\langle k\rangle r = \int_{\langle k\rangle/(1+\lambda r)}^{\langle k\rangle/(1-\lambda r)} \exp\left\{i\arcsin\left[\frac{1}{\lambda r}\left(\frac{k-\langle k\rangle}{k}\right)\right]\right\}kP(k)dk. \tag{11}$$

From the imaginary part of Eq. (11), we obtained

$$\int_{\langle k\rangle/(1+\lambda r)}^{\langle k\rangle/(1-\lambda r)} kP(k)\frac{1}{\lambda r}\left(\frac{k-\langle k\rangle}{k}\right)dk = 0, \tag{12}$$

and from the real part,

$$\langle k\rangle r = \int_{\langle k\rangle/(1+\lambda r)}^{\langle k\rangle/(1-\lambda r)} kP(k)\sqrt{1-\frac{1}{\lambda^2r^2}\left(\frac{k-\langle k\rangle}{k}\right)^2}dk. \tag{13}$$

Considering $x = (k-\langle k\rangle)/\lambda r$, we obtained

$$\langle k\rangle r = \lambda r\int_{-\langle k\rangle/(1+\lambda r)}^{\langle k\rangle/(1-\lambda r)} P(\lambda rx+\langle k\rangle)(\lambda rx+\langle k\rangle) \times\sqrt{1-\left(\frac{x}{\lambda rx+\langle k\rangle}\right)^2}dx. \tag{14}$$

For $r \neq 0$ and letting $r \to 0^+$,

$$\langle k\rangle = \lambda\int_{-\langle k\rangle}^{\langle k\rangle} P(\langle k\rangle)\langle k\rangle\sqrt{1-\left(\frac{x}{\langle k\rangle}\right)^2}dx, \tag{15}$$

we achieved the critical coupling

$$\lambda_c = \frac{2}{\pi\langle k\rangle P(\langle k\rangle)}. \tag{16}$$

Therefore, the critical coupling presents an inverse dependence on the average network degree and $P(\langle k\rangle)$. This dependence is very different from that observed when other types of frequency distribution $g(\omega)$ are taken into account. For instance, if $g(\omega)$ is symmetric in relation to a single local maximum ω_0 (e.g., $\omega_0 = 0$), the critical coupling is given as [18,19]

$$\lambda_c^{(0)} = \frac{2}{\pi g(0)} \frac{\langle k\rangle}{\langle k^2\rangle}. \tag{17}$$

Thus, for scale-free networks with $P(k) \sim k^{-\gamma}$, where $\gamma \leqslant 3$, as $N \to \infty$ the critical coupling $\lambda_c^{(0)}$ become smaller, since the ratio $\langle k\rangle/\langle k^2\rangle$ diverges. On the other hand, in the case of $g(\omega) = P(k)$ this effect for large networks should not be observed when $N \to \infty$, once the critical coupling depends only on the average degree $\langle k\rangle$ of the network. Note that in the regime of the fully connected graph and considering $g(\omega)$ not correlated with the network topology, in Eq. (16) we have $\langle k\rangle \to N-1$ and $P(\langle k\rangle) \to g(\bar{\omega})$, where $\bar{\omega}$ is the average frequency. In this case, it is recovered the result $\lambda_c^{(K)} = \frac{2}{\pi g(\bar{\omega})}\frac{1}{N-1}$ which is the same critical coupling in the fully connected graph limit, i.e., $\lambda_c = [2/\pi g(\bar{\omega})][\langle k\rangle/\langle k^2\rangle]$.

IV. NUMERICAL ANALYSIS

A numerical simulation was considered in order to check the validity of Eqs. (9) and (16). We took into account networks generated by (i) the Barabási-Albert (BA) model, which are characterized by a distribution of connections following a power law [20], (ii) the configuration model, which allows to generate networks with a given degree sequence [21], and (iii) the random graphs of Erdős-Rényi (ER). We increased the coupling strength λ adiabatically and computed the stationary value of the global coherence r for each value $\lambda_0, \lambda_0 + \delta\lambda, \ldots, \lambda_0 + n\delta\lambda$, with increments $\delta\lambda = 0.02$, as in Ref. [13]. Figure 1 shows the dispersion of phases ϕ as a function of the node's degree k for a BA network with $N = 10^3$ nodes and $\langle k\rangle = 6$. As we can see, for $\lambda = 2.0$ the system starts to present a partial synchronization, suggesting that the critical coupling is between $\lambda = 1.0$ and $\lambda = 2.0$. Note that for $\lambda = 4.0$, the numerical results of phases ϕ are in good agreement with the theoretical solutions, especially for the highly connected nodes. Figure 2 presents the dependence of phases ϕ on degree k for an Erdős-Rényi (ER) network with $N = 10^3$ and $\langle k\rangle = 6$. As in Fig. 1, we observed the same behavior for the ER network; as coupling λ becomes higher, the phases approach the theoretical solution. Therefore, our results suggest that the solution of $\rho(k;\phi)$, given by Eq. (9), is valid.

Once the validity of Eq. (9) had been verified, we estimated the critical coupling considering numerical data. An ensemble of $N_{\rm net}$ networks, $\{N_1, N_2, \ldots, N_{\rm net}\}$, with the same number of nodes N and same average degree was considered. Then, the critical coupling λ_c was estimated as an average over this ensemble by Eq. (16). Figure 3 shows the coherence diagram of r as function of λ for ER networks with $N = 10^3, 2\times10^3$, and 3×10^3 nodes considering $\langle k\rangle = 6$ and $\langle k\rangle = 8$. For each value of N we averaged the critical coupling λ_c over an ensemble

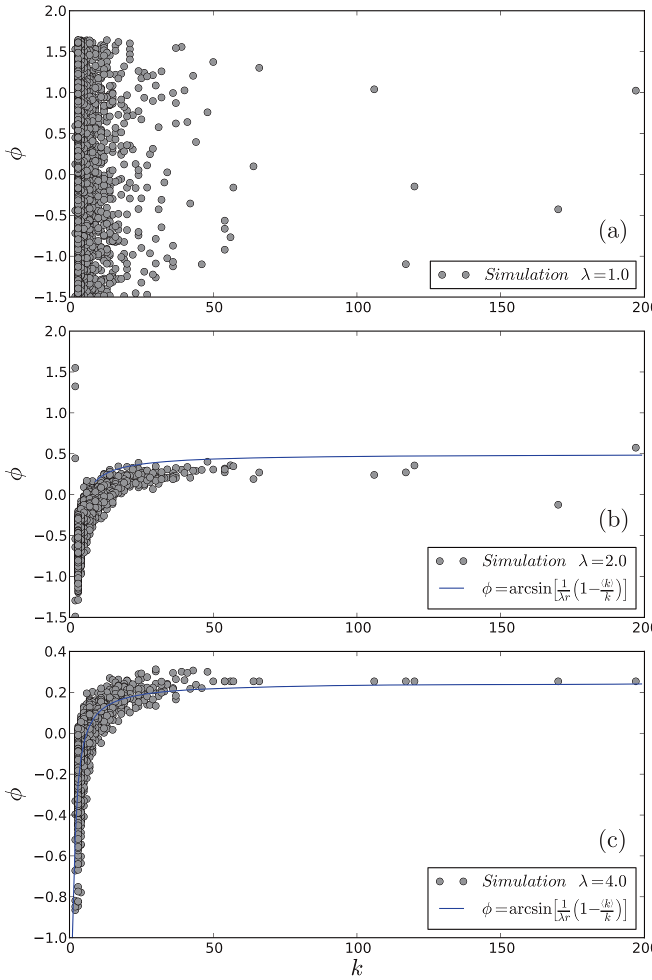

FIG. 1. (Color online) Distribution of the phases ϕ as a function of the degree k for a BA network with $N = 3\times10^3$ nodes and average degree $\langle k\rangle = 6$. The value of r considered in each theoretical curve (solid line) is calculated using the discrete form of Eq. (5), i.e., $re^{i\psi(t)} = \sum_i k_i e^{i\theta_i(t)}/\sum_i k_i$, at the given coupling λ.

of networks with the same values of $\langle k\rangle$ and N, obtaining $\lambda_c^{(N=1000)} \cong 0.65$, $\lambda_c^{(N=2000)} \cong 0.65$, and $\lambda_c^{(N=3000)} \cong 0.66$ for the ER networks with $\langle k\rangle = 6$. For networks with $\langle k\rangle = 8$, we obtained $\lambda_c \cong 0.57$ for all values of N.

Figure 4 shows the coherence diagram for the BA networks. Using the same procedure described above to estimate the critical coupling, we obtained the values $\lambda_c^{(N=1000)} \cong 1.51$, $\lambda_c^{(N=2000)} \cong 1.51$, and $\lambda_c^{(N=3000)} \cong 1.52$ for networks with $\langle k\rangle = 6$. For networks with $\langle k\rangle = 8$, we obtained $\lambda_c^{(N=1000)} \cong 1.44$, $\lambda_c^{(N=2000)} \cong 1.44$, and $\lambda_c^{(N=3000)} \cong 1.45$. These values are in good agreement with the results of numerical simulation. Also, Fig. 5 shows the explosive synchronization in scale-free networks with degree distribution $P(k) \sim k^{-\gamma}$ constructed using the configurational model [21] with $\gamma = 2.4, 2.6, 2.8$, and 3.0 and $\langle k\rangle = 8$. Note that the critical coupling for the configurational model with $\gamma = 3.0$ is the same as observed for the BA networks with the same average degree. This result agrees with Eq. (16), since λ_c depends only on $\langle k\rangle$ and $P(\langle k\rangle)$.

In order to determine the dependence of the critical coupling λ_c on the network size N and to compare the theoretical and

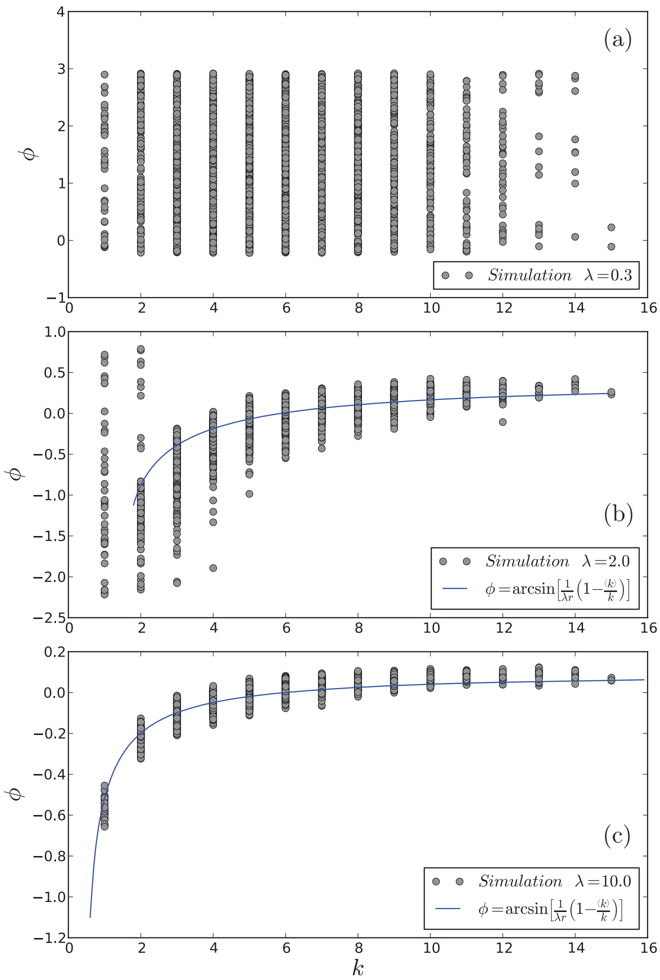

FIG. 2. (Color online) Distribution of the phases ϕ as a function of the degree k for a ER network with $N = 3 \times 10^3$ nodes and average degree $\langle k \rangle = 6$. The value of r considered in each theoretical curve (solid line) is calculated using the discrete form of Eq. (5), i.e., $re^{i\psi(t)} = \sum_i k_i e^{i\theta_i(t)} / \sum_i k_i$, at the given coupling λ.

numerical results more precisely, we considered finite-size effects. For the ER networks, we assumed the following scaling form of the order parameter [22–24],

$$r = N^{-\alpha} F[(\lambda - \lambda_c) N^{\beta}], \tag{18}$$

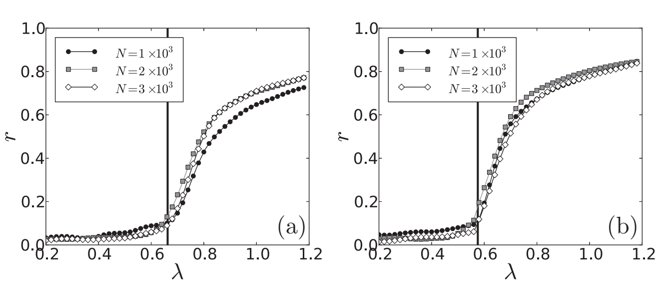

FIG. 3. Synchronization diagram for ER networks with forward continuation of the coupling strength λ with steps of $\delta\lambda = 0.02$. The networks have $N = 10^3$, 2×10^3, and 3×10^3 and the same average degree (a) $\langle k \rangle = 6$ and (b) $\langle k \rangle = 8$. Each point is an average over 30 networks. The error bars have the size of the points.

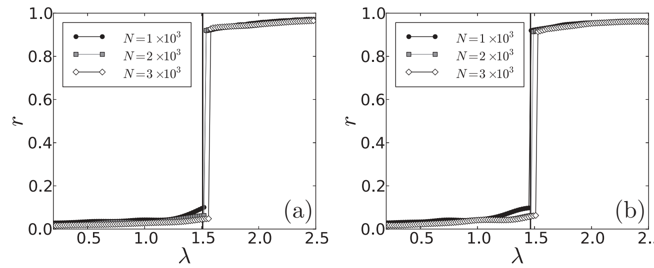

FIG. 4. Synchronization diagram for BA networks with forward continuation of the coupling strength λ with steps of $\delta\lambda = 0.02$. The networks have $N = 10^3$, 2×10^3, and 3×10^3 and the same average degree (a) $\langle k \rangle = 6$ and (b) $\langle k \rangle = 8$. Each point is an average over 30 networks. The error bars have the size of the points.

where F is the scaling function. We estimated the scaling parameters α and β through the scaling plots of rN^{α} against $(\lambda - \lambda_c)N^{\beta}$ by adjusting the parameters α and β considering different values of N in order to collapse them. Figure 6 shows the scaling plots of the order parameter r for several network sizes. As we can see, the data collapse satisfactorily for $\alpha \sim 0.02$ and $\beta \sim 0.07$ for networks with $\langle k \rangle = 6$. For networks with $\langle k \rangle = 8$, we obtained $\alpha \sim 0.02$ and $\beta \sim 0.05$. Thus, the critical coupling does not suffer significant variations with the network size.

Therefore, in contrast to the case where the frequency distribution $g(\omega)$ is unimodal and even, in which the critical coupling tends to vanish as $N \to \infty$, the consideration $\omega_i = k_i$ implies that the critical coupling does not suffer significant variations. This fact can be observed in Fig. 4. In addition, we have not verified that for the forward continuations of λ, the critical coupling increases extensively with the number of nodes. This result was obtained in Ref. [13], where authors considered a star network as an approximation of scale-free networks. Therefore, although star networks exhibit the first

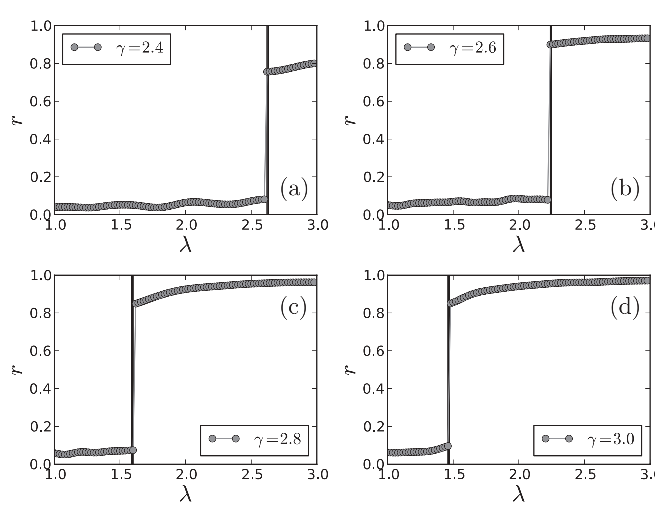

FIG. 5. Coherence diagrams for configurational models with degree distribution $P(k) \sim k^{-\gamma}$: (a) $\gamma = 2.4$ (b) $\gamma = 2.6$ (c) $\gamma = 2.8$, and (d) $\gamma = 3.0$, with forward continuation of the coupling strength λ with steps of $\delta\lambda = 0.02$.

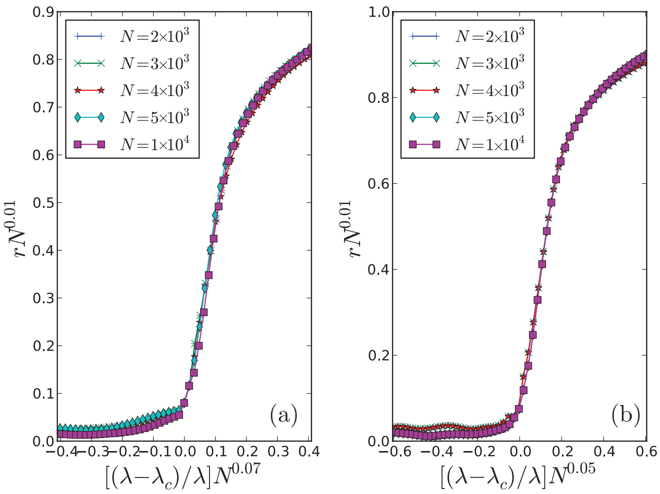

FIG. 6. (Color online) Scaling plot of r for ER networks with (a) $\langle k\rangle = 6$ and (b) $\langle k\rangle = 8$.

order phase transition, the critical coupling does not have the same behavior as verified in scale-free networks.

V. CONCLUSIONS

The analysis proposed here helps to understand the recently observed phenomena of explosive synchronization in scale-free networks. The obtained expression for the critical coupling does not depends on the ratio $\langle k\rangle/\langle k^2\rangle$, as observed in the case that $g(\omega)$ is symmetric. Indeed, the obtained critical coupling has a inverse dependence with the network average degree, $\langle k\rangle$ and $P(\langle k\rangle)$.

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