Two-actor conflict with time delay: A dynamical model

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Recent mathematical dynamical models of the conflict between two different actors, be they nations, groups, or individuals, have been developed that are capable of predicting various outcomes depending on the chosen feedback strategies, initial conditions, and the previous states of the actors. In addition to these factors, this paper examines the effect of time delayed feedback on the conflict dynamics. Our analysis shows that under certain initial and feedback conditions, a stable neutral equilibrium of conflict may destabilize for some critical values of time delay, and the two actors may evolve to new emotional states. We investigate the results by constructing critical delay surfaces for different sets of parameters and analyzing results from numerical simulations. These results provide new insights regarding conflict and conflict resolution and may help planners in adjusting and assessing their strategic decisions.

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I. INTRODUCTION

Understanding the different causes and aspects of the real-life conflicts between groups of people and/or individuals and the interactions of these causes plays a major role in the pursuit of peace and stability. Oftentimes, misunderstanding or even not considering some aspects of these conflicts could lead to the failure of peace talks or negotiations that aim to resolve conflicts or at least bring together the views of the conflicting actors (be they nations, groups, or individuals). Such failure may end up escalating the conflict and bringing it to a complicated level of instability and/or negotiation gridlock.

Mathematical modeling of a conflict may help uncover its mechanism [1] and yield insights into how the dynamics of a conflict depends on the interactions between actors. The analysis of the structure of these models can determine their logical consequences and effects on system stability. These models also provide new predictions about conflict that can motivate social and political researchers to design experiments to test them [2]. Several mathematical models of conflict were developed with a common feature of how an actor can respond to another. Some of these models consider that the reaction functions, such as cooperation and competition between actors, are logical representations of many types of conflict [3,4]. Other models developed qualitative dynamical systems metaphors [5] (see also [6]), while others [7,8] relied on qualitatively defined response functions between the actors. Linear models [9] and piecewise linear models [10–12] were also developed (for more detailed survey of the dynamical models of conflict, see [13,14] and the references therein). In recent works [2,14], Liebovitch *et al.* built on the previously mentioned models and their insights to develop a nonlinear ordinary differential equation model of the conflict between two actors. Interestingly, some results of Liebovitch's model are consistent with previously observed characteristics of conflicts. The model also provides testable predictions of a conflict outcome depending on strategies or conditions chosen by the actors.

On the other hand, over the past few decades there has been extensive work studying the effect of time delay of response functions on the stability of dynamical systems [15–19]. However, none of the dynamical conflict models considered the effect of time delayed response (feedback) between actors. Cook *et al.* [20] mentioned that time delays can cause cyclical behavior in their model of marital conflict, but no systematic analysis was presented. In sum, most models of conflict assumed that the feedback between the actors to be instantaneous and simply neglected the effect of the time elapsed between the moment an action is taken by one actor and the moment that it is received by the other.

In this paper, we modify the nonlinear dynamical model presented in [2,14] by considering nonlinear delayed response between two actors. We provide detailed analysis of the effect of time delay on the stability of a steady state for various scenarios of cooperation, competition, and mixed feedbacks between two actors. We obtain critical delay surfaces for some cases and discuss the effect of symmetry breaking on the stability of the neutral state (i.e., when the emotional states of both actors are at their "uninfluenced" levels; these are the levels that would have been reached had there been no interaction between them).

II. MODEL DESCRIPTION

In this section, we develop a model for the dynamics of the conflict between two actors. We begin by briefly discussing the notion of "emotional states." While each real-life conflict has its own details and unique historical background [2], these conflicts may share some common features. One feature is that the state of the conflict depends on its previous state. Also, each actor in the conflict interacts and responds to the other actor [2]. Such interaction among the system elements, with each element adjusting to others, gives rise to higher order coherent patterns [21]. In dynamical models of these conflicts [13], emotional states may represent beliefs, behaviors, opinions, or feelings of actors. As an example, in a dynamical model of marital conflict, the variables represent the feelings (hostility

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or friendliness) of the wife, husband, and child indicated by the number of angry facial expressions per unit time [10], which can be considered as a quantization of their emotional states. Such a state of an actor of course changes over time depending on various factors such as the actor's motives and attitude, as well as feedbacks from such states of other actors, which together drive the conflict-interaction patterns of the disputants [13]. Once viewed in this way, it seems natural that mathematical tools in dynamical systems have been applied to the problems of conflict (see the Introduction).

Now, following [2], let $x_1 = x_1(t)$ and $x_2 = x_2(t)$ represent the emotional or behavioral states at time t of the first and the second actors, respectively. The dynamics of the system can then be described by the following set of equations:

$$\frac{dx_1}{dt} = m_1(x_1 - b_1) + f_{12}(x_2,x_1), \qquad \frac{dx_2}{dt} = m_2(x_2 - b_2) + f_{21}(x_1,x_2), \tag{1}$$

where

$$f_{12}(x_2,x_1) = C_{12}\tanh(\alpha x_2), \qquad f_{21}(x_1,x_2) = C_{21}\tanh(\alpha x_1). \tag{2}$$

The above system can be written as follows:

$$\dot{x}_i = m_i(x_i - b_i) + f_{ij}(x_j,x_i), \tag{3}$$

where

$$f_{ij}(x_j,x_i) = f_{ij}(x_j) = C_{ij}\ \tanh(\alpha x_j), \tag{4}$$

with the subscripts $i,j = 1,2$ and $i \neq j$ represent the two actors, where $\dot{x}_i$ represents the time derivative of the emotional state x_i. The parameter, $m_i < 0$ is the "inertia" term which represents the tendency of an actor to remain in the same state for a period of time [10]. In fact, this term is nothing but the time constant of an exponential relaxation of the first order differential equation; we follow Gottman *et al.* and Liebovitch *et al.* [2,10,11,14] and use the same terminology. The uninfluenced state of actor i is represented with b_i. The influence function given by Eq. (4) represents the effect of response by actor j to actor i, where $\alpha > 0$ is a constant (we set $\alpha = 0.5$ for all calculations of this work), and C_{ij} denotes the response strength. Cooperation is modeled as a positive feedback while competitions is modeled as a negative feedback.

In this study, we consider cases in which the feedback between the two groups is not instantaneous. This means that there will be some significant period of time before the two groups respond to each other. For example, one can assume that the negotiation of the two groups is being mediated by a third party who carries the suggestions, requests, and responses from one group to the other. The existence of the mediation process and, probably, the internal deliberations between the members within each group itself, will take a considerable period of time for one group to get the response from the other one. This period of time can be considered as a *delay*. As a result, the system of Eqs. (3) can be modified as follows:

$$\dot{x}_i = mx_i + C_{ij}\tanh[\alpha x_j(t - \tau_j)], \tag{5}$$

where τ_j denotes the time delay defined as the time elapsed before actor x_j responds to actor x_i. Here, we consider the same conditions for the inertia (m_i) and the uninfluenced state (b_i) as discussed in [2], namely $m_1 = m_2 = m = 0$ and $b_1 = b_2 = 0$, but with $\alpha = 0.5$. Here, it is worth pointing out some properties of the hyperbolic tangent function chosen to represent the feedback. At small influence levels, this (sigmoid) function is almost linear. This enables each actor to influence the other one approximately proportionately in magnitude. Moreover, this function is bounded when emotional states are large, preventing each actor's emotional state from escaping to infinity (see [2,14] for more discussion and graphical explanation).

In the following analysis, we focus on studying the stability of the fixed point x_j^* (i.e., equilibrium) for different values of m, C_{ij}, and time delay τ_j. For simplicity, we consider only cases where both actors take equal time delay to respond to the other, that is, $\tau_1 = \tau_2 = \tau$. Performing the linear stability analysis around the fixed point x_j^* such that $x_i = x_i^* + x_i'$, the above system becomes

$$\dot{x}_i'(t) = mx_i'(t) + \alpha C_{ij}\,\text{sech}^2(\alpha x_j^*)\, x_j'(t - \tau). \tag{6}$$

III. MODEL ANALYSIS

Notice that time delay does not change the number and location of equilibrium points. Because $\text{sech}^2(\alpha x_j^*) > 0$, the above equation can be rescaled to

$$\dot{x}_i'(t) = mx_i'(t) + \widetilde{C}_{ij}\ x_j'(t - \tau), \tag{7}$$

where $\widetilde{C}_{ij} = \alpha C_{ij}\,\text{sech}^2(x_j^*)$. For notational simplicity, the symbols prime ($'$) and tilde ($\widetilde{\ }$) are dropped from this point onward. Several papers studied the stability properties of the above delay system in detail [22,23] and obtained the stability surface of the stable fixed points for various values of m, C_{ij}, and τ. In this paper, our concern is on the zero fixed point, that is, $(x_1,x_2) = (0,0)$ and we refer to it as the *neutral state* hereinafter. Following [23], Eq. (7)—the linearized system around the neutral state—can be written in the vector/matrix form as

$$\dot{\mathbf{x}}(t) = m\mathbf{x}(t) + \mathbf{C}\mathbf{x}(t - \tau), \tag{8}$$

where $\mathbf{x}(t) = [x_1(t) \quad x_2(t)]^T$ and $\mathbf{C} = [C_{ij}]$ represents the matrix of feedback strength.

At this point, it is useful to briefly discuss the notions of "cooperation" and "competition" in this model and how they relate to those in other, more established and more familiar, models of interacting populations such as predator-prey (Lotka-Volterra) [24,25] or cooperator-defector (evolutionary game) [26–29] systems. In all these models, the cooperation/competition notions are defined by the sign or direction of change in abundance, magnitude, frequency, or density (depending on the contexts) of one population caused by such changes in the other(s). As such, our conflict model can be considered conceptually similar to these models for some choices of elements C_{ij} of $\mathbf{C}$. The difference, however, lies in the dependence of the interaction or feedback, or in our terminology influence function, on the state of actor of interest. In the predator-prey or cooperator-defector systems, the interaction depends on the densities or frequencies of *both* populations, typically in a mass-action manner (i.e., product of two densities), reflecting the probability that the two interacting agents come from different populations. In contrast, our influence function [Eq. (4)] depends *only* on the state of the other actor; this is because there are only two agents in the system, who interact with each other with certainty by construction. Recognizing these conceptual similarity and difference would allow us to draw lessons and tools from these more established models to apply to the further analysis of conflict. Following [22,23], we can decompose $\mathbf{C}$ according to $\mathbf{C} = \mathbf{E}^{-1}\Lambda\mathbf{E}$, where Λ is the Jordan form with eigenvalues $\lambda = \pm\sqrt{C_{12}C_{21}}$ and matrix $\mathbf{E}$ contains the corresponding eigenvectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$. Multiplying Eq. (8) on the left by $\mathbf{E}$, we obtain

$$\mathbf{E}\dot{\mathbf{x}}(t) = m\mathbf{E}\mathbf{x}(t) + \Lambda\mathbf{E}\mathbf{x}(t-\tau). \tag{9}$$

If we project $\mathbf{x}(t)$ onto the ith eigenvector $\hat{\mathbf{e}}_i$, we obtain their time-dependent coefficients $u_i(t) = \hat{\mathbf{e}}_i \cdot \mathbf{x}(t)$, and as a result, Eq. (9) reduces to the following decoupled representation of the system dynamics:

$$\dot{u}(t) = m\,u(t) + \lambda\,u(t-\tau), \tag{10}$$

where we have dropped the index i for clarity. Assuming a solution of the form $u(t) = e^{zt}$, $z \in \mathcal{C}$, the stability condition is determined by the following characteristic equation:

$$H(z) = z - m - \lambda e^{-z\tau} = 0. \tag{11}$$

The solution of Eq. (8) is stable if all solutions of Eq. (11) satisfy $\mathrm{Re}[z] < 0$. When $\mathrm{Re}[z]$ is positive, the system destabilizes, and the critical value is obtained when $\mathrm{Re}[z] = 0$. According to a theorem of Datko [30], the possibility of a change of sign of $\mathrm{Re}[z]$ by way of $\mathrm{Re}[z] = \infty$ is excluded. Hence, the change of sign of $\mathrm{Re}[z]$ must occur at $z = i\omega$.

We consider two examples: (i) a symmetric response case ($C_{12} = C = C_{21}$) with the real eigenvalues $\lambda = \pm\sqrt{C^2} = \pm|C|$; and (ii) an asymmetric response case ($C_{12} = C = -C_{21}$) with the purely imaginary eigenvalues $\lambda = \pm i\sqrt{C^2} = \pm i|C|$, for any value of $m < 0$. The reason for choosing these two cases will be clear in the next sections. For these two cases, it is worth noting that λ is either real or purely imaginary but cannot be complex (having both real and imaginary parts). On the critical delay surface parameterized by m, C, and τ, the solution z of the characteristic equation is purely imaginary, that is, $z = i\omega$, $\omega \in \mathcal{R}_0^+$. This surface represents the boundary at which the system bifurcates; that is, the neutral state $(x_1,x_2) = (0,0)$ changes its stability. Substituting in Eq. (11) and separating the real and imaginary parts, we obtain

$$-m = \mathrm{Re}[\lambda]\ \cos(\omega\tau) + \mathrm{Im}[\lambda]\ \sin(\omega\tau), \tag{12}$$

$$\omega = \mathrm{Im}[\lambda]\ \cos(\omega\tau) - \mathrm{Re}[\lambda]\ \sin(\omega\tau), \tag{13}$$

where $\omega^2 = C^2 - m^2$. The above equations provide the condition for the smallest positive value of τ and the critical values of the response strength for which a change in stability may occur [18]. In the symmetric case, there is no solution of the characteristic equation in the parameter subspace for $\tau > 0$; hence, the system will remain as it is for $\tau = 0$, as shown in Fig. 1(a). In the asymmetric case, the system is stable at $\tau = 0$ for all values of c and for all values of m (where $m < 0$). The system destabilizes for $\tau > \tau^* = \omega^{-1}\arccos(\omega/m)$ in the region of $|C| > |m|$ and remains stable in the region of $|C| < |m|$, where Eq. (11) has no solution. The stability surface for the asymmetric case is illustrated in Fig. 1(b).

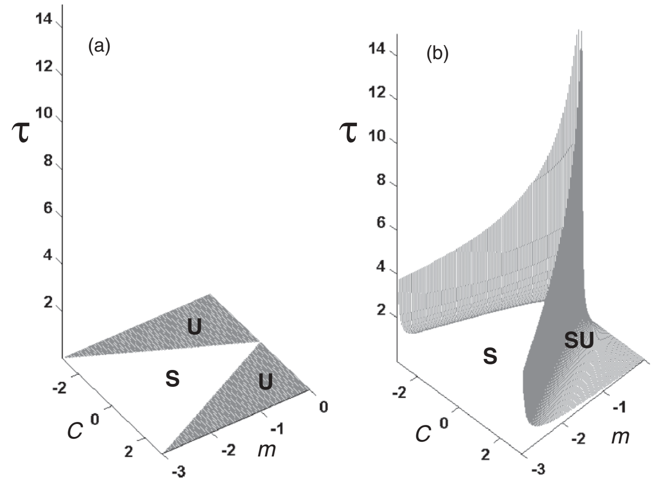

FIG. 1. Stability surfaces for the system described by Eq. (7). The critical stability surfaces define the minimal delay value separating stable and unstable regions of the neutral state ($x_1 = 0, x_2 = 0$) as a function of the feedback strength c and the intrinsic inertia parameter m for (a) symmetric response strengths ($C_{12} = C = C_{21}$) representing the cooperation and competition feedback and (b) asymmetric response strengths ($C_{12} = C = -C_{21}$) representing the mixed (positive-negative) feedback. The region denoted by **S** shows the stable areas while **U** shows the regions where the system is always unstable independent of time delay τ. **SU** shows the regions where the system is unstable above the critical surface and stable below it.

These two particular cases of symmetric and asymmetric feedbacks provide insights for the general case with any combination of feedback strengths. For a fixed value of $m < 0$, we obtain the stability surface parameterized by the two different feedback strengths C_{12} and C_{21} with time delay τ, as shown in Fig. 2. The cases of symmetric and asymmetric

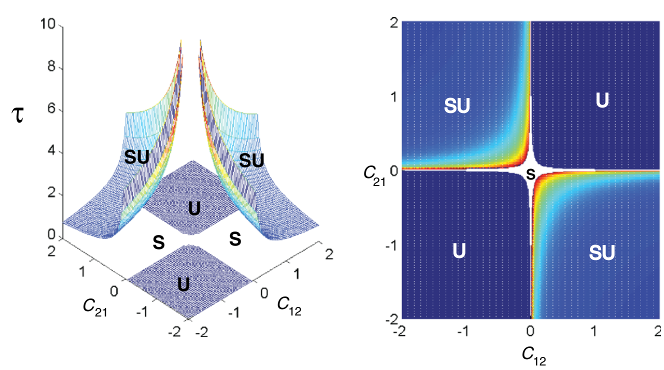

FIG. 2. (Color online) Stability surface for the neutral state of the system described by Eq. (7) with delay at a fixed value of the inertia parameter $m = -0.1$ (this value is chosen only for a better visual representation although all values $m < 0$ return similar results). The surface defines the minimal delay value as a function of the feedback strengths of the two actors, C_{12} and C_{21}. The region denoted by **S** shows the stable regions while **U** shows the regions where the system is always unstable. **SU** shows the regions where the system is unstable above the critical surface and stable below it. The right panel is the top view of the surface shown in the left panel.

feedbacks considered above correspond to diagonal lines of this figure.

In addition, we can show, using the characteristic polynomial $H(z)$ for $z = i\omega$, that

$$\frac{d\mathrm{Re}[z]}{d\tau} = -\mathrm{Re}\left[\frac{\partial H/\partial \tau}{\partial H/\partial z}\right] = \frac{\omega^2}{|K|^2} > 0, \qquad (14)$$

where $K = \lambda\tau + e^{i\omega\tau}$ [see [23] for proof of the result obtained in Eq. (14)]. This result implies that as τ increases across the critical surface from below, the equilibrium, once destabilized, will remain unstable for all τ larger than the critical value τ^*. In other words, for $\tau > \tau^*$, one has to change other parameters (such as feedback strengths or the inertia) for the stability of the system to be maintained.

IV. RESULTS AND DISCUSSION

A. Weak feedback

Although there are many combinations of parameters that one can investigate, we limit ourselves to those discussed in [2]. That is, the two actors have the same inertia ($m_1 = m_2 = m = -0.9$) and the same uninfluenced state ($b_1 = b_2 = b = 0$). On the other hand, we consider various combinations of feedback strengths between the actors (C_{12} and C_{21}) with different initial conditions and with different values of time delay. However, we assume that the two actors will have identical time delay, that is, $\tau_1 = \tau_2 = \tau$. The feedback/response combination types include positive-positive feedback (cooperation), negative-negative feedback (competition), and the mixed case with positive-negative feedback. It is worth noting that the first two types of feedback represent the symmetric case, while the mixed feedback type represents the asymmetric case discussed in the previous section.

For all three types of feedback (cooperation, competition, and mixed), if the response strength is *weak* (i.e., the magnitudes of feedback strengths $|C_{ij}|$ are below a threshold equal to the inertia to change $|m|$), the two actors will evolve to the neutral state $(x_1,x_2) = (0,0)$ for any value of time delay $\tau \geqslant 0$ and regardless of their initial conditions. This result can be observed by considering the points corresponding to parameter values $|C_{ij}| = 0.5 < |m| = 0.9$ in Fig. 1(a) for the symmetric feedback types (cooperation and competition) and Fig. 1(b) for the asymmetric (mixed) type (see also Fig. 2 for a specific inertia value $m < 0$, where the former cases of symmetric and asymmetric feedbacks correspond to the different diagonals of this general case). Numerical integrations in time for weak feedback without and with time delay for the three feedback types are shown in Fig. 3. The results show that even for large values of time delay, the two actors will evolve to the neutral state, just as they do when they respond instantaneously (i.e., without time delay), but with longer relaxation times.

B. Strong positive-positive (cooperation) feedback

In this case, as the feedback strength exceeds the inertia ($C_{ij} > |m|$), there is a bifurcation in the dynamical behavior for all time delay values and any initial emotional states of the two actors. However, as shown in Fig. 4, when the initial values of the two actors are of different signs, both actors will oscillate out of phase in their approach to a new non-neutral state [i.e., $(x_1,x_2) \neq (0,0)$]. As time delay increases, the period with oscillatory behavior persists longer before the actors finally settle onto the new state. It is worth noting that

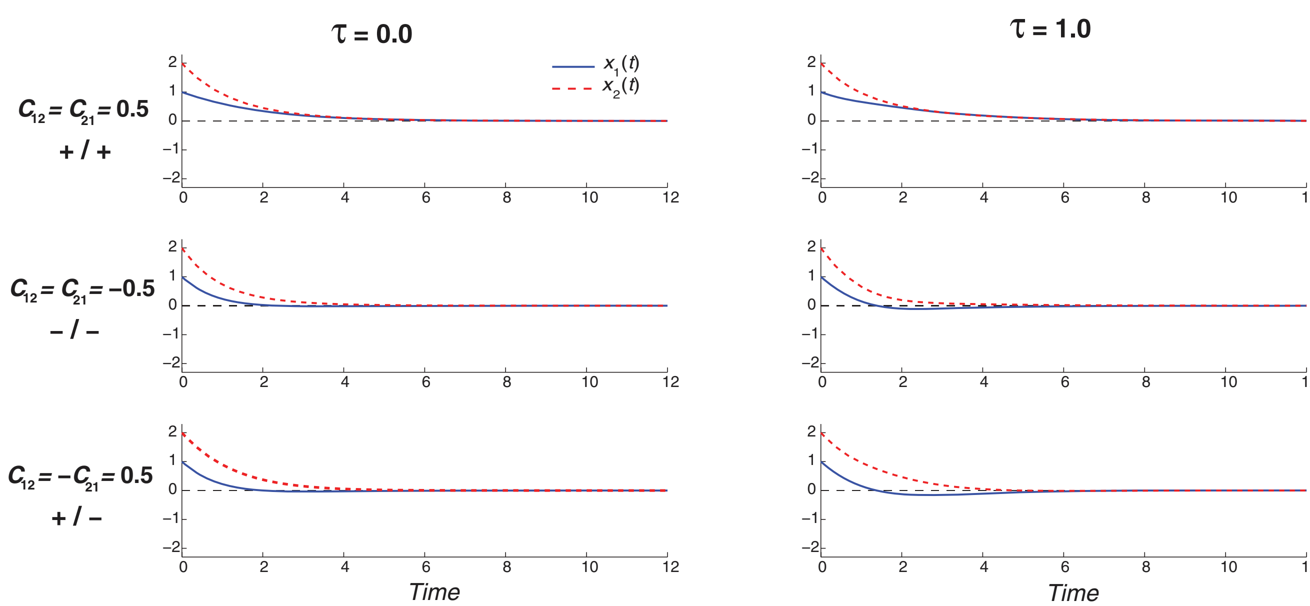

FIG. 3. (Color online) Time series of emotional states of the two actors, $x_1(t)$ (solid line) and $x_2(t)$ (dashed line), with weak feedback strengths ($|C_{12}| = |C| = |C_{21}| = 0.5 < |m| = 0.9$) for all types of feedback. Numerical integration is used to calculate $x_1(t)$ and $x_2(t)$ as the emotional states of the two actors evolve from their initial values of $x_1(0) = 1$ and $x_2(0) = 2$ for the three feedback types (rows) when the feedback strength ($|C|$) is less than the threshold equal to the absolute value of the inertia to change, that is, $|C| < |m|$. The first column shows the results when there is no time delay and the second column with time delay, $\tau = 1$. Note that the two actors will evolve to the neutral state $x_1(t) = 0$ and $x_2(t) = 0$ in all feedback types for any value of time delay.

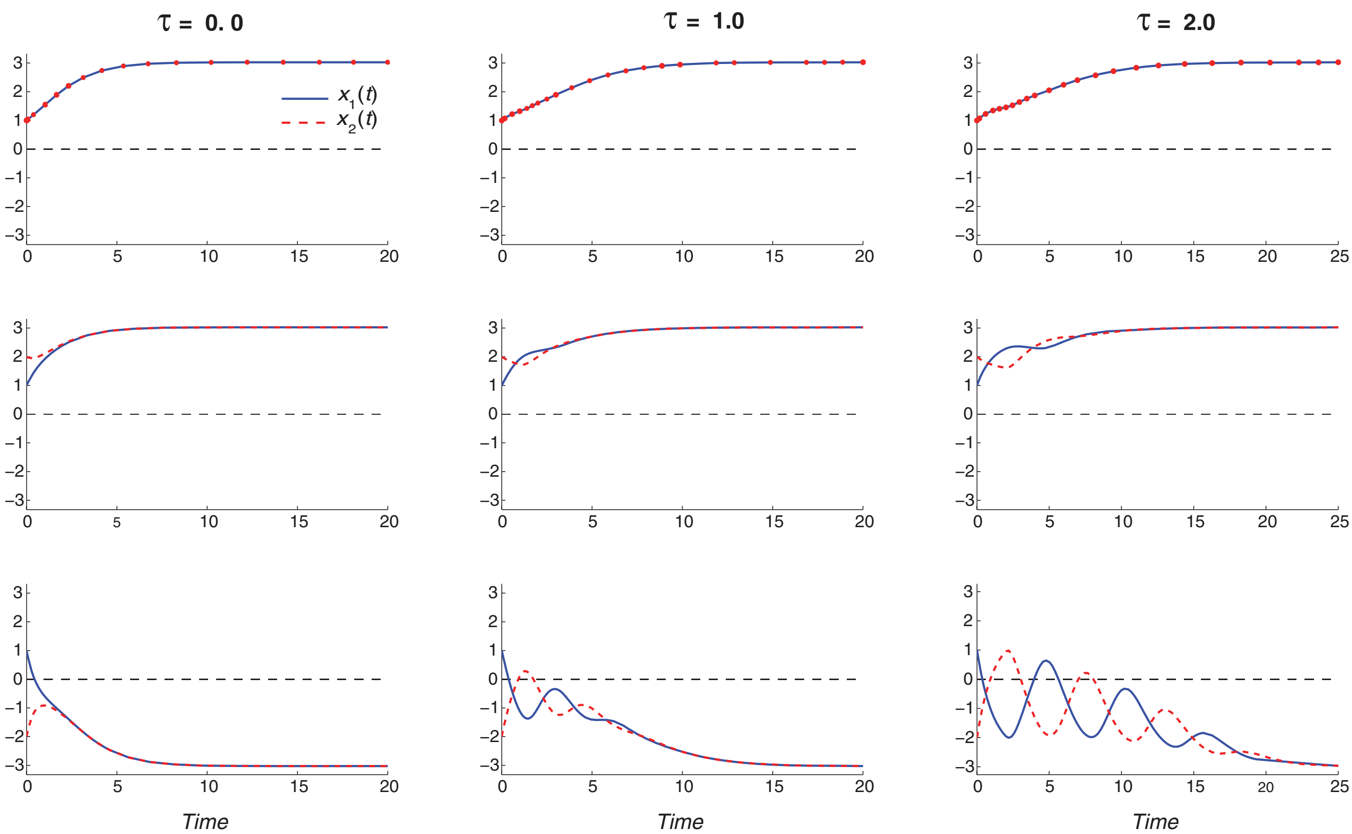

FIG. 4. (Color online) Time series of emotional states of the two actors, $x_1(t)$ (solid line) and $x_2(t)$ (dashed line), with strong positive-positive feedbacks ($C_{12} = C_{21} = 3$) at different time delay values (columns) with $m = -0.9$. The values of $x_1(t)$ and $x_2(t)$ are shown as the states of the two actors evolve from different initial values (rows). The values of $x_1(t)$ and $x_2(t)$ always evolve toward the stable states where either both are positive or both are negative for any value of time delay $\tau \geqslant 0$. When the initial conditions are different, both actors will oscillate out of phase as they evolve towards the same final state. The actor of stronger initial emotional state (larger magnitude) will drag the other towards the new state that is closer to its initial side.

the actor with an initial value of larger magnitude, regardless of its sign, will attract the other to its side and evolve together to a new state other than the neutral state.

C. Strong negative-negative (competition) feedback

In the case of a strong negative-negative feedback ($C_{ij} < m < 0$), numerical simulations in Fig. 5 show that the two actors of different initial conditions will evolve to two different states other than the neutral state [$(x_1,x_2) = (0,0)$] for any value of time delay, $\tau \geqslant 0$. We call the actor with a positive final state "winner" and that with a negative final state the "loser." These two final states can be of different signs even if the initial emotional states are of the same sign. Surprisingly, when the two actors have different initial conditions, although of the same sign, oscillatory behavior is detected with the existence of time delay. This oscillatory behavior will last before the two actors split and evolve to their new and opposite emotional states. The period with significant oscillatory behavior increases as the time delay increases.

When the actors have exactly the same parameters and identical initial conditions, the scenario will be quite different. In this case, in the absence of time delay the two actors will evolve monotonically towards the neutral state as shown in [2]. When time delay increases, but remains less than some critical value, the two actors will oscillate around the neutral state before they finally settle onto it (i.e., damped oscillation). As time delay increases, the two actors will destabilize, oscillating around the neutral state with growing amplitudes towards a final fixed amplitude.

D. Strong positive-negative (mixed) feedback

When there is a positive feedback from one actor while the other actor responds negatively (i.e., asymmetric response strengths with purely imaginary eigenvalues λ), numerical integrations in Fig. 6 show that the emotional states of both actors oscillate with decaying amplitude as they evolve toward their neutral state if the time delay is less than a critical value, $\tau < \tau^*$. If the time delay exceeds this particular critical value for given parameters, the neutral state will destabilize and approach an oscillation with a fixed amplitude. According to Fig. 1(b), this scenario means that the system crosses the critical delay surface and becomes unstable for $\tau > \tau^*$. It is worth noting that there will be some phase shift between the two actors although they may have identical initial emotional state and time delay. This implies that the out-of-phase

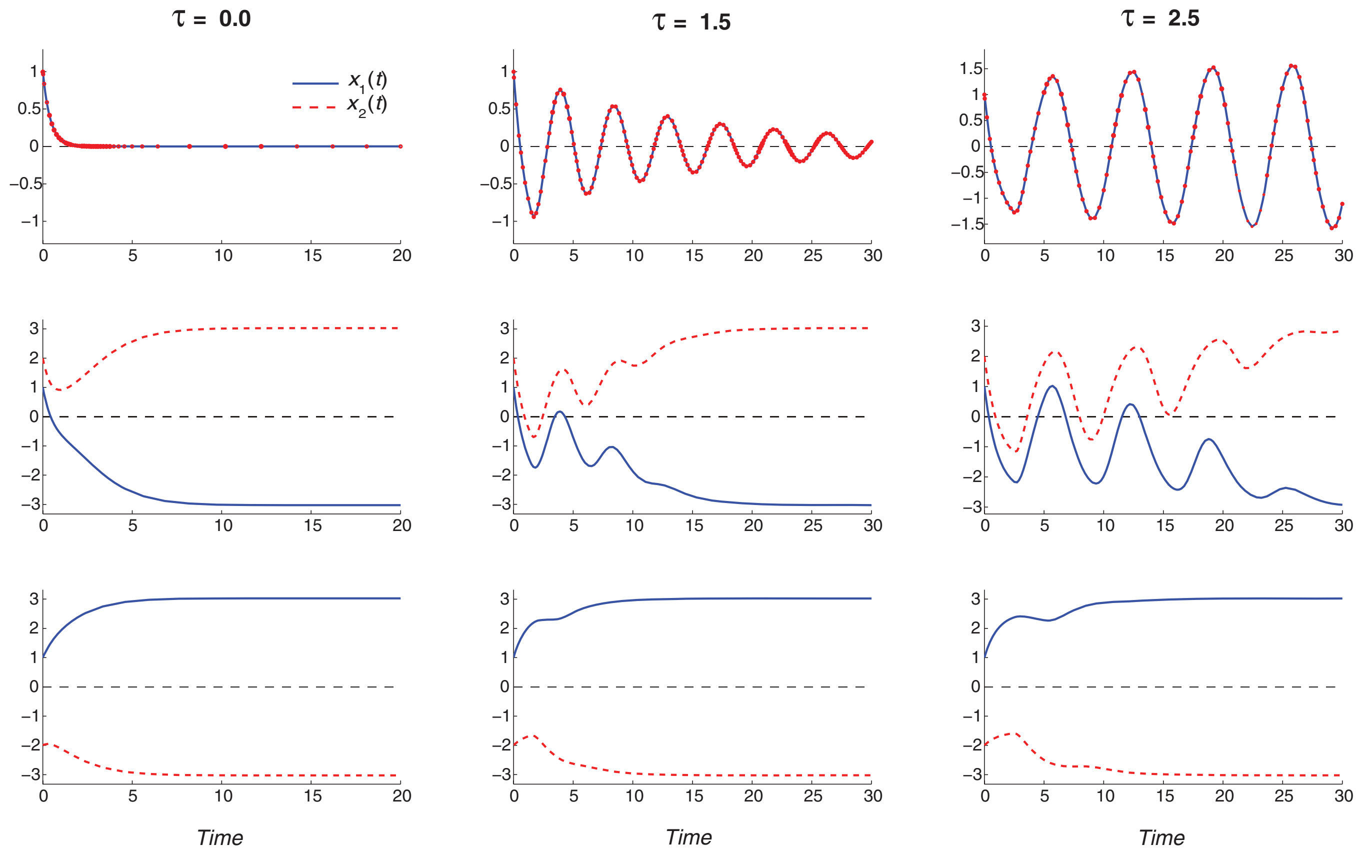

FIG. 5. (Color online) Time series of emotional states of the two actors, $x_1(t)$ (solid line) and $x_2(t)$ (dashed line), with strong negative-negative feedbacks ($C_{12} = C_{21} = -3$) at different time delay values (columns) with $m = -0.9$. The values of $x_1(t)$ and $x_2(t)$ are shown as the states of the two actors evolve from different initial values (rows). In the absence of time delay, both actors, with the same parameters and the same initial conditions, will evolve to the neutral state. If time delay is less than a critical value, the two actors will oscillate about the neutral state with decaying amplitude. As time delay exceeds the critical value, both actors will oscillate in phase about the neutral state but with growing amplitudes. When the initial conditions are not the same, the two actors will oscillate for a short period of time and move away from each other toward two opposite states. The larger the time delay, the longer the period with significant oscillatory behavior.

oscillatory behavior of the two actors is not sensitive to the initial conditions of the two actors.

E. The effect of one actor unilaterally temporarily reversing its feedback

In the absence of time delay, Figs. 4 and 5 show that both systems with the positive-positive (cooperation) feedback and negative-negative (competition) feedback types reach fixed points. Liebovitch *et al.* [2,14] showed that these stable outcomes can be reversed by one actor alone (call it *the controlling actor*) unilaterally switching its feedback for a duration of time. Switching the controlling actor's feedback from positive to negative in the positive-positive feedback type (or from negative to positive in the negative-negative feedback type) will give a rise to the previously discussed type of positive-negative feedback (see Sec. IV D) which will lead to an out-of-phase oscillation of the emotional states. As the controlling actor switches its feedback for a second time to its original value after a specific duration of time, the system will be back to the initial scenario but effectively with new initial conditions defined at the moment of the second switch. These new initial conditions will depend on the duration of time between the two switches. As a result, and since the outcomes of the cooperation and competition cases (Secs. IV B and IV C) are sensitive to initial conditions, we may see a change in the final configuration of both actors' emotional states. Therefore, the duration of time between the first and the second switch, denoted by D, is the factor that determines the transient and final emotional states of the two actors. The panels in the first column of Fig. 7 show the results of such switching for a negative-negative case without time delay for two different D's. Hence, with a proper choice of D, one actor will be able to reverse the role of winner or loser in a conflict [2,14] (also noted by [10,11]).

However, time delay can alter the proper choice of D. Time delay itself will cause oscillations depending on the initial conditions of both actors. As a result, it is not only the duration between the first and the second switches, but also the time delay, that affects the new initial conditions at the time of the second switch. In the top row panels of Fig. 7 ($D = 0.75$), we see that increasing time delay is not enough for one actor to reverse the roles of winner and loser in a conflict. On the other hand, for duration $D = 1.5$ and without time delay (Fig. 7, bottom-left panel), both actors move to the opposite fixed points, and the winner/loser roles are

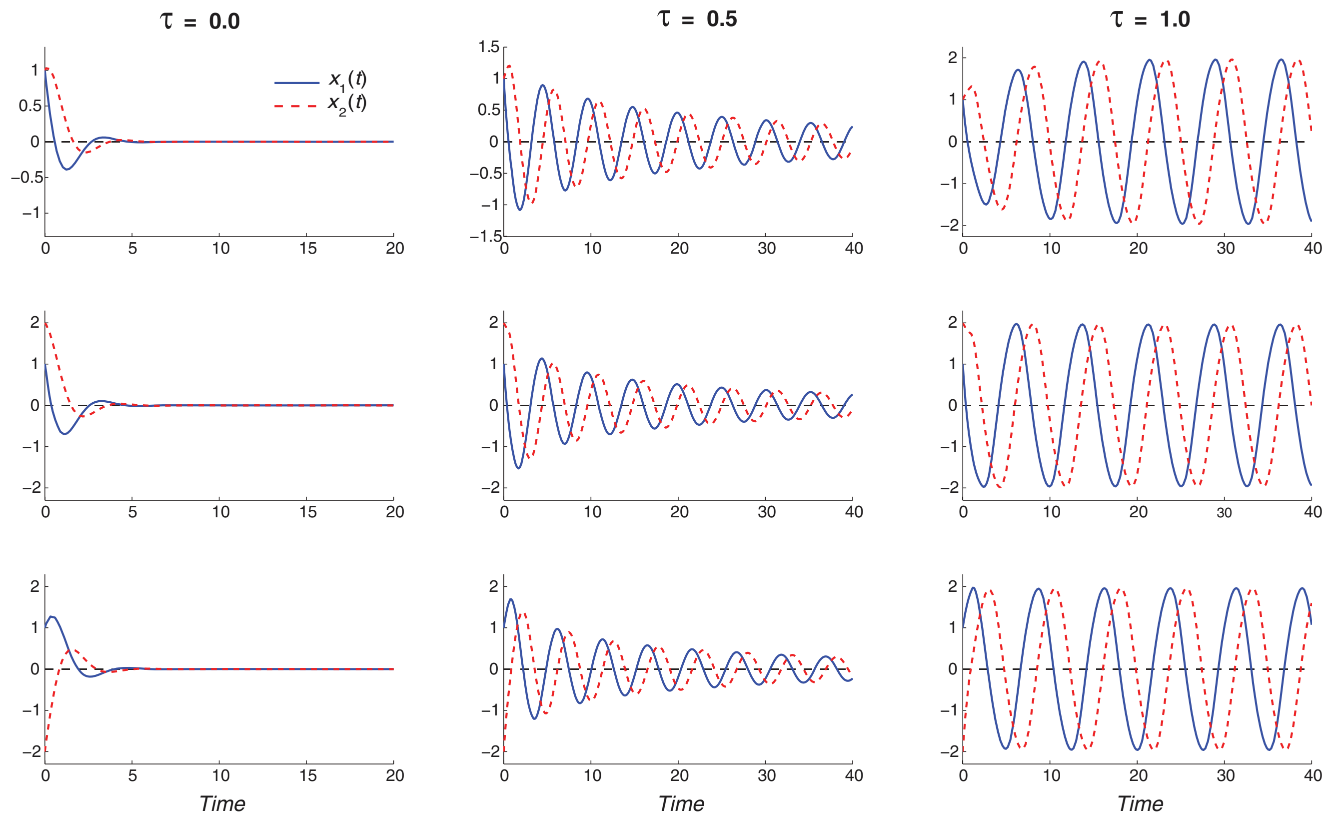

FIG. 6. (Color online) Time series of emotional states of the two actors, $x_1(t)$ (solid line) and $x_2(t)$ (dashed line), with strong positive-negative feedbacks ($C_{12} = 3$, $C_{21} = -3$) at different time delay values (columns) with $m = -0.9$. The values of $x_1(t)$ and $x_2(t)$ exhibit out-of-phase oscillation about the neutral state as they evolve from different initial values (rows). As time delay increases, the amplitude of the out-of-phase oscillation becomes larger, and eventually changes from being damped to undamped.

reversed. However, increasing time delay results in oscillatory behavior and consequently different new initial conditions at the second switch. As a result, the controlling actor may not be able to reverse the winner/loser roles in the conflict with the same switching duration. Instead, other parameters must also be changed, for example, increasing the switched feedback strength in order to retain the reverse scenario for the same duration or changing the moment at which the switch starts. It is worth noting here that these patterns are sensitive to the initial conditions of the both actors as well as the feedback type.

F. The effect of one actor unilaterally responding with delay and delay symmetry breaking

In the previous sections, we assume that time delays for both actors are identical. In other words, each actor is given (or takes) the same amount of time given to (or taken by) the other actor to respond. This "fairness" (or symmetry) of equal time delay can be understood in the presence of a fair mediator controlling peace talks or negotiations. However, in real-world conflicts, these time periods required to respond may differ from one actor to another. Interestingly, time delay may be used by one actor as a tactical parameter to change the whole outcome (emotional states) of both actors. Comparing the third-row with the first-row panels of Fig. 8, we see how one *slow* actor (i.e., with delay) alone could completely change the emotional states of both actors assuming that the other actor responds instantaneously. For the positive-positive (cooperative) case, both actors will leave the neutral state towards a new state in the side of the controlling actor (compare the first and third rows in the first column of Fig. 8). This is as if the slower actor drags the other actor toward its state. For the negative-negative (competition) type, the delayed response of one actor alone will also drive both actors to evolve away from the neutral steady state towards two new opposite (positive and negative) emotional states (compare the first and the third rows of the middle column of Fig. 8). On the other hand, in the case of mixed (positive-negative) feedback, both delay and instantaneous actors keep oscillating with decaying amplitudes around the neutral state. However, these oscillations are of larger amplitudes and persist longer than those when both actors respond instantaneously.

In fact, the model shows that even very small symmetry breaking of the time delay values may result in dramatic change of the emotional states of the actors. The last row of Fig. 8 demonstrates the effect of this symmetry breaking in time delay for the different feedback types. For the cooperative feedback with opposite initial conditions, even a small difference in time delay values of both actors results in driving both actors to evolve with oscillation away from the neutral state and stay in a new state on the side of the actor with larger, although slightly, time delay (compare the fourth and the second rows of the first column of Fig. 8). In competition

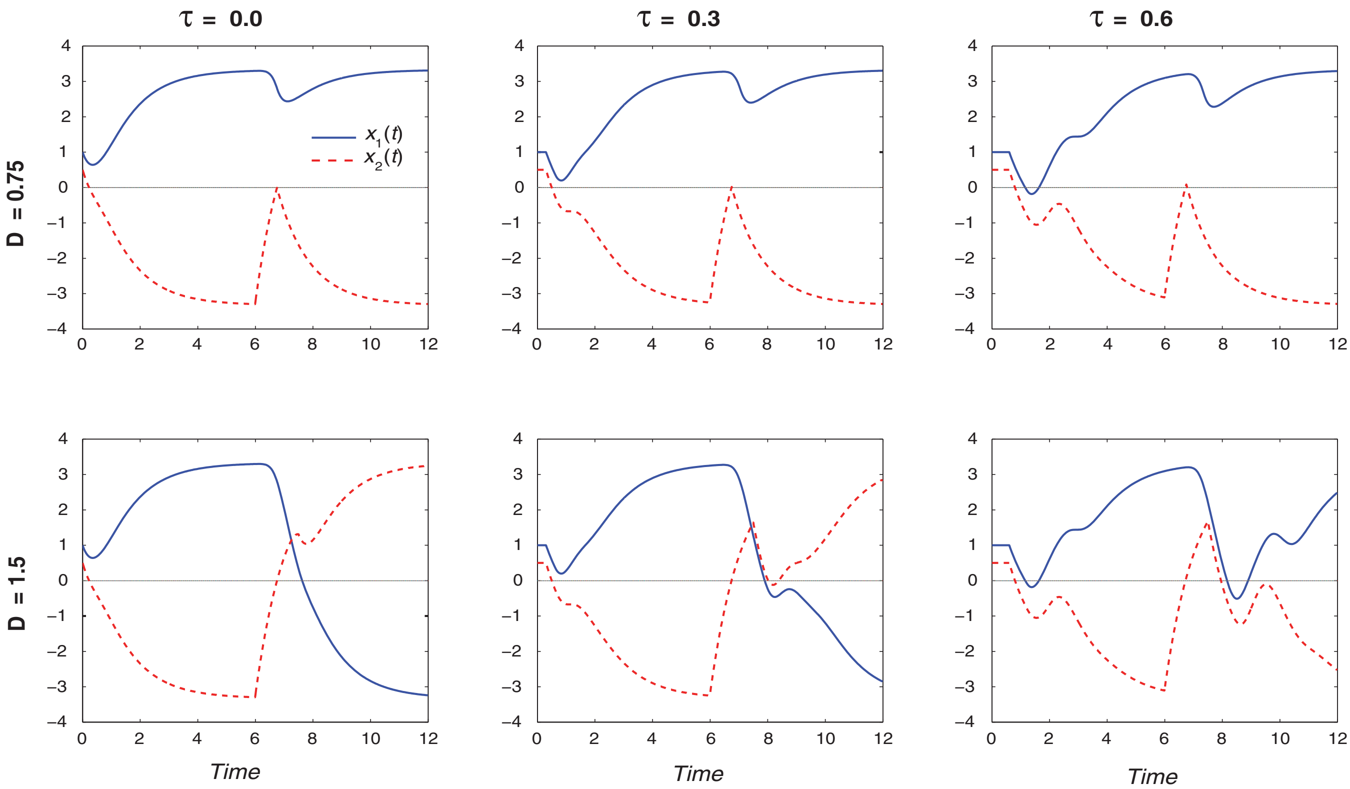

FIG. 7. (Color online) The effect of time delay when one actor unilaterally changes the sign of its feedback in a conflict with negative-negative feedback. In the first row: at time $t = 6$, actor x_1 temporarily switches its strategy to positive feedback for duration $D = 0.75$. For all time delay values, both actors will evolve to their previous fixed states. In the second row, the duration is larger: $D = 1.5$. In this case, for any time delay less than a certain value, the two actors will evolve to two states opposite their respective signs before the switch. As time delay increases, the reverse of the states cannot be achieved. This effect is highly sensitive to the initial conditions, moment of switch, duration of switch, and the time delay. The initial conditions of the two states shown are $x_1(0) = 1.0$, $x_2(0) = 0.5$, and the inertia $m = -0.9$.

feedback type, the effect of such small symmetry breaking in time delay results in the two actors oscillating away from the neutral state toward two new opposite emotional states (compare the bottom and the second panels of the middle column of Fig. 8). Clearly, the larger the difference in time delay values, the earlier the split occurs (comparing with the third plot of the same column panel). In the mixed feedback case, small differences of time delay values will not be enough to make notable changes of the outcomes. Again, these results may differ depending on the values of initial conditions.

V. SUMMARY AND CONCLUSION

In this paper, we use analytical and numerical methods to study the effects of time delay on a simple dynamical model of conflict under different feedback configurations. We find that the inclusion of time delay results in a rich set of predictions and insights. Our results show that when the feedback strength is less than the absolute value of the inertia, time delay cannot destabilize the neutral state: Time delay will only extend the relaxation time for both actors to reach it. Our analysis shows that time delay plays an important role in controlling the transient oscillatory dynamics not only in the positive-negative feedback type, but also in the other types (with strong negative-negative and positive-negative feedback). This result confirms a previous prediction by [20]. For the cooperative feedback, time delay can cause both actors to leave the neutral state (with or without oscillation, depending on the initial values) and evolve towards a new state that takes the side of the actor of a stronger initial state. In a strong competitive feedback type, time delay increases the period with significant oscillatory behavior before the two actors evolve towards two opposite states if their initial emotional states are not identical. However, when the two initial states are identical, both actors will oscillate in phase around the neutral state but with decaying amplitudes if time delay is less than a threshold, and with undamped amplitudes if time delay exceeds the threshold. Similar observations are obtained for the mixed feedback type, but with out-of-phase oscillations.

We show that time delay may enable a tactical switch of feedback by a single controlling actor. Liebovitch *et al.* [2,14] showed that a single actor can unilaterally swap the loser and winner roles in a conflict by switching the nature of its feedback with proper timing. However, in the presence of time delay, we show that a strategic timing of the switch entails the consideration of the magnitude and duration of the switch necessary to reverse the conflict outcomes. This is because time delay itself may cause oscillations (depending on time delay values and initial conditions) which in turn may alter the new initial emotional conditions at the moment of switch.

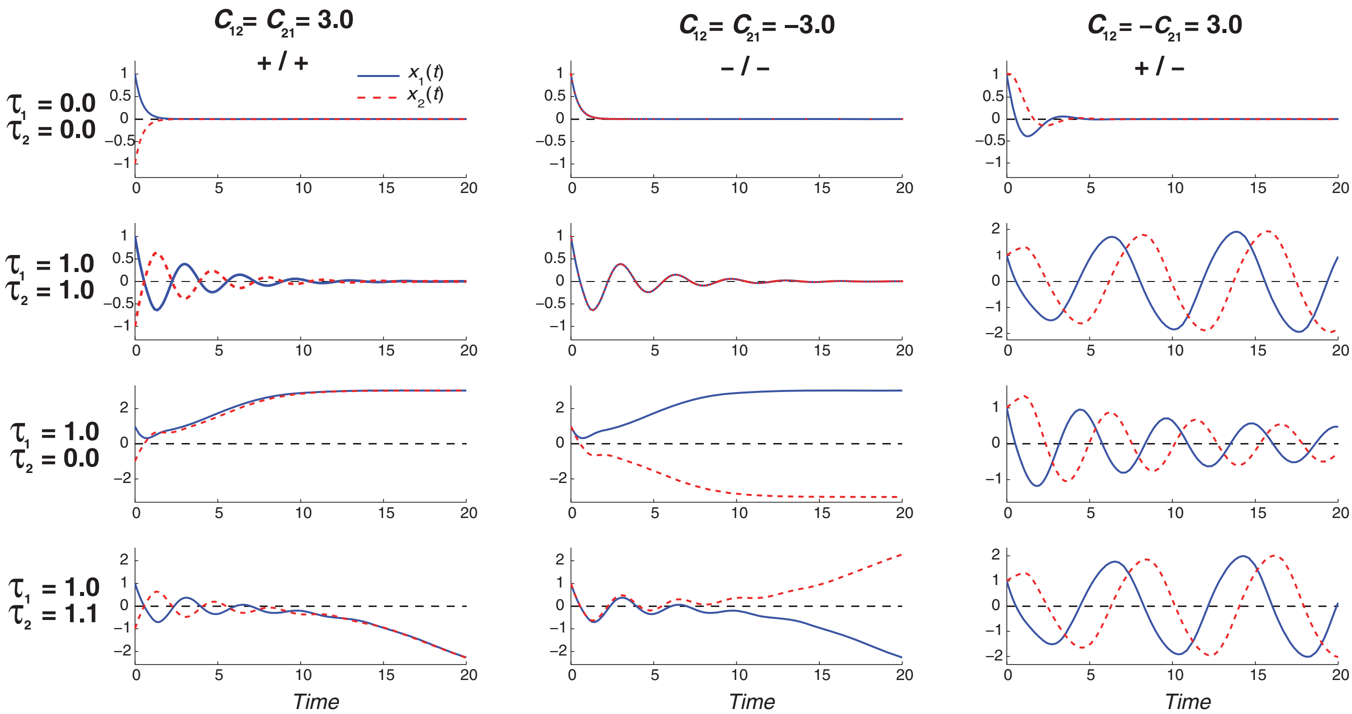

FIG. 8. (Color online) The effect of one actor responding unilaterally with delay and the effect of symmetry breaking in time delay values on the stability of the neutral state. The three columns represent the three different types of feedback: cooperation, competition, and mixed, respectively. The first row shows the previously obtained results of the time series of the emotional states of the two actors, $x_1(t)$ (solid line) and $x_2(t)$ (dashed line), with instantaneous (undelayed) responses. The second row is the time series of $x_1(t)$ and $x_2(t)$ with the same nonzero delay values, $\tau_1 = \tau_2 = 1$. The third row shows the effect of one actor who unilaterally responds with delay while the other actor responds instantaneously. The last row shows the effect of a small symmetry breaking between time delay values on the stability of the neutral state of the two actors. The initial conditions are the same for each type. The inertia value $m = -0.9$.

Indeed, potential future work may consider the duration of switching and time delay together as a strategy with some associated cost in conjunction with the benefit associated with reversing the winner/loser roles. The insights of these predictions can be tested by scholars dealing with conflict resolutions strategies. They may also provide material for social psychology laboratory scientists to design and perform experiments to test them.

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