Phase diagram for a zero-temperature Glauber dynamics under partially synchronous updates

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We consider generalized zero-temperature Glauber dynamics under a partially synchronous updating mode for a one-dimensional system. Using Monte Carlo simulations, we calculate the phase diagram and show that the system exhibits phase transition between the ferromagnetic and active antiferromagnetic phases. Moreover, we provide analytical calculations that allow us to understand the origin of the phase transition and confirm simulation results obtained earlier for synchronous updates.

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I. INTRODUCTION

In the last decade renewed interest in Glauber dynamics [1] has been observed, especially at zero temperature [2–14]. This is partially caused by recent experiments with so-called single-chain magnets (for a recent review, see [15]) but is also due to the development of the nonequilibrium statistical physics. From this point of view one-dimensional systems at zero temperature are especially interesting [9].

The dynamical rules of stochastic models, such as Glauber dynamics, can be defined in terms of various update schemes, the most important ones being parallel (synchronous) and random-sequential (asynchronous) updates [16]. Although Glauber dynamics was originally introduced as a sequential updating process, interesting theoretical results can be obtained also using a synchronous updating mode [4,8,10,12, 13,17]. Moreover, clear evidence of a relaxation mechanism which involves the simultaneous reversal of spins has been shown experimentally for magnetic chains at low temperatures [18]. In computer simulations under the synchronous updating mode all units of the system are updated at the same time. However, in real systems one can expect that simultaneous reversal of spins concerns only a part of the system. From this point of view partially synchronous updates are the most realistic.

We introduced such a partially synchronous updating scheme in 2006 [12] to investigate the differences between Glauber and Sznajd dynamics for a chain of L Ising spins. Within such an update in each elementary time step we visit all sites and select each of them with probability c as a candidate to get flipped, i.e., on average, cL randomly chosen spins are considered in a single time step [12]. Of course $c = 1$ corresponds to the synchronous updating scheme and $c = 1/L$ corresponds to random sequential updates. Partially synchronous updates were also used in 2007 by Radicchi *et al.* [13] to investigate the Ising spin chain at zero temperature for the Metropolis algorithm [19]. They observed, as a function of c, a critical phase transition between two phases: a ferromagnetic phase and the so-called active phase. A similar phase transition had already been observed earlier for the generalized zero-temperature Glauber dynamics by Menyhard and Odor in the case of a synchronous updating scheme [4].

It should be noticed that the Metropolis algorithm at zero temperature is a special case of a broader class of zero-temperature Glauber dynamics. Within the Glauber dynamics for Ising spins with a spin $s = 1/2$, in a broad sense, each spin is flipped $S_i(t) \to -S_i(t+1)$ with a rate $W(\delta E)$ per unit time, and this rate is assumed to depend only on the energy difference implied in the flip. At zero temperature it can be defined as [9]

$$W(\delta E) = \begin{cases} 1 & \text{if} \quad \delta E < 0, \\ W_0 & \text{if} \quad \delta E = 0, \\ 0 & \text{if} \quad \delta E > 0. \end{cases} \tag{1}$$

The zero-temperature limits of the original Glauber dynamics [1] and Metropolis rates [19] (two of the most popular choices) are respectively $W_0^G = 1/2$ and $W_0^M = 1$.

Very recently, generalized Glauber dynamics defined by (1) under a synchronous updating mode have been studied [17]. It has been shown that the system exhibits a phase transition for $W_0 = W_c = 1/2$ between ferromagnetic and antiferromagnetic phases. As an order parameter, the density ρ of active bonds has been used:

$$\rho = \frac{1}{2L} \sum_{i=1}^{L} (1 - \sigma_i \sigma_{i+1}), \tag{2}$$

where L is the number of spins and $\sigma_i = \pm 1$ is the Ising spin variable at the ith site on the one-dimensional chain with the periodic boundary condition. Starting from a randomly disordered initial state (high-temperature situation), the system eventually approaches one of two steady states: fully ferromagnetic, $\rho_{st} = 0$, or fully antiferromagnetic, $\rho_{st} = 1$. In a previous paper [17] it has been suggested that for $W_c = 1/2$, in the case of synchronous updating, the system undergoes a discontinuous phase transition between two types of order. However, very recently, it has been claimed that the observed phase transition is rather continuous [20]. It has been shown that the dependence between the mean value of ρ_{st} and the control parameter W_0 scales with the system size L with scaling exponents $\beta = 0$ and $\nu = 1$ [20]. Moreover, the mean exit time needed to reach the stationary state also scales with the system size with the dynamical scaling exponent $z = 2$. According to [20], both scaling laws indicate continuous phase transition, contrary to the suggestion made in [17]. However, it should be noticed that trivial scaling exponents $\beta = 0$ and $\nu = 1/d$ (where d denotes spatial dimension, i.e., $d = 1$ in our case) are typical for the first order phase transitions, as shown both analytically by Fisher and Berker [21] and using Monte Carlo simulations by Binder and Landau [22]. We will come back to this problem in Sec. III.

In this paper we consider zero-temperature Glauber dynamics defined by (1) under a partially synchronous updating mode. We show that both parameters W_0 and c are responsible for the phase transition between ferromagnetic and antiferromagnetic phases. We construct the phase diagram in (c, W_0) space based on the Monte Carlo simulations. Moreover, we provide exact analytical calculations for a simple case with only three active bonds. Such a simple approach allows us to understand the origin of the phase transition and shows that, indeed, for $c = 1$ the critical value $W_0 = 1/2$, which confirms the results obtained in [17,20].

II. THE MODEL

As mentioned above, we consider a one-dimensional chain of L Ising spins $\sigma = \pm 1$ with the periodic boundary condition, described by the Hamiltonian

$$H = -J \sum_{i=0}^{L} \sigma_i \sigma_{i+1}, \tag{3}$$

where $J > 0$, which means that we are dealing with a ferromagnetic system. We consider the system at temperature $T = 0$, and therefore we use the generalized Glauber dynamics defined by (1).

In our computer simulations we use partially synchronous updates, parametrized by $c \in [1/L, 1]$, which allows us to tune the algorithm from a sequential ($c = 1/L$) to synchronous ($c = 1$) updating scheme. At time t we visit all sites of the chain and select each of them with probability c as a candidate to get flipped. Each of the selected sites is then updated according to the zero-temperature Glauber dynamics defined by (1). After one step of the algorithm, the time increases as $t \to t + c$. As usual, one Monte Carlo step (MCS) passes when the average number of update events equals the total number of sites L. We investigate quench from $T = \infty$ to $T = 0$, i.e., an initial state is disordered: at each site i there is a randomly chosen value of spin $\sigma_i = \pm 1$, and both values $\sigma_i = +1$ and $\sigma_i = -1$ are equally probable.

III. MONTE CARLO RESULTS

In the Monte Carlo simulations, relaxation processes in magnetic or reaction-diffusion systems are usually investigated by measuring the time evolution of so-called active bonds (domain walls) [16,23–25]. As already mentioned, under a synchronous updating scheme ($c = 1$), the system described by dynamical rule (1) eventually approaches one of two steady states: fully ferromagnetic, $\rho_{\rm st} = 0$, or fully antiferromagnetic, $\rho_{\rm st} = 1$. We start by clarifying the problem of the type of the phase transition between ferromagnetic and antiferromagnetic orders that occurs at $W_0 = 1/2$. As written above, very recently, it has been claimed that the observed phase transition is continuous [20], contrary to what has been suggested in [17]. It is true that discontinuous phase transitions are rare in one-dimensional, even nonequilibrium systems, but there are several lattice models that exhibit discontinuous absorbing phase transition in one dimension [16]. There are several phenomena attributed to discontinuous phase transitions, such as phase coexistence, hysteresis cycles, and trivial critical exponents, in particular, $\beta = 0$, which indicates a jump of an order parameter. As has been shown for $c = 1$, there is a phase coexistence at $W_0 = 1/2$ [17]. Moreover, it has been shown that critical exponents $\beta = 0$ and $\nu = 1$ [20], which is typical for the first order phase transitions [21,22].

To distinguish ultimately between continuous and discontinuous phase transitions, the hysteresis loop should be observed. To measure the hysteresis we have decided to start with two types of initial conditions: (1) for a disordered ferromagnet, we disturb the ferromagnetic order by flipping one spin: $\cdots\uparrow\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow\cdots$. (2) For a disordered antiferromagnet, we disturb the antiferromagnetic order by flipping one spin: $\cdots\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\cdots$.

In Fig. 1 the dependence between $\langle\rho_{\rm st}\rangle$ and control parameter W_0 is shown for two values of c. It is clear that the phase transition for $c = 1$ is qualitatively different from the phase transition in the case of $c < 1$. In the latter case the initial condition does not influence significantly the asymptotic state, while in the case of synchronous updating a hysteresis loop can be observed.

Now we are ready to discuss results for $c < 1$. As we have already seen in Fig. 1, there is no hysteresis loop for $c < 1$. This result suggests that in this case the continuous phase transition probably occurs. In such a case we should observe the continuous change of order parameter $\langle\rho_{\rm st}\rangle$, and therefore the antiferromagnetic state should not be an absorbing state for $c < 1$. To check these predictions let us start by presenting the time evolution of the average density of active bonds $\langle\rho(t)\rangle$. From Fig. 2 we see that starting from disordered initial conditions ($\langle\rho(0)\rangle = 0.5$), the number of active bonds rapidly decreases to zero below some threshold value W_c, and for $W_0 > W_c$ it increases to a certain stationary value $\langle\rho_{\rm st}\rangle$ that depends on both c and W_0. This means that for $c < 1$ there is a phase transition between the ferromagnetic order and the so-called active phase [16]. To check if the phase transition is indeed continuous even for $c \lesssim 1$ we have conducted detailed simulations for $c = 0.9, 0.99, 0.999, 1$.

In Figs. 3 and 4 results for $c = 0.9$ and $c = 0.99$ are presented. The phase transition between ferromagnetically

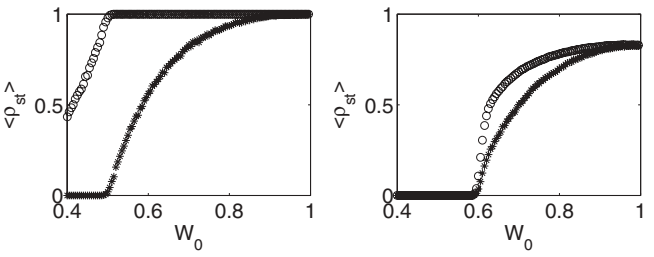

FIG. 1. The dependence between $\langle\rho_{\rm st}\rangle$ and $W_0 \in [0,1]$ in the case of synchronous updating (left) $c = 1$ and (right) $c = 0.95$ for the lattice size $L = 100$ from two different initial states: open circles denote the antiferromagnetic initial state disturbed by flipping one spin, and stars denote the ferromagnetic initial state disturbed by flipping one spin. It is seen that for synchronous updating (left) there is a hysteresis loop: different steady states are reached for different initial conditions. For $c < 1$ (right) there is no hysteresis loop: for $W_0 < 0.6$ the ferromagnetic steady state is reached independently of the initial state, and for $W_0 > 0.6$ there is an active steady state with $\rho_{\rm st} > 0$.

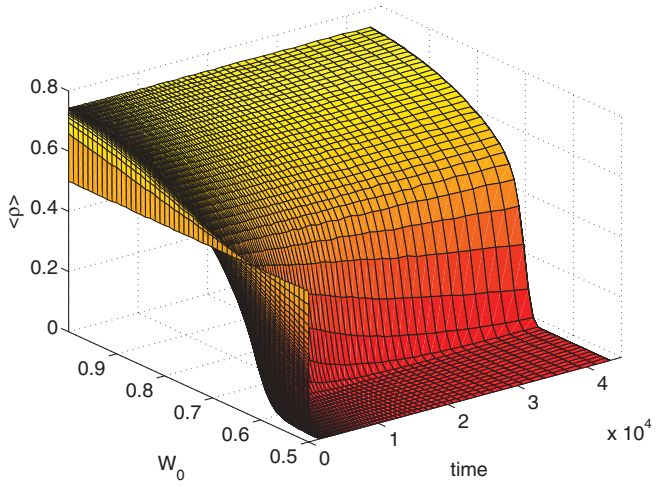

FIG. 2. (Color online) The average density of active bonds $\langle\rho\rangle$ as a function of time and W_0 for $c = 0.9$. It is seen that starting from disordered initial conditions ($\langle\rho(0)\rangle = 0.5$), the number of active bonds rapidly decreases to zero below some threshold value of W_0, and above this threshold value it increases to a certain stationary value $\langle\rho_{st}\rangle$ that depends both on c and W_0.

ordered and active phases is clearly visible, and the critical value of $W_0 = W_0(c)$, as well as the scaling exponents, can be estimated from the finite size scaling (see Table I). For all values of c the critical exponent $\nu = 1$, whereas $\beta = \beta(c)$ and for $c \to 1$ decreases with increasing c.

Finding precise values of critical exponents for all values of $c \in [0,1]$ is tedious but could be done. Here we were more interested in answering whether $c = 1$ is the only point at which the transition is discontinuous, and therefore we investigated $c \to 1$. According to our results, indeed, the discontinuous phase transition is observed only for $c = 1$, where generated clusters become compact (see Fig. 8). For $c < 1$ the transition is continuous, and β increases with the distance from the upper terminal point $c = 1$.

The exceptional behavior at the terminal point is due to the symmetry between ferromagnetic and antiferromagnetic states. Similar behavior is observed also in the Domany-Kinzel (DK) model and is usually referred to as compact directed percolation, which may be, in fact, misleading because the dynamics at this special point is the same as in the

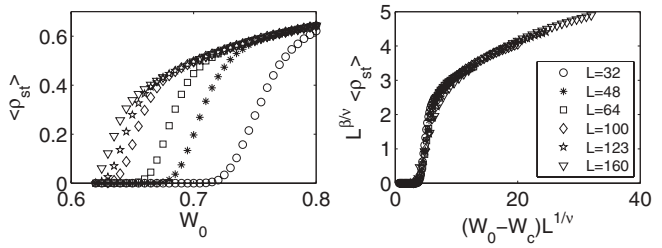

FIG. 3. (left) The average density of active bonds in the stationary state $\langle\rho_{st}\rangle$ as a function of W_0 for $c = 0.9$ and several lattice sizes L. The phase transition is clearly seen, and the critical value of W_0 can be found from the finite size scaling $W_c \approx 0.6$. (right) Results from the left panel are rescaled, showing clearly critical behavior with critical exponents $\nu = 1$ and $\beta \approx 0.4$.

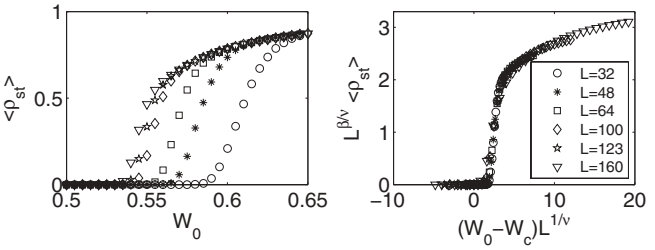

FIG. 4. (left) The average density of active bonds in the stationary state $\langle\rho_{st}\rangle$ as a function of W_0 for $c = 0.99$ and several lattice sizes L. The phase transition is clearly seen, and the critical value of W_0 can be found from the finite size scaling $W_c \approx 0.53$. (right) Results from the left panel are rescaled, showing clearly critical behavior with critical exponents $\nu = 1$ and $\beta \approx 0.25$.

$(1 + 1)$-dimensional Glauber-Ising model at zero temperature, or, equivalently, the voter model [16]. It should be recalled here that a DK model is a stochastic cellular automaton, and therefore it evolves by parallel updates, which for our model corresponds to $c = 1$, whereas the Glauber-Ising and voter model evolves by random sequential updating ($c = 1/L$). Therefore it is much easier to find direct correspondence between DK and our model with $c = 1$ than between our model and, e.g., the voter model. The DK model is characterized by two parameters, p_1 and p_2; p_1 is the probability that a site is activated if only one of two neighboring sites is active, and p_2 is the probability that the site is activated if both neighboring sites are active. In our model $p_2 = 1$ and p_1 corresponds to W_0. In the DK model for $p_2 = 1$ there is a discontinuous phase transition at $p_1 = 1/2$, which agrees exactly with the results obtained for our model with $c = 1$.

As we have written, the average density of active bonds in the stationary state $\langle\rho_{st}\rangle$ depends both on c and W_0. Up to now we have presented only the dependence between $\langle\rho_{st}\rangle$ and W_0 for several values of $c \to 1$. The average density of active bonds in the stationary state $\langle\rho_{st}\rangle$ as a function of W_0 and c is presented in Fig. 5. The transition line between ferromagnetically ordered ($\langle\rho_{st}\rangle = 0$) and active phases ($\langle\rho_{st}\rangle > 0$) is clearly visible. We have presented here results for a relatively small lattice size $L = 64$, although simulations were conducted also for larger systems, as presented in Figs. 3 and 4. Simulating smaller lattices allows us to measure the first passage time to one of the fully ordered states, i.e., with $\rho = 0$ or $\rho = 1$. As indicated, these two states are absorbing only for $c = 1$, and for $c < 1$ only $\rho = 0$ is an absorbing steady state. However, the small system still has nonzero probability to enter the antiferromagnetic state, although after it escapes from this state. Therefore we have decided to measure the mean time to enter one of the fully ordered states for the first

TABLE I. Approximate values of critical flipping probabilities and critical exponent β for several values of c.

c	W_c	β	ν
0.9	0.6	0.4	1.0
0.99	0.53	0.25	1.0
0.999	0.51	0.1	1.0
1	0.5	0	1.0

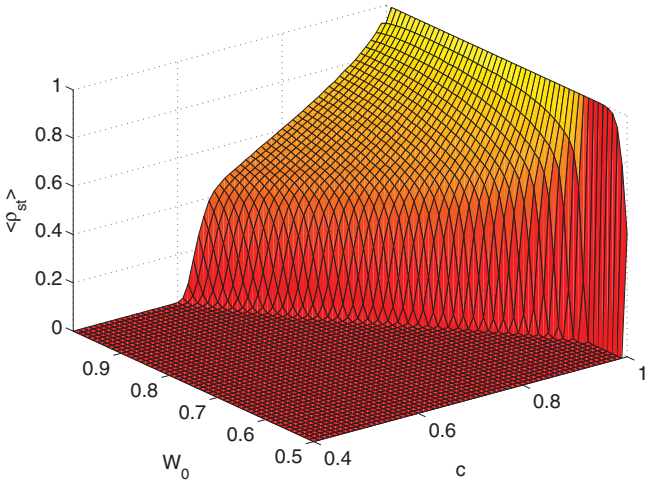

FIG. 5. (Color online) The average density of active bonds in the stationary state $\langle \rho_{st} \rangle$ as a function of W_0 and c for lattice size $L = 64$. Simulations were conducted for 5×10^5 MCS, and averaging was done over 5×10^3 samples.

time $\langle \tau \rangle$ and see if any interesting behavior related to $\langle \tau \rangle$ will be seen along the transition line.

The mean first passage time $\langle \tau \rangle$ to reach one of the two types of fully ordered states (the so-called exit time [26]), ferromagnetic ($\rho = 0$) or antiferromagnetic ($\rho = 1$), as a function of W_0 and c for the lattice size $L = 64$ is presented in Fig. 6. It is seen that $\langle \tau \rangle$ dramatically increases approaching the transition line, which is an expected behavior. However,

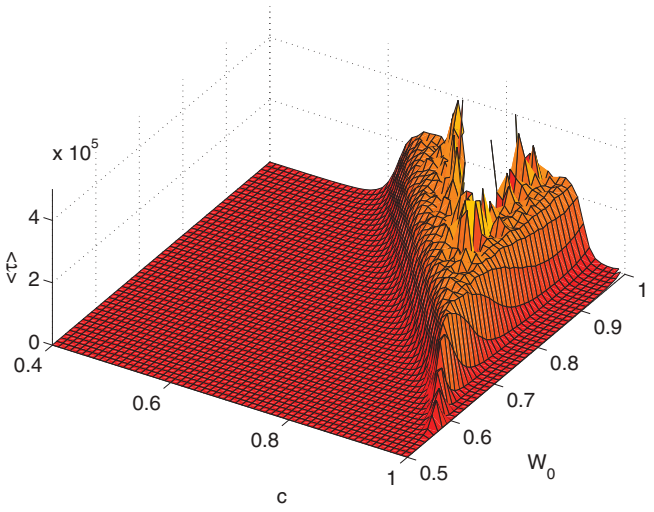

FIG. 6. (Color online) The mean exit time $\langle \tau \rangle$ to reach one of the two types of fully ordered states, ferromagnetic ($\rho = 0$) or antiferromagnetic ($\rho = 1$), as a function of W_0 and c for lattice size $L = 64$. Simulations were conducted for 5×10^5 MCS, and averaging was done over 5×10^3 samples. It is seen that below the transition line the system reaches the ordered ferromagnetic state quickly. Similarly, significantly above the transition line (large values of c) the system quickly reaches the ordered antiferromagnetic state, although for $c < 1$ this is not an absorbing state. The shape of a triangle, in which $\langle \tau \rangle$ dramatically increases, is seen. The hole inside the triangle indicates that none of the ordered states have been reached in 10^6 Monte Carlo steps.

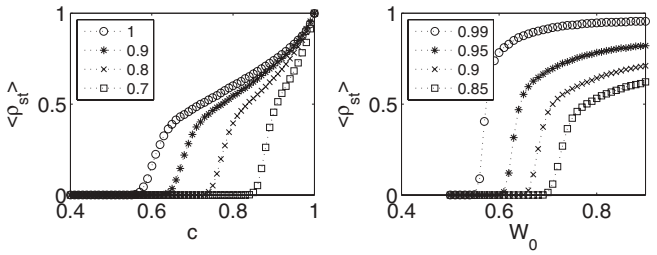

FIG. 7. The average density of active bonds in the stationary state $\langle \rho_{st} \rangle$ (left) as a function of c for several values of W_0 and (right) as a function of W_0 for several values of c. The difference between the transitions with respect to c and W_0 is visible.

what is even more interesting is that it increases also along the transition line. For $c \to 1$ the mean first passage time $\langle \tau \rangle$ is relatively short, and it increases with the distance from the upper terminal point $c = 1$. Let us recall here that the same behavior is related to a critical exponent β. Colloquially speaking, the exit time increases with an increase in the continuity (β) of the transition. Therefore, although the antiferromagnetic state is not absorbing any longer for $c < 1$, the mean exit time $\langle \tau \rangle$ is a useful characteristic of an observed phase transition.

The last interesting feature connected to the phase transition seen in Fig. 5 is a difference between the transition along axis W_0 and c. The differences between the transitions with respect to c and W_0 are visible also in Fig. 7. For $c = 1$ the average density of active bonds in the stationary state $\langle \rho_{st} \rangle$ is 1 for any $W_0 > 0.5$, whereas for $W_0 = 1$ the average density of active bonds in the stationary state $\langle \rho_{st} \rangle$ depends on c. The transition with respect to c is much more gentle than the transition with respect to W_0.

The difference between phase transitions with respect to c and W_0 can also be seen from the time evolution of active bonds presented in Fig. 8. The phase transition for the Metropolis algorithm, i.e., $W_0 = 1$, which is induced by changing c, reminds us of typical annihilation: a branching process (right panel in Fig. 8). On the other hand, in the case of synchronous updating the growth of the antiferromagnetic domain from a single active bond can be observed (left panel in Fig. 8). In this case a kind of phase coexistence can be observed: ferromagnetic and antiferromagnetic clusters are present in the system.

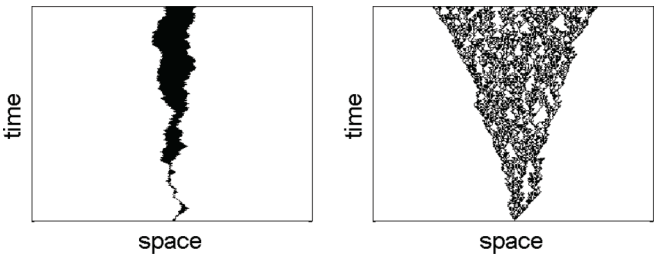

FIG. 8. Time evolution of active bonds near the phase transition (left) in the case of synchronous updating $c = 1$ induced by W_0 and (right) in the case of the Metropolis algorithm $W_0 = 1$ induced by c. In the initial state only one bond was active.

IV. THE ORIGIN OF THE PHASE TRANSITION

We have shown in the previous section that a one-dimensional system of Ising spins with generalized Glauber dynamics under partially synchronous updates exhibits well-defined phase transition between a stable ferromagnetic phase and an active phase. Moreover, it has been shown that for the synchronous updating scheme ($c = 1$) the system exhibits phase transition for $W_0 = 1/2$ between two absorbing stable states: ferromagnetically and antiferromagnetically ordered [17,20]. Although some mean field calculation has been provided [17], the origin of the investigated phase transition has not yet been.

The model considered here belongs to a broad class of so-called branching-annihilating random walks [16]; that is, three processes are possible: diffusion, branching, and annihilation of active bonds. It should also be mentioned that these processes conserve the number of active bonds modulo 2, which is usually called *parity conserving*. However, it has been shown that the conservation of the parity is not very relevant [16,27].

Let us first consider the simplest case, a chain of length L with a single active bond, i.e., $\cdots\uparrow\uparrow\downarrow\downarrow\cdots$, at time t. Since we deal with a zero-temperature situation, changes are possible only on the domain wall (active bond). Therefore at time $t + c$ the single-bond system can evolve to

$$\begin{aligned}
&\cdots\uparrow\Downarrow\Uparrow\downarrow\cdots \text{ with probability } c^2W_0^2P_{L-2},\\
&\cdots\uparrow\uparrow\Uparrow\downarrow\cdots \text{ with probability } [c(1-c)+c^2(1-W_0)]P_{L-2},\\
&\cdots\uparrow\Downarrow\downarrow\downarrow\cdots \text{ with probability } [c(1-c)+c^2(1-W_0)]P_{L-2},\\
&\cdots\uparrow\uparrow\downarrow\downarrow\cdots \text{ with probability } [(1-c)^2+c^2(1-W_0)^2]P_{L-2},
\end{aligned}$$

where $\Uparrow$ and $\Downarrow$ denote spins that were flipped and

$$P_{L-2}=\sum_{k=0}^{L-2}\frac{(L-2)!}{k!(L-2-k)!}c^k(1-c)^{L-2-k} \tag{4}$$

denotes the sum of probabilities of all possible choices of remaining $L-2$ spins.

Clearly, only the first process, which occurs with the probability $P_b = c^2W_0^2P_{L-2}$, leads to the growth of antiferromagnetic domains. The remaining three situations do not change the number of active bonds since the annihilation of a single active bond is impossible. Therefore the single-bond system can either remain unchanged or evolve to the system that consists of three neighboring active bonds, $\cdots\uparrow\downarrow\uparrow\downarrow\downarrow\cdots$. Analyzing all possible transitions in such a system (see Table II), we can calculate the probability of annihilation (P_a), branching (P_b), and diffusion (P_d):

$$\begin{aligned}
P_a &= c^4W_0^2 - c^2(1+2W_0)+2c, \quad P_b = c^4W_0^2,\\
P_d &= -c^4W_0(2+W_0) - 2c^3\left(W_0^2-2W_0-1\right)\\
&\quad + c^2\left(W_0^2-4\right)+2c.
\end{aligned} \tag{5}$$

Of course there is also the possibility of no change in the system:

$$P_{no} = 1-(P_a+P_b+P_d). \tag{6}$$

TABLE II. All possible outcome configurations from initial state $\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow$. Here $\Uparrow$ and $\Downarrow$ denote spins that were flipped. The constant factor P_{L-4}, which multiplies the right sides of the Eqs. (5), has been omitted to simplify notation. The initial state has three bonds.

After flip	Bonds	Probability
$\uparrow\uparrow\Uparrow\uparrow\downarrow\downarrow$	1	$c(1-c)^3+c^3(1-W_0)^2(1-c)$ $+2c^2(1-c)^2(1-W_0)$
$\uparrow\uparrow\downarrow\Downarrow\downarrow\downarrow$	1	$c(1-c)^3+c^3(1-W_0)^2(1-c)$ $+2c^2(1-c)^2(1-W_0)$
$\uparrow\Downarrow\downarrow\uparrow\downarrow\downarrow$	3	$cW_0(1-c)^3+c^2W_0(1-c)^2(1-W_0)$
$\uparrow\uparrow\downarrow\uparrow\Uparrow\downarrow$	3	$cW_0(1-c)^3+c^2W_0(1-c)^2(1-W_0)$
$\uparrow\Downarrow\Uparrow\uparrow\downarrow\downarrow$	3	$c^2W_0(1-c)^2+c^3W_0(1-c)(1-W_0)$
$\uparrow\uparrow\downarrow\Downarrow\Uparrow\downarrow$	3	$c^2W_0(1-c)^2+c^3W_0(1-c)(1-W_0)$
$\uparrow\Downarrow\downarrow\Downarrow\downarrow\downarrow$	1	$c^2W_0(1-c)^2+c^3W_0(1-c)(1-W_0)$
$\uparrow\uparrow\Uparrow\uparrow\Uparrow\downarrow$	1	$c^2W_0(1-c)^2+c^3W_0(1-c)(1-W_0)$
$\uparrow\uparrow\Uparrow\Downarrow\downarrow\downarrow$	1	$c^2(1-c)^2+c^4(1-W_0)^2$
$\uparrow\Downarrow\downarrow\uparrow\Uparrow\downarrow$	3	$c^2W_0^2(1-c)^2$
$\uparrow\Downarrow\Uparrow\Downarrow\downarrow\downarrow$	3	$c^3W_0(1-c)+c^4W_0(1-W_0)$
$\uparrow\uparrow\Uparrow\Downarrow\Uparrow\downarrow$	3	$c^3W_0(1-c)+c^4W_0(1-W_0)$
$\uparrow\Downarrow\downarrow\Downarrow\Uparrow\downarrow$	3	$c^3W_0^2(1-c)$
$\uparrow\Downarrow\Uparrow\uparrow\Uparrow\downarrow$	3	$c^3W_0^2(1-c)$
$\uparrow\Downarrow\Uparrow\Downarrow\Uparrow\downarrow$	5	$c^4W_0^2$

The constant factor

$$P_{L-4}=\sum_{k=0}^{L-4}\frac{(L-4)!}{k!(L-4-k)!}c^k(1-c)^{L-4-k}, \tag{7}$$

which multiplies the right sides of the above equations, has been omitted to simplify the notation.

Now we can ask what the dependence is between parameters c and W_0 for which annihilation and branching are equally probable:

$$P_a = P_b \to -c^2(1+2W_0)+2c = 0. \tag{8}$$

This means that annihilation and branching are equally probable for $c = 0$ or

$$c=\frac{2}{1+2W_0}. \tag{9}$$

From Eq. (9) we find that for synchronous updating, i.e., $c = 1$, the critical value of $W_0 = 1/2$, which confirms results obtained recently in [17,20]. Moreover, for $W_0 = 1$ we obtain the critical value of $c = 2/3$, which is also very close to the value obtained from Monte Carlo simulations (see Fig. 5). Therefore it seems that the phase transition between the ferromagnetic phase and antiferromagnetic active phase appears when annihilation and branching are equally probable.

Let us now present the dependence between probabilities (5) and parameter c for a given value of W_0. We focus on the Metropolis algorithm, i.e., $W_0 = 1$ (the case considered also in [13]). Results are presented in Fig. 9. Several interesting features of our system are visible.

(1) The value of c for which annihilation and branching are equally probable, i.e., $P_a = P_b$, agrees quite well with the critical value of c obtained from Monte Carlo simulations.

(2) The probability of diffusion has a maximum for the same value of c, for which $P_a = P_b$.

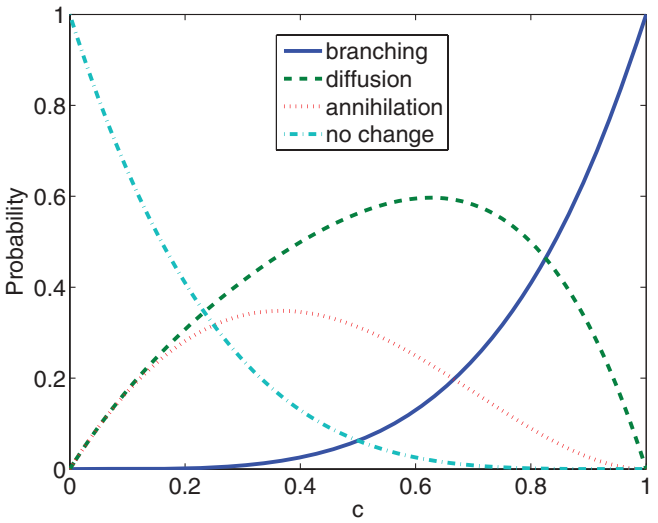

FIG. 9. (Color online) Probabilities of annihilation (P_a), branching (P_b), diffusion (P_d), and no change (P_{no}) as a function of updating scheme c for $W_0 = 1$.

(3) The probability of annihilation grows with c to a certain value, $c \sim 0.4$, and for $c > 0.4$ it decays. Simultaneously with decreasing P_a, the probability of branching grows, although it is still smaller then annihilation for $c < 2/3$. Therefore, one expects that eventually the system will still reach a ferromagnetic state, although branching of active bonds should be visible during time evolution. It should be mentioned here that in [13] the case of $W_0 = 1$ has also been studied, and the authors claimed that a system with partially synchronous updates exhibits phase transition for $c \sim 0.4$. However, our results (both types of Monte Carlo simulations and the simple analytical approach) suggest that the value of $c \sim 0.4$ corresponds merely to the situation in which the probability of annihilation starts to decay and branching appears.

V. SUMMARY

In this paper we have investigated one-dimensional systems of Ising spins driven by the generalized zero-temperature Glauber dynamics with a partially synchronous updating mode (tuned from sequential to synchronous by parameter c). It has been shown that for the synchronous updating mode, which corresponds to $c = 1$, there is a discontinuous phase transition between two ordered phases (ferromagnetic and antiferromagnetic). Three signatures of a discontinuous phase transition have been found in this case: (1) jump of an order parameter ($\beta = 0$), (2) phase coexistence, and (3) hysteresis cycles. Similar behavior has been observed in a one-dimensional Glauber-Ising model at zero temperature in a magnetic field, which is also known as compact directed percolation [16]. On the other hand, finding the precise values of critical exponents for $c < 1$ turned out to not be so easy a task. Nevertheless, the results obtained in this paper suggest that for any value of $c < 1$ there is a continuous order-disorder transition (between the ferromagnetic and so-called active phases). Using the finite scaling technique, we have shown that the critical exponent β has no single value along the transition line, i.e., $\beta = \beta(c)$, and it increases with the distance from the upper terminal point $c = 1$, at least for $c \to 1$. Finding the dependence between critical exponent β and c along the whole line, i.e., for $c \in [0,1]$, is quite tedious. Moreover, we were more interested in answering the question of whether $c = 1$ is the only point at which the transition is discontinuous and what the type of transition is for $c < 1$. Therefore we investigated $c \to 1$. The numerical findings of critical exponents are often difficult, and one should be careful when drawing conclusions only from simulations. However, it seems that the discontinuous phase transition for $c = 1$, similar to the Domany-Kinzel model [16,28], is exceptional due to an additional symmetry between active and inactive bonds.

Another interesting problem that could be investigated, but was not the subject of this paper, is the phase transition with respect to c for a given W_0. We have presented the general dependence between an order parameter $\langle \rho_{st} \rangle$ and parameters W_0 and c. We have also discussed briefly the differences between transitions with respect to W_0 and c. However, a detailed analysis has not been provided. The only results connected to this issue were obtained for $W_0 = 1$ in [13]. In this paper it has been shown that the phase transition can also be observed for any other value of $W_0 > 0.5$. It would be interesting to investigate this problem more precisely in the future.

To understand the origin of the phase transition we have provided a simple analytical approach and showed that transition occurs when branching and annihilation are equally probable, which is fulfilled for $W_0 = (2 - c)/2c$. Again, this confirms results from [17,20] since for $c = 1, W_0 = 1/2$, which was obtained earlier by Monte Carlo simulations and a simple mean field approach.

To conclude this work we would like to highlight one important issue that justifies the subject of the paper. As mentioned in the Introduction, clear evidence of a relaxation mechanism which involves the simultaneous reversal of spins has been shown experimentally for magnetic chains at low temperatures [18]. However, in [18] it has been suggested that the probability of simultaneous reversal of L spins scales as q^L (with certain parameter $q < 1$), which is not the same kind of macroscopic reversal which is assumed in this paper. Moreover, in [18] the simultaneous reversal of spins in a single segment has been considered, which is also very different from our approach. To be honest, we were not able to find any other example of a physical experiment that shows an evidence of simultaneous changes. One should also remember that Glauber dynamics, which has been introduced as a sequential updating process, satisfies the detailed balance condition and therefore ensures the existence of an equilibrium. There is thus a natural question of whether the model with partially simultaneous updating is merely another mathematical toy. Let us stress here that we strongly believe in toy models. They help to explore new regions and develop new fields even without meeting any reality. On the other hand, we understand the skepticism of people who would like to have even the smallest hope that the model would turn into something useful. We are not sure if partially synchronous or fully synchronous updating can describe a real physical experiment. On the other hand, the problem of updating methods is widely discussed in a recent work on cellular automata, Boolean networks, neural networks, and the so-called agent-based modeling in

ecology and sociology [29–32]. It has been shown that the updating scheme can have an enormous influence on the model output [30]. It is also suggested that "the updating effects will be particularly marked in models with increasing interaction complexity such as models of interaction between many trophic levels." In this paper we show that the effect of the type of updating is clearly visible even within extremely simple model, which might be instructive, taking into account that many models of opinion dynamics are inspired by the Ising model [33]. In a world of agent-based modeling, asynchronous and synchronous updating are treated as two contrasting methods [33], and we see no reason why either of these two would be better than partially synchronous updating. As stated in [13], "Probably neither a completely synchronous nor a random asynchronous update is realistic for natural systems."

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