Quasiconservation laws for compressible three-dimensional Navier-Stokes flow

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We formulate the quasi-Lagrangian fluid transport dynamics of mass density ρ and the projection $q = \boldsymbol{\omega} \cdot \nabla \rho$ of the vorticity $\boldsymbol{\omega}$ onto the density gradient, as determined by the three-dimensional compressible Navier-Stokes equations for an ideal gas, although the results apply for an arbitrary equation of state. It turns out that the quasi-Lagrangian transport of q cannot cross a level set of ρ. That is, in this formulation, level sets of ρ (isopycnals) are impermeable to the transport of the projection q.

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The aim of this Brief Report is to formulate the fluid conservation dynamics for mass density ρ and the projection $q = \boldsymbol{\omega} \cdot \nabla \rho$ of the vorticity $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$ onto the mass density gradient, as determined by the three-dimensional compressible Navier-Stokes equations [1]:

$$\rho \frac{D\boldsymbol{u}}{Dt} = \mu \Delta \boldsymbol{u} - \nabla \varpi, \quad \varpi = p - (\mu/3 + \mu^v)\operatorname{div} \boldsymbol{u}, \quad (1)$$

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \boldsymbol{u} = 0, \quad \frac{D}{Dt} = \partial_t + \boldsymbol{u} \cdot \nabla, \quad (2)$$

$$c_v \frac{D\theta}{Dt} = \frac{p}{\rho} \operatorname{div} \boldsymbol{u} + Q, \quad p = R\rho\theta. \quad (3)$$

Here, $\boldsymbol{u}$ denotes the spatial fluid velocity, and μ is the shear viscosity and μ^v is the volume viscosity, both of which are taken as constitutive constants of the fluid.

For definiteness, we have chosen an ideal gas equation of state to relate pressure, p, temperature, θ, and mass density, ρ. In addition, R is the gas constant, c_v is the specific heat constant, and Q is the heating rate, which we may assume is known. The pressure depends on the two thermodynamic variables ρ and θ. It is noteworthy that the dynamics of the temperature and the choice of equation of state in Eq. (3) do not affect our considerations below of the transport dynamics of the projection $q = \boldsymbol{\omega} \cdot \nabla \rho$. This means the geometric considerations that follow are universal for any viscous compressible fluid flow.

According to Eqs. (1) and (2) the vorticity $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$ evolves according to a stretching and folding equation,

$$\partial_t \boldsymbol{\omega} - \operatorname{curl}(\boldsymbol{u} \times \boldsymbol{\omega}) = \mu\rho^{-1}\Delta\boldsymbol{\omega} + \nabla\rho^{-1} \times \left[\mu\Delta\boldsymbol{u} - \nabla\left(\varpi - \frac{u^2}{2}\right)\right], \quad (4)$$

driven by the right-hand side. The form of the last term invites a projection against $\nabla\rho$. By the product rule, the projection $q = \boldsymbol{\omega} \cdot \nabla\rho$ satisfies a geometric relation reminiscent of Ertel's theorem [2]:

$$\frac{Dq}{Dt} = \left(\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}\right) \cdot \nabla\rho + \boldsymbol{\omega} \cdot \nabla\left(\frac{D\rho}{Dt}\right). \quad (5)$$

Ertel's theorem, important in atmospheric dynamics, has the same form as Eq. (5), with mass density replaced by potential temperature. Substituting mass conservation (2) and Navier-Stokes vorticity dynamics (4) into the purely geometric equation (5) yields the following equation for q written in a suggestive divergence form,

$$\partial_t q + \operatorname{div} q\boldsymbol{u} + \operatorname{div}[\boldsymbol{\omega}\rho \operatorname{div} \boldsymbol{u} - \mu\Delta\boldsymbol{u} \times \nabla(\ln\rho)] = 0. \quad (6)$$

Now we apply an observation of Haynes and McIntyre [3] that first arose in atmospheric physics and allows one to define the current density $\mathbf{J}$, assuming that solutions exist for Eqs. (1)–(3), as

$$\mathbf{J} = q\boldsymbol{u} + \boldsymbol{\omega}\rho \operatorname{div} \boldsymbol{u} - \mu\Delta\boldsymbol{u} \times \nabla(\ln\rho) + \nabla\phi \times \nabla f(\rho), \quad (7)$$

where ϕ is an undetermined gauge potential and f is an arbitrary differentiable function of ρ. Then Eqs. (6) and (2) can be rewritten in the *quasiconservative* form as

$$\partial_t q + \operatorname{div} \mathbf{J} = 0 \quad \text{and} \quad q\,\partial_t\rho + \mathbf{J} \cdot \nabla\rho = 0. \quad (8)$$

The relation $\mathbf{J} \cdot \nabla\rho = q \operatorname{div} \rho\boldsymbol{u}$ allows zero projection, $q = 0$, by the second equation in Eqs. (8). Thus, the projection q may vanish anywhere in the flow, but it cannot be maintained, because $\operatorname{div} \mathbf{J} \neq 0$. Together, Eqs. (8) imply a family of conserved quantities, since

$$\partial_t(q\Phi'(\rho)) + \operatorname{div}(\mathbf{J}\Phi'(\rho)) = 0, \quad (9)$$

for any function $\Phi'(\rho) = d\Phi/d\rho$.

The conserved densities $q\Phi'(\rho) = \operatorname{div}(\Phi(\rho)\boldsymbol{\omega})$ in Eq. (9) possess quite different flow properties from those of mass, energy, and momentum. The question of the physical interpretation of these quantities may now be raised. We know that Eqs. (8) are purely kinematic, because the projection taken against $\nabla\rho$ removed any dependence on the dynamics of the temperature and the choice of equation of state in Eq. (3). Nonetheless, Eqs. (8) have been derived *without approximation* from the Navier-Stokes fluid equations for compressible motion and mass transport.

Moreover, their analogs also occur and have been found useful in other areas of fluid dynamics, particularly in the dynamics of the ocean and the atmosphere [3,4]. (In the oceanic context, ρ denotes buoyancy and the motion is usually taken to be incompressible.)

Indeed, a similar calculation may be performed to derive the quasiconservative form in Eqs. (8) of the dynamics of the projection of vorticity on temperature gradient, $q' = \boldsymbol{\omega} \cdot \nabla\theta$

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from Eq. (5) with $\nabla\rho$ replaced by $\nabla\theta$. The projection q' with θ denoting the potential temperature is known as the *potential vorticity density* and has been useful in guiding thinking about nonlinear convective processes in atmospheric science [3,4].

In addition, the corresponding forms of Eqs. (7) and (8) may be written in the absence of viscosity. One hopes that with the proper interpretation Eqs. (8) may also be useful in considerations of compressible Navier-Stokes flows.

Following Refs. [3,4] in the atmospheric context for which the projection q is replaced by potential vorticity q', we propose to interpret Eqs. (7) and (8) locally as a type of *impermeability theorem* for the *quasi-Lagrangian* transport of the projection q and the mass density ρ by the pseudovelocity $\mathcal{U} = \mathbf{J}/q$. Here, impermeability means that the projection $q = \boldsymbol{\omega}\cdot\nabla\rho$ cannot be transported by the pseudovelocity $\mathcal{U}$ across level sets of the mass density, ρ [5]. This is remarkable because the two quantities q and ρ are transported by *different velocities*.

We hope that the quasiconservative form of compressible Navier-Stokes fluid dynamics in Eqs. (8) may find some future use, perhaps in analogy with the use of potential vorticity in the atmospheric context. In particular, these equations may be useful in the study of stretching and folding in compressible fluid flows, just as has been recently investigated in the atmospheric context in Refs. [6,7]. Although previous studies of compressible flows have often focused on shock formation, we close this note with a remark about how the gradient ∇q of the projection $q = \boldsymbol{\omega}\cdot\nabla\rho$ participates in stretching, folding, and expansion of higher-order gradients in compressible Navier-Stokes fluid flows. In preparation, we rewrite Eqs. (2) and (8) in terms of the pseudovelocity $\mathcal{U} = \mathbf{J}/q$ as

$$\partial_t q + \text{div}\,(q\,\mathcal{U}) = 0 \quad \text{and} \quad \partial_t\rho + \mathcal{U}\cdot\nabla\rho = 0. \qquad (10)$$

Note however that $\text{div}\,\mathcal{U} \neq 0$. We define

$$\mathcal{B} = \nabla q \times \nabla\rho \qquad (11)$$

and discover from a direct computation that $\mathcal{B}$ satisfies

$$\partial_t\mathcal{B} - \text{curl}\,(\mathcal{U}\times\mathcal{B}) = \boldsymbol{\mathcal{D}}, \qquad (12)$$

where $\boldsymbol{\mathcal{D}} = -\nabla(q\,\text{div}\,\mathcal{U})\times\nabla\rho$. The proof of the corresponding relation in an atmospheric physics context can be found in Refs. [6,7].

Based on the transport velocity defined in Eq. (7), the left-hand side of Eq. (12) makes it clear that the vector $\mathcal{B}$ undergoes the same type of stretching, folding, and expansion processes driven by the $\boldsymbol{\mathcal{D}}$ vector on the right-hand side as occurs in the vorticity equation (4). Explicitly, the stretching, folding, and expansion processes are given by

$$\partial_t\mathcal{B} + \underbrace{\mathcal{U}\cdot\nabla\mathcal{B} - \mathcal{B}\cdot\nabla\mathcal{U} + \mathcal{B}\,\text{div}\,\mathcal{U}}_{\text{stretch, fold, and expand }\mathcal{B}} = \boldsymbol{\mathcal{D}}. \qquad (13)$$

This is remarkable, because $\mathcal{B} = \nabla q \times \nabla\rho$ contains information not only about $\nabla\boldsymbol{\omega}$ but also about $\nabla\rho$ and even $\nabla\nabla\rho$. Thus, the stretching and folding in the original vorticity equation (4) compounds itself in the same form in the $\mathcal{B}$ equation (12), but with higher spatial derivatives.

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[1] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Elmsford, NY, 1959).
[2] H. Ertel, Met. Z. **59**, 271 (1942).
[3] P. Haynes and M. McIntyre, J. Atmos. Sci. **44**, 828 (1987); **47**, 2021 (1990).
[4] B. J. Hoskins, M. E. McIntyre, and A. W. Robertson, Q. J. R. Meteorol. Soc. **111**, 877 (1985).
[5] By the definition of $\mathbf{J}$ in Eq. (7) the pseudovelocity $\mathcal{U} = \mathbf{J}/q$ is determined only up to the addition of the curl of a vector proportional to $\nabla\rho$.
[6] J. D. Gibbon and D. D. Holm, J. Phys. A **43**, 17200 (2010).
[7] J. D. Gibbon and D. D. Holm, in *Mathematical Aspects of Fluid Mechanics*, edited by J. C. Robinson, J. L. Rodrigo, and W. Sadowski (Cambridge University Press, Cambridge, UK, 2012), pp. 201–220.