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Time scales of turbulent relative dispersion

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Tracers in a turbulent flow separate according to the celebrated $t^{3/2}$ Richardson-Obukhov law, which is usually explained by a scale-dependent effective diffusivity. Here, supported by state-of-the-art numerics, we revisit this argument. The Lagrangian correlation time of velocity differences increases too quickly for validating this approach, but acceleration differences decorrelate on dissipative time scales. Phenomenological arguments are used to relate the behavior of separations to that of a "local energy dissipation," defined as the average ratio between the cube of the longitudinal velocity difference and the distance between the two tracers. This quantity is shown to stabilize on short time scales and this results in an asymptotic diffusion $\propto t^{1/2}$ of velocity differences. The time of convergence to this regime is shown to be that of deviations from Batchelor's initial ballistic regime, given by a scale-dependent energy dissipation time rather than the usual turnover time. It is finally demonstrated that the fluid flow intermittency should not affect this long-time behavior of the relative motion.

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Turbulence has the feature of strongly enhancing the dispersion and mixing of the species it transports. It is known since the work of Richardson [1] that tracer particles separate in an explosive manner, $\propto t^{3/2}$. While little doubt remains about its validity in three-dimensional homogeneous isotropic turbulence, observations of this law in numerics and experiments are difficult, as they require a huge scale separation between the dissipative lengths, the initial separation of tracers, the observation range, and the integral scale of the flow [2,3]. Much effort has been devoted to test the universality of this law, which was actually retrieved in various turbulent settings, such as the two-dimensional inverse cascade [4], buoyancy-driven flows [5], and magnetohydrodynamics [6]. At the same time, breakthroughs on transport by time-uncorrelated scale-invariant flows have strengthened the original idea of Richardson that this law originates from the diffusion of tracer separation in a scale-dependent environment [7]. As a result, the physical mechanisms leading to the Richardson-Obukhov $t^{3/2}$ law are still rather poorly understood and many questions remain open on the nature of subleading terms, the rate of convergence, and on the effects of the intermittent nature of turbulent velocity fluctuations [8,9].

Turbulent relative dispersion consists in understanding the evolution of the separation $\delta\boldsymbol{x}(t) = \boldsymbol{X}_1(t) - \boldsymbol{X}_2(t)$ between two tracers. Richardson's argument can be reinterpreted by assuming that the velocity difference $\delta\boldsymbol{u}(t) = \boldsymbol{u}(\boldsymbol{X}_1,t) - \boldsymbol{u}(\boldsymbol{X}_2,t)$ has a short correlation time. This means that the central-limit theorem applies and that, for sufficiently large time scales,

$$\frac{d\delta\boldsymbol{x}}{dt} = \delta\boldsymbol{u} \simeq \sqrt{\tau_L}\,\boldsymbol{U}(\delta\boldsymbol{x})\boldsymbol{\xi}(t), \qquad (1)$$

where $\boldsymbol{\xi}$ is the standard three-dimensional white noise, $\boldsymbol{U}^{\mathsf{T}}\boldsymbol{U} = \langle\delta\boldsymbol{u}\otimes\delta\boldsymbol{u}\rangle$ the Eulerian velocity difference correlation tensor, and τ_L the Lagrangian correlation time of velocity differences between pair separated by $\delta x = |\delta\boldsymbol{x}|$. As stressed by Obukhov [10], when assuming Kolmogorov 1941 scaling, $\tau_L \sim \delta x^{2/3}$, $\boldsymbol{U} \sim \delta x^{1/3}$, and the Fokker-Planck equation associated to (1) exactly corresponds to that derived by Richardson for the probability density $p(\delta x,t)$. It predicts in particular that the squared distance $\langle|\delta\boldsymbol{x}(t)|^2\rangle_{r_0}$ averaged over all pairs that are initially at a distance $|\delta\boldsymbol{x}(0)| = r_0$ has a long-time behavior $\propto t^3$ that is independent of r_0. This loss of memory on the initial separation can only occur on time scales longer than the correlation time $\tau_L(r_0) \sim r_0^{2/3}$ of the initial velocity difference. For times $t \ll \tau_L(r_0)$, one cannot make use of the approximation (1) as the velocity difference almost keeps its initial value. This corresponds to the ballistic regime $\langle|\delta\boldsymbol{x}(t) - \delta\boldsymbol{x}(0)|^2\rangle_{r_0} \simeq t^2 S_2(r_0)$, where $S_2(r) = \langle|\delta\boldsymbol{u}|^2\rangle$ is the Eulerian second-order structure function over a separation r, introduced by Batchelor [11]. The diffusive approach (1) can, however, be modified to account for the ballistic regime [12]. Nevertheless a short-time correlation of velocity differences can be hardly justified from first principles and seems to contradict turbulence phenomenology. Indeed, as stressed in Ref. [7], if δx grows as $t^{3/2}$, the Lagrangian correlation time τ_L is of the order of $\delta x^{2/3} \sim t$, so that the velocity difference correlation time is always of the order of the observation time. Relative dispersion strongly depends on flow time correlations as evidenced in Ref. [13]. Despite such apparent contradictions, the Richardson diffusive approach is relevant to describe some intermediate regime valid for large enough times and typical separations. Several measurements show that the separations distribute with a probability that is fairly close to that obtained from an eddy-diffusivity approach [9,14,15].

To clarify when and where Richardson's approach might be valid, it is important to understand the time scale of convergence to the explosive t^3 law. Recently, much work has been devoted to this issue: It was, for instance, proposed to make use of fractional diffusion with memory [16], to introduce random delay times of convergence to Richardson scaling [17], or to estimate the influence of extreme events in particle separation [18]. All these approaches consider as granted that the final behavior of separations is diffusive. As we will see here, many aspects of the convergence to Richardson's law for pair dispersion can be clarified in terms of a diffusive behavior of the velocity differences.

To address such issues, we make use of direct numerical simulations. For this, the Navier-Stokes equation with a

TABLE I. Parameters of the numerical simulations in arbitrary units where the box size is 2π. N is the number of grid points, R_λ the Taylor-based Reynolds number, ν the kinematic viscosity, ϵ the averaged energy dissipation rate, $u_{\rm rms}$ the root-mean-square velocity, $\eta = (\nu^3/\epsilon)^{1/4}$ the Kolmogorov dissipative scale, $\tau_\eta = (\nu/\epsilon)^{1/2}$ the associated turnover time, $L = u_{\rm rms}^3/\epsilon$ the integral scale, and $T = L/u_{\rm rms}$ the associated large-scale turnover time.

N	R_λ	ν	ϵ	$u_{\rm rms}$	η	τ_η	L	T
2048^3	460	2.5×10^{-5}	3.6×10^{-3}	0.19	1.4×10^{-3}	0.083	1.85	9.9
4096^3	730	1.0×10^{-5}	3.8×10^{-3}	0.19	7.2×10^{-4}	0.05	1.85	9.6

large-scale forcing is integrated in a periodic domain using a massively parallel spectral solver at two different resolutions. Table I summarizes the parameters of the simulations (see Ref. [19] for more details). In each case, the flow is seeded with 10^7 Lagrangian tracers. Their positions, velocities, and accelerations are then stored with enough frequency to study relative motion.

We first report results on the behavior of the separation $\delta\boldsymbol{x}(t)$ as a function of time. Following Ref. [14], a Taylor expansion at short times leads to

$$\langle|\delta\boldsymbol{x}(t) - \delta\boldsymbol{x}(0)|^2\rangle_{r_0} = t^2 S_2(r_0) + t^3\,\langle\delta\boldsymbol{u}\cdot\delta\boldsymbol{a}\rangle + O(t^4), \quad (2)$$

where $S_2(r) = \langle|\delta\boldsymbol{u}|^2\rangle$ is the second-order structure function, $\langle\cdot\rangle$ denote Eulerian averages, and $\delta\boldsymbol{a}(t) = \boldsymbol{a}(\boldsymbol{X}_1,t) - \boldsymbol{a}(\boldsymbol{X}_2,t)$ is the difference of the fluid acceleration sampled by the two tracers (using the notation $\boldsymbol{a} = \partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}$). As long as the term $\propto t^2$ is dominant, the tracers separate ballistically. Expansion (2) clearly fails for $t \approx t_0 = S_2(r_0)/|\langle\delta\boldsymbol{u}\cdot\delta\boldsymbol{a}\rangle|$. It is known [7,20] that for separations in the inertial range, $\langle\delta\boldsymbol{u}\cdot\delta\boldsymbol{a}\rangle = -2\epsilon$, which is nothing but a Lagrangian version of the 4/5 law. This implies that the ballistic regime ends at times of the order of

$$t_0 = S_2(r_0)/(2\epsilon). \quad (3)$$

This time scale can be interpreted as the time required to dissipate the kinetic energy contained at the scale r_0. We thus expect it to be equal to the correlation time of the initial velocity difference. t_0 differs from the turnover time $\tau(r_0) = r_0/[S_2(r_0)]^{1/2}$ defined as the ratio between the separation r_0 and the typical turbulent velocity at that scale. When Kolmogorov 1941 scaling is assumed, these two time scales have the same dependency on r_0. However, usual estimates of the Kolmogorov constant lead to $t_0/\tau(r_0) \approx 20$. Also, note that intermittency corrections to the scaling behavior of S_2 should in principle decrease this ratio. We indeed have $t_0 \propto r_0^{\zeta_2}$ and $\tau(r_0) \propto r_0^{1-\zeta_2/2}$, where ζ_2 denotes the scaling exponent of the second-order structure function; this is evidenced in the inset of Fig. 1. The main body of this figure represents the mean-squared displacement rescaled by $t_0^2 S_2(r_0)$ as a function of t/t_0, for various values of the initial separation r_0. In such units and when r_0 is far in the inertial range, all measurements collapse onto a single curve. The subleading term $\propto t^3$ in (2) is relevant for times $t \lesssim 0.01 t_0$. Note that we have checked that the same data using the turnover time $\tau(r_0)$ instead of t_0 does not display such a collapse.

The data collapse extends to times larger than t_0 when the mean-squared separation tends to Richardson t^3 regime. This unexpected fact implies that t_0 is not only the time scale of departure from the ballistic regime, but also that of convergence to Richardson's law. More precisely, numerical data suggest that for $t \gg t_0$

$$\langle|\delta\boldsymbol{x}(t) - \delta\boldsymbol{x}(0)|^2\rangle_{r_0} = g\epsilon t^3\,[1 + C t_0/t] + \text{h.o.t.} \quad (4)$$

C does not strongly depend on the Reynolds number. Systematic measurements as a function of the initial separation show that C is negative when r_0 is of the order of the Kolmogorov scale η. The convergence to Richardson law is then from below and is thus contaminated by tracer pairs which spend long times close together before sampling the inertial range; this is consistent with the findings of Ref. [18]. When r_0 is far enough in the inertial range, $C \approx 1.6$ becomes independent of the initial separation and the convergence to Richardson law is from above. One finds that $C = 0$ for $r_0 \approx 4\eta$; the only subleading terms in (4) are then of higher order, so that the mean-squared separation converges faster to the Richardson regime. Such an initial separation could be an "optimal choice" to observe the t^3 behavior in experimental settings.

To understand why the time scale of convergence to Richardson law is of the order of t_0, let us examine the time scales entering the relative dispersion process. As already stated, the velocity difference $\delta\boldsymbol{u}$ between the two tracers stays correlated over a time that increases too fast with the separation, making it difficult to justify the diffusive approach (1). However, it is known that turbulent acceleration, which is a small-scale quantity, is correlated over times that are of the

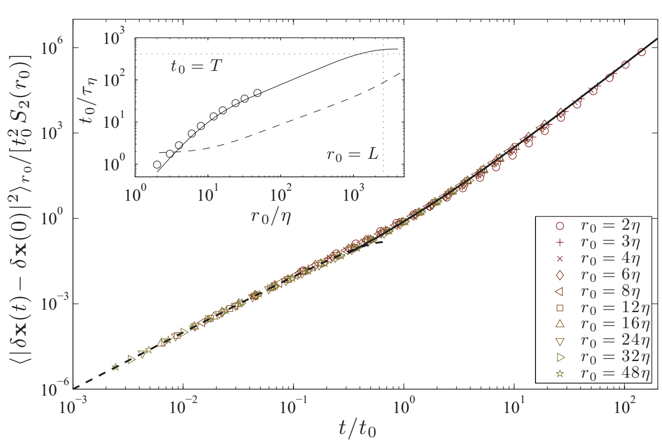

FIG. 1. (Color online) Time evolution of the mean-square separation for $R_\lambda = 730$ and various initial separations. The dashed line represents the behavior (2). The solid line is a fit to the Richardson regime (4) with $g = 0.52$ and $C = 1.6$. Inset: t_0 as a function of r_0 in dissipative-scale units. The solid line is an Eulerian average, the circles are Lagrangian measurements, and the dashed line is the turnover time $\tau(r_0)$.

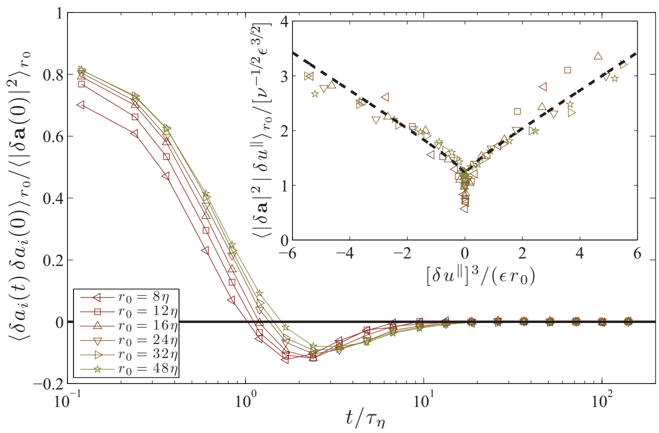

FIG. 2. (Color online) Lagrangian time autocorrelation of the acceleration difference $\delta\boldsymbol{a}$ for various r_0 and $R_\lambda = 730$. Inset: For the same separations r_0, variance of the acceleration difference amplitude conditioned on the longitudinal velocity difference $\delta u^\parallel$ as a function of the local dissipation rate $[\delta u^\parallel]^3/r_0$.

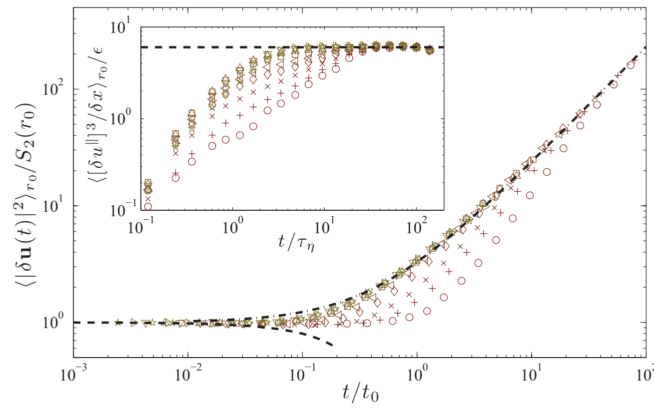

FIG. 3. (Color online) Time evolution of the averaged squared modulus of the velocity difference for $R_\lambda = 730$ and different r_0 (same symbols as in Fig. 1). The short-time prediction (7) is shown as a dashed line. The diffusive behavior $\langle|\delta\boldsymbol{u}|^2\rangle_{r_0} \simeq S_2(r_0) + 2.3\epsilon t$ is represented as a dashed-dotted line. Inset: Time evolution of $\langle[\delta u^\parallel]^3/\delta x\rangle_{r_0}$; the dashed line corresponds to the value 6ϵ.

order of τ_η, the Kolmogorov turnover time [21]. Its amplitude is rather correlated on times of the order of the forcing correlation time, but this does not alter the argument below.

Figure 2 represents the Lagrangian autocorrelation of the difference of acceleration $\delta\boldsymbol{a}$ between two tracers. We clearly see that the components of this quantity decorrelate on times of the order of τ_η. This suggests applying the central-limit theorem, so that for separations in the inertial range and on time scales much longer than the τ_η, the difference of acceleration between two tracers can be approximated by a delta-correlated-in-time random process. We thus have

$$\frac{d\delta\boldsymbol{x}}{dt} = \delta\boldsymbol{u}, \quad \text{with } \frac{d\delta\boldsymbol{u}}{dt} = \delta\boldsymbol{a} \simeq \sqrt{\tau_\eta^{\text{loc}}}\,\boldsymbol{A}(\delta\boldsymbol{x},\delta\boldsymbol{u})\boldsymbol{\xi}(t), \quad (5)$$

where $\boldsymbol{A}$ is defined as $\boldsymbol{A}^\mathsf{T}\boldsymbol{A} = \langle\delta\boldsymbol{a}\otimes\delta\boldsymbol{a}|\delta\boldsymbol{x},\delta\boldsymbol{u}\rangle$, $\boldsymbol{\xi}$ and the product is here understood in the Stratonovich sense. The idea of assuming uncorrelated accelerations is common to many stochastic models for turbulent dispersion (see, e.g., Refs. [2,22]). However, the aim here is not to derive a new model and contrast it to other approaches but to make use of (5) for phenomenological purposes. The local Kolmogorov time τ_η^{loc} and the acceleration amplitude $A = |\boldsymbol{A}|$ are expected to only depend on the viscosity ν and on the local energy dissipation rate ϵ_{loc}, as $\tau_\eta^{\text{loc}} \sim \nu^{1/2}\epsilon_{\text{loc}}^{-1/2}$ and $A \sim \nu^{-1/4}\epsilon_{\text{loc}}^{3/4}$. The multiplicative term in (5) then behaves as $[\tau_\eta^{\text{loc}}]^{1/2}A \sim \epsilon_{\text{loc}}^{1/2}$. Interestingly, this quantity is independent of ν and is thus expected to have a finite limit when $R_\lambda \to \infty$. Phenomenology suggests that for typical values of the velocity difference $\delta\boldsymbol{u}$, the local dissipation rate can be written as $\epsilon_{\text{loc}} \sim [\delta u^\parallel]^3/\delta x$, where $\delta u^\parallel = \delta\boldsymbol{x}\cdot\delta\boldsymbol{u}/\delta x$ is the longitudinal velocity difference between the tracers. When $\delta u^\parallel = 0$, the local dissipation rate does not vanish but can be estimated through an averaged contribution of larger eddies, leading to $\epsilon_{\text{loc}} \simeq \epsilon$, the averaged energy dissipation rate. These estimations have been tested against numerical simulations: The inset of Fig. 2 shows the variance of the acceleration differences conditioned on the longitudinal velocity difference for various separations. Up to statistical errors, data are in good agreement with the phenomenological prediction shown as a dashed line. Finally such dimensional considerations suggest approaching the large-time evolution of tracer separation as

$$\frac{d\delta x}{dt} = \delta u^\parallel, \quad \frac{d\delta u^\parallel}{dt} \sim \left[\epsilon + \alpha\frac{[\delta u^\parallel]^3}{\delta x}\right]^{1/2}\xi(t), \quad (6)$$

where α is a positive parameter. Again here the stochastic integration involves the Stratonovich convention. When rewriting it in the Itō sense, the additional drift that appears introduces a "correlation time" equal to the instantaneous turnover time $\delta x/\delta u^\parallel$. Preliminary studies of (6) showed that its solutions follow a ballistic regime at short times and behave according to Richardson law, i.e., $\langle\delta x^2\rangle \sim t^3$ at large times. Also, the local dissipation $[\delta u^\parallel]^3/\delta x$ tends to a constant, so that at large times the velocity difference obeys an equation of the form $d\delta u^\parallel/dt \propto \xi(t)$ and thus diffuses. So far, we have only used (6) for phenomenological purposes. Extending this approach to derive a functional stochastic model for relative dispersion requires generalizing it to higher dimensions to account for incompressibility.

To address the relevance of such an approach to real flows, we turn back to the analysis of simulation data. Figure 3 shows the time evolution of $\langle|\delta\boldsymbol{u}(t)|^2\rangle_{r_0}$ for various values of r_0. At small times this quantity slightly decreases because the subleading term is negative. We indeed have $\delta\boldsymbol{u}(t) \simeq \delta\boldsymbol{u}(0) + t\,\delta\boldsymbol{a}(0)$, so that the ballistic regime reads

$$\langle|\delta\boldsymbol{u}(t)|^2\rangle_{r_0} = S_2(r_0)(1 - 2t/t_0) + \text{h.o.t.} \quad (7)$$

Again, the subleading terms are relevant for times $t \lesssim 0.01t_0$. Figure 3 also shows that at large times the mean-squared velocity difference loses dependence on r_0 and grows $\propto\epsilon t$. In addition, as seen from the inset of Fig. 3, the averaged local dissipation rate $\langle[\delta u^\parallel]^3/\delta x\rangle_{r_0}$ along particle pairs approaches a positive constant $\simeq 6\epsilon$ (independently of R_λ) on times of the order of τ_η. This confirms the relevance of the mechanisms described above in terms of a stochastic equation for the velocity differences.

Numerical results indicate that the time t_0 controls the convergence to a diffusive regime for initial separations r_0

far enough in the inertial range. This can be explained by the following argument. As $\langle[\delta u^{\parallel}]^3/\delta x\rangle_{r_0}$ becomes constant on a short time scale (of the order of τ_η), one expects that

$$\langle|\delta \boldsymbol{u}(t)|^2\rangle_{r_0} \simeq S_2(r_0) + D\epsilon t \quad \text{for} \quad t \gg \tau_\eta, \tag{8}$$

where D is a positive constant (for both Reynolds numbers, we observe $D \approx 2.1$). By balancing the diffusive term with the initial mean-squared velocity difference $\langle|\delta \boldsymbol{u}(0)|^2\rangle_{r_0} = S_2(r_0)$, we find again that the former is dominant for times t much larger than t_0. The diffusive behavior of velocity differences is thus reached at times of the order of t_0 and this explains in turn why this time scale is that of convergence to Richardson's regime.

We have thus obtained evidence that the Richardson explosive regime $\langle|\delta \boldsymbol{x}|^2\rangle \propto t^3$ for the separation between two tracers originates from a diffusive behavior of their velocity difference rather than from dimensional arguments or, equivalently, a scale-dependent eddy diffusivity for their distance. This leads on to reinterpret the t^3 law as that of the integral of Brownian motion. Such an argument is supported by two observations. First, the acceleration difference has a short correlation time (of the order of the Kolmogorov dissipative time scale) and can be approximated as a white noise. Second, the amplitude of this noise solely depends on the local dissipation rate $\langle[\delta u^{\parallel}]^3/\delta x\rangle_{r_0}$, which becomes constant also on short time scales. These considerations allow us to show that the time t_0 of convergence to Richardson's law is equal to that of deviations from Batchelor's ballistic regime. This time, which reads $t_0 = S_2(r_0)/(2\epsilon)$, is the time required to dissipate the kinetic energy contained at a scale equal to the initial separation between tracers.

The interpretation of Richardson's law as the diffusion of velocity differences strongly questions possible effects of fluid-flow intermittency on the separation of trajectories. Indeed, considerations on velocity scaling, which are primordial in approaches based on eddy diffusivity, are absent from the arguments leading to a diffusive behavior of $\delta \boldsymbol{u}$. We find that, for this reason, the separation $\delta \boldsymbol{x}$ follows an almost self-similar evolution in time, independently of the order of the statistics. Indeed, as seen in Fig. 4, the fourth- and sixth-order moments of the displacement are, up to statistical errors, proportional to the square and the cube of the second-order moment. Any intermittent correction might only be visible in much higher-order moments or in statistics related to slowly separating pairs. We nevertheless predict that intermittency will affect the time of convergence to such a regime. More frequent violent events (of tracer pairs approaching or fleeing away in an anomalously strong manner) will result in longer times for being absorbed by the average. Such arguments do not rule out the possibility of having intermittency corrections when interested in other observables than moments of the separation, as it is, for instance, the case for exit times [8]. Such issues will certainly gain much from a systematic study of multidimensional stochastic models that are using the approach described here.

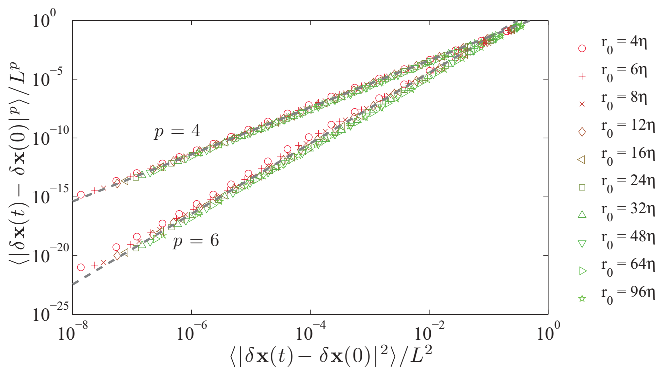

FIG. 4. (Color online) Fourth- and sixth-order moments of the displacement $|\delta \boldsymbol{x}(t) - \delta \boldsymbol{x}(0)|$ for $R_\lambda = 730$. Following ideas from Ref. [23], they are represented as a function of the second-order moment. The two gray dashed lines show scale-invariant behaviors of the form $\langle|\delta \boldsymbol{x}(t) - \delta \boldsymbol{x}(0)|^p\rangle \propto \langle|\delta \boldsymbol{x}(t) - \delta \boldsymbol{x}(0)|^2\rangle^{p/2}$.

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[1] L. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926).
[2] B. Sawford, Annu. Rev. Fluid Mech. **33**, 289 (2001).
[3] J. Salazar and L. Collins, Annu. Rev. Fluid Mech. **41**, 405 (2009).
[4] M.-C. Jullien, J. Paret, and P. Tabeling, Phys. Rev. Lett. **82**, 2872 (1999).
[5] J. Schumacher, Phys. Rev. Lett. **100**, 134502 (2008).
[6] A. Busse and W. Müller, Astron. Nachr. **329**, 714 (2008).
[7] G. Falkovich, K. Gawędzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).
[8] G. Boffetta and I. M. Sokolov, Phys. Rev. Lett. **88**, 094501 (2002).
[9] L. Biferale, G. Boffetta, A. Celani, B. Devenish, A. Lanotte, and F. Toschi, Phys. Fluids **17**, 115101 (2005).
[10] A. Obukhov, Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz. **5**, 453 (1941).
[11] G. Batchelor, Q. J. R. Meteorol. Soc. **76**, 133 (1950).
[12] R. Kraichnan, Phys. Fluids **9**, 1937 (1966).
[13] M. Chaves, K. Gawedzki, P. Horvai, A. Kupiainen, and M. Vergassola, J. Stat. Phys. **113**, 643 (2003).
[14] N. Ouellette, H. Xu, M. Bourgoin, and E. Bodenschatz, New J. Phys. **8**, 109 (2006).
[15] G. L. Eyink, Phys. Rev. E **83**, 056405 (2011).
[16] V. Ilyin, I. Procaccia, and A. Zagorodny, Phys. Rev. E **81**, 030105 (2010).

[17] M. P. Rast and J.-F. Pinton, Phys. Rev. Lett. **107**, 214501 (2011).
[18] R. Scatamacchia, L. Biferale, and F. Toschi, Phys. Rev. Lett. **109**, 144501 (2012).
[19] R. Grauer, H. Homann, and J.-F. Pinton, New J. Phys. **14**, 063016 (2012).
[20] S. Ott and J. Mann, J. Fluid Mech. **422**, 207 (2000).
[21] N. Mordant, A. M. Crawford, and E. Bodenschatz, Phys. Rev. Lett. **93**, 214501 (2004).
[22] O. Kurbanmuradov and K. Sabelfeld, Monte Carlo Methods Appl. **1**, 101 (1995).
[23] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli, and S. Succi, Phys. Rev. E **48**, R29 (1993).