Phase-locking-level statistics of coupled random fiber lasers

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We measure the statistics of phase locking levels of coupled fiber lasers with fluctuating cavity lengths. We found that the measured distribution of the phase locking level of such coupled lasers can be described by the generalized extreme value distribution. For large number of lasers the distribution of the phase locking level can be approximated by a Gumbel distribution. We present a simple model, based on the spectral response of coupled lasers, and the calculated results are in good agreement with the experimental results.

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I. INTRODUCTION

Phase locking of coupled oscillators has been studied over the years in many different contexts, including chemical oscillators with mutual coherence [1], arrays of Josephson junctions that are frequency locked [2], and arrays of coupled lasers that are phase locked [3–5]. In coupled oscillators, complete phase locking occurs when all the oscillators have at least one common frequency. When there is no common frequency, the oscillators group in several clusters, where each cluster oscillates at a different frequency [6]. Recently, we measured power fluctuations in coupled fiber lasers and showed the distribution fits the distribution of the maximum eigenvalue of a random Wishart matrix ensemble. However, this was done close to threshold, where the lasing was not efficient [7].

Here, we study the statistical aspects of coupled oscillators that do not share a common frequency. Specifically, we study the phase locking level of an array of fiber lasers without a common frequency operating far above threshold and show that the distribution of the phase locking level can be described by the generalized extreme value (GEV) distribution function. The GEV distribution of a random variable depends on three parameters: the mean μ, the standard deviation σ, and a single shape parameter ξ [8], which control the asymmetry of the distribution due to the boundaries of the phase locking level. For $\xi = 0$, the GEV distribution reduces to the Gumbel distribution, which is the extreme value statistics of a Gaussian process [9,10]. Although each fiber laser supports 100 000 eigenfrequencies, the probability of finding a common frequency for all the lasers in the array is very small ($<10^{-5}$) [11–13], so the lasers group in several clusters, each with its own frequency [14,15]. Due to thermal and acoustic fluctuations, the length of each fiber laser and its corresponding eigenfrequencies changes rapidly and randomly.

The phase locking level for different frequencies is random and bounded by 1 [11]. Since phase locking minimizes loss in the array, mode competition favors frequencies that maximize the phase locking level of the array at each moment [16]. Therefore, the lasing frequency is the one with the highest phase locking levels, so the measured phase locking level is the maximum phase locking level out of all the possible frequencies. Hence, its distribution can be described by the GEV distribution function. The three parameters of the GEV distribution function can be directly estimated from the data using the maximum-likelihood method. Thus, a simple method to check that our reasoning is correct is to plot the GEV distribution with the maximum-likelihood-estimated parameters and compare it to the measured distribution. If, indeed, the fluctuations obey extreme value statistics, there should be good agreement without any fitting parameters.

II. EXPERIMENTAL CONFIGURATION AND RESULTS

The experimental configuration that we used for measuring the phase locking level of an array of fiber lasers is presented in Fig. 1 and is described in detail in [15]. Briefly, each fiber laser was comprised of a ytterbium doped fiber, a rear high reflection (>99%) fiber Bragg grating (FBG), and a front low reflecting (5%) FBG, both with a 10 nm bandwidth. Each laser was pumped through the rear FBG with a 975 nm diode laser at 200 mW, and after the front FBG we attached a collimator to obtain a 0.4 mm diameter beam. The collimators of all 25 lasers were accurately aligned in a 5×5 square array of parallel beams with parallelism better than 0.1 mrad. The separation between adjacent beams was 3.6 mm. A representative near-field intensity distribution, measured close to the output coupler when all 25 fiber lasers are operating, is presented in the inset in Fig. 1. We determined the length of each fiber laser by measuring the longitudinal mode beating frequency at the output by means of a fast photodetector connected to a rf spectrum analyzer. We found that the distribution of the lasers lengths is Gaussian, with a mean value of 3 m and a width of 0.5 m. The intensity of each fiber laser was about 100 mW, which is much above threshold but still low enough that nonlinear effects were negligible [17]. We also measured the intensity fluctuations of a single laser and found them to be less than 2%.

The coupling between the fiber lasers was achieved by means of four coupling mirrors denoted as r_1, r_2, r_3, and r_4 with reflectivity of 40% for r_1 and r_3 and reflectivity of 100% for r_2 and r_4. All the coupling mirrors were located close to the focal plane of a focusing lens with 500 mm focal length, forming a self-imaging cavity with the array. Since there was only enough space for one pair of mirrors within the Rayleigh range of the focusing lens, we inserted a 50% beam splitter to obtain another focal plane where we placed another pair of mirrors. By controlling the orientations of the coupling mirrors we could realize a variety of connections for the fiber lasers in

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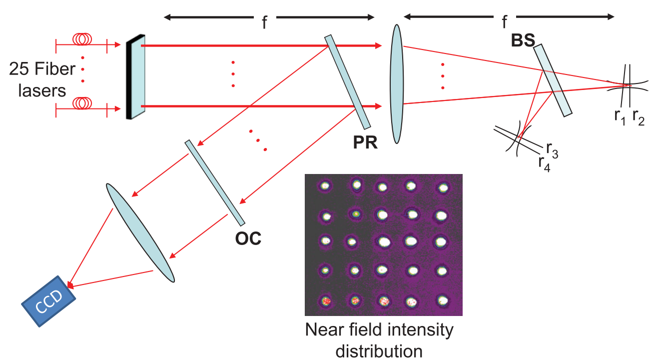

FIG. 1. (Color online) Experimental configuration for phase locking an array of fiber lasers and for determining their phase locking level. OC, output coupler; PR, partial reflector; BS, 50% beam splitter. The inset shows the near field intensity distribution when all 25 fiber lasers are operating.

the array, and in our experiments we concentrated on the one- and two-dimensional nearest neighbor connectivities [15]. Finally, we directed about 10% of the light with a partially reflecting mirror (PR) towards an output coupler (OC) of 99% reflectivity. The output coupler was placed at a distance of $2f$ from the collimator array and reflected part of the light from each laser back onto itself with the same delay as the light that is coupled from the other lasers [18].

The lengths of the fibers are changing due to temperature fluctuations and acoustic noises, so the phase locking level continuously changes. We measured the phase locking level as a function of time for different number of lasers in the array. This was done by continuously detecting the far-field intensity distribution of the interference pattern of all the light from the array with a CCD camera, determining the maxima and minima intensities, and calculating the average fringe visibility along the x and y directions. The fringe visibility provides a direct measure for the phase locking level that ranges from 0 to 1. The correlation time of the phase locking level was measured to be shorter than 100 ms, so over a 10 h period we acquired about 370 000 uncorrelated measurements of the fringe visibility. Representative experimental results of the fringe visibility as a function of time for a 10 s interval are presented in Fig. 2. The insets show two typical far-field intensity distributions, one with low fringe visibility and the other with high fringe visibility. We note that when coupling only two fiber lasers, the fluctuations of the phase locking level were less than 1% [19].

To check for possible correlations in the experimental results we fitted the phase-locking-level distribution with the Bramwell-Holdsworth-Pinton (BHP) distribution using a single fitting parameter c [20]. In the two-dimensional XY model at low temperatures, $c \approx \pi/2$, indicating a highly correlated process, while for uncorrelated systems the parameter c has an integer value. In particular, when $c = 1$, the BHP distribution reduces to the Gumbel extreme value distribution. The functional form of the BHP distribution is given by

$$P(x) = e^{c(\frac{x-\mu}{\sigma} - e^{\frac{x-\mu}{\sigma}})}, \qquad (1)$$

where μ denotes the mean value, σ is the standard deviation of the distribution, and c is the measure for correlations [20]. After fitting the measured probability distribution of the phase

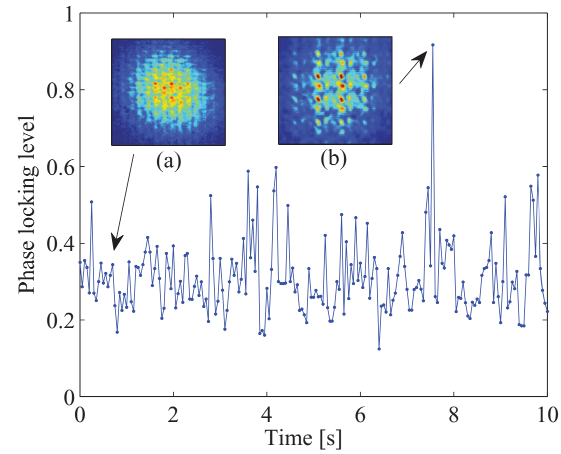

FIG. 2. (Color online) Typical experimental results of the phase locking level as a function of time over a 10 s interval. The phase locking level was determined from the far-field intensity distribution of the output. The insets show typical far field intensity distributions: (a) low fringe visibility, where the phase locking level is low, and (b) high fringe visibility, where the phase locking level is high.

locking level of 25 coupled fiber lasers to the BHP distribution, we obtained $c = 1.03$, indicating a regular extreme value distribution of a noncorrelated Gaussian process, namely, the Gumbel distribution.

Fitting the Gumbel distribution to the experimental results for 12, 16, and 20 fiber lasers was not as good as for the 25 fiber lasers. The distribution of the experimental results for a low number of lasers is clamped because the phase locking level is bound between 0 and 1 and μ and σ (mean and standard deviation, respectively) are high. Therefore, we resort to the GEV distribution, which contains one extra parameter, the shape parameter ξ [8]. When $\xi = 0$, the GEV distribution reduces to the Gumbel distribution, but as $\xi < 0$, the GEV distribution is clamped and approaches the Weibull distribution. The GEV distribution is given by

$$P(x) = \frac{1}{\sigma}\left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{(-1/\xi)-1} e^{-[1+\xi(\frac{x-\mu}{\sigma})]^{-1/\xi}}, \qquad (2)$$

where μ, σ, and ξ were calculated from the experimental data by maximum-likelihood parameter estimation. We expect that as the number of fiber lasers in the array decreases, the probability of having a common frequency between the lasers increases, so the mean value and standard deviation of the phase locking level will also increase, and the ξ parameter should vary from zero to a negative value.

The measured phase-locking-level distribution for laser arrays with 12, 16, 20, and 25 fiber lasers and the corresponding GEV distribution with the calculated parameters and their 95% confidence intervals are presented in Fig. 3. As is evident, there is a very good agreement between the experimental results and the GEV distribution extending over three decades. As the number of fiber lasers increases, the values of μ and σ decrease, and the ξ parameter approaches zero. For 25 fiber lasers the ξ parameter reaches -0.01, close to the expected zero value where the GEV distribution reduces to the Gumbel distribution.

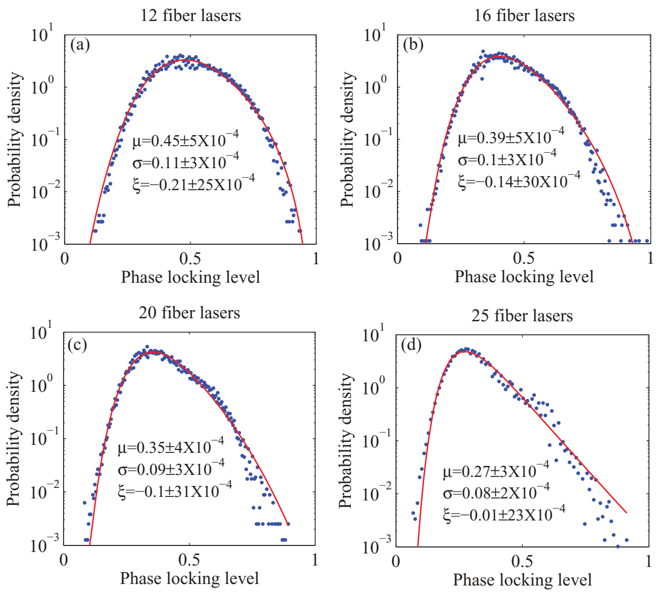

FIG. 3. (Color online) Measured phase-locking-level distribution for four different arrays of fiber lasers: (a) 12 fiber lasers, (b) 16 fiber lasers, (c) 20 fiber lasers, and (d) 25 fiber lasers. The curves are associated GEV distributions without any fitting parameters. The parameters were estimated from the measured phase locking level using the maximum likelihood. The 95% confidence intervals are also shown.

By changing the orientation of the coupling mirrors we also realized a one-dimensional connectivity of the 25 fiber lasers and measured the phase locking level as a function of time. Again, we found good agreement between the measured phase-locking-level distribution and the Gumbel distribution. The experimental results of the one-dimensional connectivity and the two-dimensional connectivity, together with calculated Gumbel distributions, are presented in Fig. 4. An array with one-dimensional connectivity is far more sensitive to misalignments, which reduce the coupling and the phase locking levels [6,15]; indeed, the average phase locking level was even lower than the 25 lasers with the two-dimensional connectivity. This explains why the phase-locking level for the one-dimensional connectivity is also distributed to a good approximation like a Gumbel distribution.

The connection between the phase locking level for large number of lasers and the Gumbel distribution can be qualitatively explained as follows. The number of lasers in each phase-locked cluster changes rapidly and randomly but always while maximizing the phase locking level in the array. This maximum phase locking level occurs at a specific frequency out of all the possible frequencies within the FBG bandwidth where the lasers losses are minimal. By considering the spectral response of the coupled lasers it was shown that the distribution of the phase locking levels for all the available frequencies is an uncorrelated Gaussian distribution [11,19]. Since the distributions of maxima of uncorrelated Gaussian processes are described by the Gumbel distribution function [9,10], the probability distribution of the phase locking level should be the same. As long as the mean phase locking level is far from unity, the influence of the boundaries is negligible.

III. THEORETICAL MODEL AND CALCULATIONS

Next, we present a simple quantitative model that relates the phase locking level to an extreme value in the spectral response of the array of coupled lasers. We start by assuming no gain and determining the spectral response of each laser cavity when we replace all components that couple light into the laser cavity with an effective front mirror. The reflectivity of this effective mirror depends on the frequency [19]. For example, the effective reflectivity of the ith laser in a one-dimensional array of N coupled lasers when considering the coupling to its two nearest neighbors is

$$R_i = \frac{1}{1 - r(1-2\kappa) + \frac{\kappa^2}{1-R_{i-1}^{(u)}e^{2\imath k l_{i-1}}} + \frac{\kappa^2}{1-R_{i+1}^{(d)}e^{2\imath k l_{i+1}}}}, \tag{3}$$

where κ denotes the coupling to the two neighbors, r is the reflectivity of the output coupler, l_i is the length of the ith laser, k is the propagation vector, and $R_i^{(u)}$ and $R_i^{(d)}$ are the effective reflectivities from all the lasers above and below the ith laser,

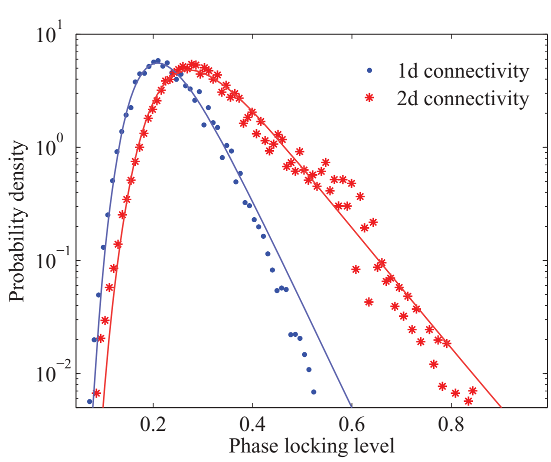

FIG. 4. (Color online) Measured phase-locking-level distribution for arrays of 25 fiber lasers with two different connectivities. (a) One-dimensional connectivity and (b) two-dimensional connectivity. The curves are associated Gumbel distributions.

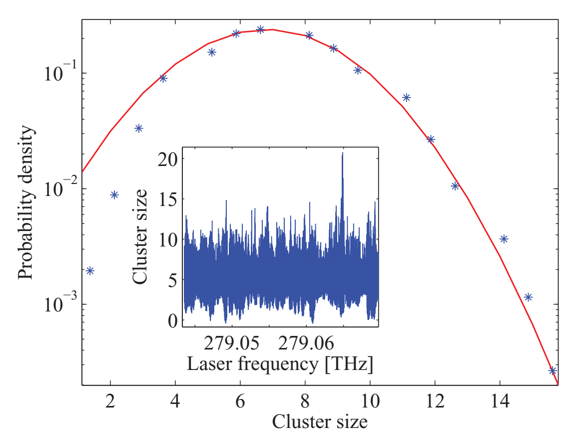

FIG. 5. (Color online) Calculated probability distribution of the size of the main cluster together with a Gaussian fit for clusters larger than the mean size. The inset shows representative calculated results of the main cluster size as a function of the laser frequency.

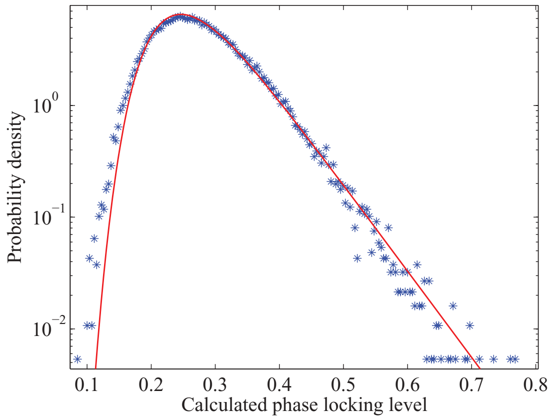

FIG. 6. (Color online) Calculated probability distribution of the phase locking level from 50 000 different arrays of fiber lasers and a fit to a Gumbel distribution.

given by the recursive relations

$$R_i^{(u)} = \frac{1}{1 - r(1-2\kappa) + \frac{\kappa^2}{1 - R_{i-1}^{(u)} e^{2\iota k l_{i-1}}}}, \tag{4}$$

$$R_i^{(d)} = \frac{1}{1 - r(1-2\kappa) + \frac{\kappa^2}{1 - R_{i+1}^{(d)} e^{2\iota k l_{i+1}}}}, \tag{5}$$

and $R_1^{(u)} = R_N^{(d)} = 0$. Now, introducing the laser gain together with a mode competition results in amplifications only at frequencies with high effective reflectivity.

We solve Eqs. (3)–(5) for an array of 25 fiber lasers and obtain the effective reflectivity of each laser as a function of frequency. Since the effective reflectivity takes into account the coupling between the lasers, the effective reflectivity is higher and the losses are lower when more lasers have a common frequency, which means that more lasers are phase locked. By counting the number of lasers with effective reflectivity above a certain threshold as a function of frequency, we obtain the number of lasers in the main cluster, which provides a direct measure for the phase locking level of the array as a function of frequency [12,13,15]. Representative results for one realization of the random fiber laser lengths are shown in Fig. 5, which shows the probability distribution of the size of the main cluster together with a Gaussian fit for the tail of the distribution. The inset shows the size of the main cluster as a function of the frequency. As is evident, the calculated probability distribution above the mean size of the main cluster is Gaussian. Then we select the maximum size of the main cluster within the bandwidth of the FBG (10 nm) and determine the resulting phase locking level as the ratio of the size of the main cluster over the size of the array.

We repeat these calculations for 50 000 different arrays, each with a different random fiber length. The fiber length of the ith laser in each array was chosen to be $l_i + \Delta l_i$, where l_i is the measured length of the ith fiber and Δl_i is a random length taken from a normal distribution with zero mean and 10 μm width. The results are presented in Fig. 6, which shows the probability distribution of the calculated phase locking level together with a fit to a Gumbel distribution. As is evident, there is a very good agreement between the distribution of the calculated results and the Gumbel distribution, indicating that the effective reflectivity is suitable for modeling arrays of coupled lasers and that the underlying Gaussian process is the number of lasers in the main cluster.

IV. CONCLUSIONS

To conclude, we measured the distribution of phase locking levels for arrays of coupled fiber lasers with fluctuating cavity lengths and showed that they are well described by generalized extreme value distributions without any fitting parameters. Fitting BHP distribution to the experimental results yields no correlation in the data. We found that for both one- and two-dimensional arrays of 25 lasers, the statistics of the phase locking level can be described by the Gumbel distribution and that as the number of lasers decreases it approaches the Weibull distribution. We presented a simple theoretical model to explain this observation. The model demonstrates that the level of phase locking is determined mostly by the size of the largest phase-locked cluster. We calculated the distribution of the number of lasers in such a cluster and found it to be asymptotically Gaussian. Our results can be used to calculate the probability of obtaining a specific phase locking level for a given number of coupled lasers. Finally, by operating the fiber lasers close to threshold we also observed strong fluctuations in the total power of the array which are related to the statistics of extreme eigenvalue of random Wishart matrices [7].

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