Poissonian steady states: From stationary densities to stationary intensities

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Markov dynamics are the most elemental and omnipresent form of stochastic dynamics in the sciences, with applications ranging from physics to chemistry, from biology to evolution, and from economics to finance. Markov dynamics can be either stationary or nonstationary. Stationary Markov dynamics represent statistical steady states and are quantified by stationary densities. In this paper, we generalize the notion of steady state to the case of general Markov dynamics. Considering an ensemble of independent motions governed by common Markov dynamics, we establish that the entire ensemble attains Poissonian steady states which are quantified by stationary Poissonian intensities and which hold valid also in the case of nonstationary Markov dynamics. The methodology is applied to a host of Markov dynamics, including Brownian motion, birth-death processes, random walks, geometric random walks, renewal processes, growth-collapse dynamics, decay-surge dynamics, Ito diffusions, and Langevin dynamics.

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I. INTRODUCTION

Markov dynamics are the stochastic counterpart of ordinary differential equations. Motions governed by Markov dynamics are the most commonly applied class of stochastic processes in the sciences [1–3], and their applications range from physics and chemistry to biology and economics [4–6]. Motions governed by Markov dynamics—henceforth termed *Markov motions*—can be either stationary or nonstationary. The notion of stationarity is of the utmost importance, as it represents the statistical steady state of stochastic systems and processes whose evolution is Markov.

The steady states of a given Markov motion are quantified by its *stationary densities*. The computation of the stationary densities of a given Markov motion is carried out via the analysis of the steady-state solutions of the *master equation* of the motion's Markov dynamics [4,7]. If the master equation has positive-valued and normalized steady-state solutions, then these solutions are the motion's stationary densities. On the other hand, if the master equation has positive-valued and steady-state solutions which are not normalizable, then the motion is nonstationary. A quintessential example of the non-normalizable scenario is Brownian motion, the archetypal model of diffusion [4,8].

The goal of this paper is to extend the notion of steady state to the case of Markov motions whose master equations have positive-valued and non-normalizable steady-state solutions. Describing the steady states in the context of nonstationary Markov motions is oxymoronic. To attain our oxymoronic goal, we shall consider not just one Markov motion but countably many motions. Specifically, we shall consider a countable ensemble of independent Markov motions with common Markov dynamics and address the stochastic evolution of the entire ensemble.

In this paper we establish that the positions of ensembles of independent Markov motions (with common Markov dynamics) attain steady states even in the case of nonstationary Markov dynamics. Considering given Markov dynamics we will show that, if the corresponding master equation has positive-valued steady-state solutions, then:

(i) these solutions characterize the steady states of an ensemble of independent motions governed by the Markov dynamics;

(ii) if the solutions are normalizable, then each motion attains a steady state, and the ensemble's steady states are equivalent to the steady states of its composing motions;

(iii) if the solutions are not normalizable, then each motion is nonstationary and does not attain a steady state but the entire ensemble does attain steady states.

The steady states of ensembles of independent Markov motions are based on the notion of *Poisson processes*, the common statistical methodology to model the scattering of points in general domains [9–11]. Poisson processes have a wide spectrum of applications ranging from insurance and finance [12] to queueing systems [13], from anomalous diffusion [14] to statistical diversity [15], and from fractal processes [16] to central limit theorems [17] and power laws [18]. The distribution of a random variable, taking values in a given state space, is characterized by a probability density function defined on the state space; the density is a positive-valued and normalized function. On the other hand, the distribution of a Poisson process, scattered in a given state space, is characterized by a Poissonian intensity function defined on the state space; the intensity is a positive-valued function which can be either integrable or nonintegrable.

We will show that the positive-valued steady-state solutions of a master equation—of given Markov dynamics—represent Poissonian intensity functions. In turn, these intensity functions characterize Poisson processes which are the steady states of an ensemble of independent Markov motions governed by the common Markov dynamics. Integrable intensity functions represent the case where each motion attains a steady state, whereas nonintegrable intensity functions represent the case where each motion is nonstationary but the entire ensemble attains a steady state. Thus, shifting from steady states to "Poissonian steady states" and from stationary densities to "stationary Poissonian intensities" enables us to attain the oxymoronic goal of describing the steady states of

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nonstationary Markov motions. For a concise exposition of part of the results presented in this paper, addressed to a general scientific audience, the readers are referred to Ref. [19].

This paper is organized as follows. We begin with a brief review of Markov dynamics (Sec. II), followed by an illustrative example of Brownian motion (Sec. III). Introducing, within the context of Markov dynamics, an underlying Poissonian setting (Sec. IV), we establish the notions of Poissonian steady states and stationary intensities (Sec. V). The stationary intensities are then rederived via the notion of Poissonian fluxes (Sec. VI), and the notion of Poissonian correlations is further established (Sec. VII). The application of Poissonian steady states is exemplified in the context of (Sec. VIII) birth-death processes, random walks, geometric random walks, renewal processes, growth-collapse dynamics, decay-surge dynamics, Ito diffusions, and Langevin dynamics.

A note about notation. Throughout the paper, IID is the acronym for "independent and identically distributed," $\mathbf{E}[\cdot]$ denotes the operation of mathematical expectation, and $\langle\cdot,\cdot\rangle$ denotes the operation of scalar product in the d-dimensional Euclidean space $\mathbb{R}^d$.

II. MARKOV DYNAMICS

Consider a general *Markov motion* X taking place in a general state space $\mathcal{S}$, and let $X(t)$ denote the motion's position at time t ($t \geqslant 0$). The motion's Markov dynamics are analytically characterized by the *infinitesimal generator*

$$[\mathbf{G}\phi](s) = \lim_{\delta\to 0}\frac{1}{\delta}\mathbf{E}[\phi(X(t+\delta)) - \phi(X(t))|X(t) = s] \quad (1)$$

($s \in \mathcal{S}$), where $\phi(s)$ is an arbitrary real-valued test function defined on the state space [20]. The infinitesimal generator $\mathbf{G}$ is a linear operator and, in a sense, is the "derivative" of the Markov motion X.

The *adjoint operator* $\mathbf{G}^*$ of the infinitesimal generator $\mathbf{G}$ is given implicitly by

$$\int_{\mathcal{S}}[\mathbf{G}\phi](s)\psi(s)m(ds) = \int_{\mathcal{S}}\phi(s)[\mathbf{G}^*\psi](s)m(ds), \quad (2)$$

where $\phi(s)$ and $\psi(s)$ are real-valued test functions defined on the state space and where $m(\cdot)$ is the space's "natural measure."[1] The adjoint operator $\mathbf{G}^*$ is a linear operator which governs the dynamics of the distributions of the motion's positions. Indeed, let $P(t,s)$ ($t \geqslant 0, s \in \mathcal{S}$) denote the probability density function [with respect to the measure $m(\cdot)$] of the random variable $X(t)$, the motion's position at time t. The motion's *master equation* then is given by

$$\frac{\partial}{\partial t}P = \mathbf{G}^*P, \quad (3)$$

where the initial condition $P(0,s)$ is an arbitrary probability density function defined on the state space [4,7,8].

The Markov motion X may or may not attain steady states. In case a steady state is attained, then it is characterized by a *stationary density*, a probability density function $P_{\mathrm{SD}}(s)$ [with respect to the measure $m(\cdot)$] defined on the state space which satisfies the following stationarity condition: If $P(0,s) = P_{\mathrm{SD}}(s)$, then $P(t,s) = P_{\mathrm{SD}}(s)$ for all $t > 0$. The stationary densities $P_{\mathrm{SD}}(s)$ are the "fixed points" of the underlying Markov dynamics. Substituting $P(t,s) = P_{\mathrm{SD}}(s)$ into Eq. (3) implies that $\mathbf{G}^*P_{\mathrm{SD}} = 0$. Thus, in order to compute the stationary densities $P_{\mathrm{SD}}(s)$ of the Markov motion X, one needs to solve the equation

$$\mathbf{G}^*\psi = 0. \quad (4)$$

If the "steady-state equation" (4) has solutions which are positive-valued [$\psi(s) > 0$] and normalized [$\int_{\mathcal{S}}\psi(s)m(ds) = 1$], then these solutions are the stationary densities $P_{\mathrm{SD}}(s)$ of the Markov motion X. On the other hand, if the "steady-state equation" (4) has no solutions which are positive-valued and normalized, then the Markov motion X is nonstationary and does not attain steady states.

III. BROWNIAN MOTION

As an illustrative example of Markov dynamics, consider *Brownian motion*, the archetypal model of diffusion in the physical sciences [4,8]. Brownian motion was discovered by Brown in 1827 [21], was first applied by Bachelier to model stock prices in 1900 [22], was established by Einstein and Smoluchowski as the prototypical model of diffusion in 1905 [23,24], and was mathematically constructed by Wiener in 1923 [25].

In the case of d-dimensional Brownian motion, the state space is the d-dimensional Euclidean space $\mathcal{S} = \mathbb{R}^d$, and the "natural measure" is the Lebesgue measure $m(ds) = ds$. Brownian motion X is a random process whose increments are stationary, independent, and Gauss distributed [26]. The compact characterization of Brownian motion is via the Fourier transforms of its increments,

$$\mathbf{E}[\exp(i\langle\theta, X(t+\delta) - X(t)\rangle)] = \exp\left(-\tfrac{1}{2}\langle\theta,\theta\rangle\delta\right) \quad (5)$$

($\theta \in \mathbb{R}^d$ being the Fourier variable; $t,\delta > 0$).

The infinitesimal generator of Brownian motion is the Laplacian

$$[\mathbf{G}\phi](s) = \tfrac{1}{2}[\Delta\phi](s), \quad (6)$$

and the adjoint operator of Brownian motion is also the Laplacian

$$[\mathbf{G}^*\psi](s) = \tfrac{1}{2}[\Delta\psi](s) \quad (7)$$

[8,26]. In turn, the corresponding "steady-state equation" (4) is given by

$$[\Delta\psi](s) = 0. \quad (8)$$

In the one-dimensional case ($\mathcal{S} = \mathbb{R}$) the solutions of the "steady-state equation" (8) are affine functions: $\psi(s) = as + b$ ($a,b \in \mathbb{R}$). Consequently, the positive-valued solutions of the "steady-state equation" (8) are constant functions. In the general d-dimensional case ($\mathcal{S} = \mathbb{R}^d$), the solutions of the "steady-state equation" (8) are more complex, but the positive-valued solutions are—as in the one-dimensional case—constant functions:

$$\psi(s) = c, \quad (9)$$

[1]For example, (i) if the state space is a countable set, then $m(\cdot)$ is the counting measure and $\int_{\mathcal{S}}\varphi(s)m\,(ds) = \sum_{s\in\mathcal{S}}\varphi(s)$; (ii) if the state space is an Euclidean domain, then $m(\cdot)$ is the Lebesgue measure and $\int_{\mathcal{S}}\varphi(s)m\,(ds) = \int_{\mathcal{S}}\varphi(s)ds$.

where c is a positive constant. Since constant functions are not integrable over the d-dimensional Euclidean space, the "steady-state equation" (8) has no solutions which are positive-valued and normalized. Consequently, we obtain that Brownian motion is nonstationary and does not attain a steady state.

IV. POISSONIAN SETTING

As noted in the Introduction, *Poisson processes* are the common statistical methodology to model the scattering of points in general domains [9–11]. In this section, we present a Poissonian setting in the context of general Markov dynamics.

A random ensemble of points $\mathcal{E}$ scattered across the state space $\mathcal{S}$ is a *Poisson process* with intensity $\lambda(s)$ ($s \in \mathcal{S}$) if the following pair of conditions hold [9]: (i) the number of points residing in the domain $D \subset \mathcal{S}$ is a Poisson-distributed random variable with mean $\int_D \lambda(s)m(ds)$ [$m(\cdot)$ being the space's "natural measure"] and (ii) the numbers of points residing in disjoint domains are independent random variables. Recall that an integer-valued random variable N is Poisson distributed with mean μ if

$$\mathrm{Prob}(N = n) = \exp(-\mu)\frac{\mu^n}{n!} \tag{10}$$

($n = 0,1,2,\ldots$).

The Poissonian intensity $\lambda(s)$ is positive valued [$\lambda(s) > 0$], and it may be either integrable or nonintegrable [$I = \int_{\mathcal{S}} \lambda(s)m(ds) \leqslant \infty$]. If the Poissonian intensity $\lambda(s)$ is integrable ($I < \infty$), then the ensemble $\mathcal{E}$ is finite, and it admits the stochastic representation

$$\mathcal{E} = \{S_1,\ldots,S_N\}, \tag{11}$$

where (i) $\{S_n\}_{n=1}^{\infty}$ is a sequence of IID random variables, taking values in the state space $\mathcal{S}$, whose distribution is governed by the probability density function $\lambda(s)/I$ ($s \in \mathcal{S}$)[2] and (ii) N is a Poisson-distributed random variable with mean I, which is independent of the sequence $\{S_n\}_{n=1}^{\infty}$. Namely, to construct the ensemble $\mathcal{E}$, we, first, simulate the Poisson-distributed random variable N; thereafter, given the realization of N, we simulate N IID random variables $S_1,\ldots,S_N$. On the other hand, if the Poissonian intensity $\lambda(s)$ is not integrable ($I = \infty$), then the ensemble $\mathcal{E}$ is infinite and cannot be represented as a sequence of IID random variables.

The best known example of Poisson processes is the "standard" Poisson process. In this example, the state space is the positive half-line $\mathcal{S} = (0,\infty)$, the "natural measure" is the Lebesgue measure $m(ds) = ds$, and the Poissonian intensity is constant: $\lambda(s) = r$, where r is a positive parameter. The parameter r represents the *rate* at which the random points of the "standard" Poisson process occur. Commonly, the standard Poisson process is applied to model the occurrence of random temporal events such as the arrivals of particles to a physical system and the arrivals of customers to a queueing system [13,16]. Note that in this example the Poissonian intensity $\lambda(s)$ is nonintegrable ($I = \infty$).

[2]This probability density is with respect to the underlying measure $m(\cdot)$.

In effect, Poisson processes with constant intensities are the generalization of *uniform* probability distributions. Indeed, if $\mathcal{E}$ is a Poisson process with constant intensity, and the state space $\mathcal{S}$ has a finite measure [$m(\mathcal{S}) < \infty$], then the stochastic representation of Eq. (11) holds, and the IID random variables $\{S_n\}_{n=1}^{\infty}$ are uniformly distributed over the state space $\mathcal{S}$. On the other hand, if the state space $\mathcal{S}$ has an infinite measure [$m(\mathcal{S}) = \infty$], then it supports no uniform probability distribution, but it does support Poisson processes with constant intensities, which represent a uniform scattering of points across the state space. In other words, Poisson processes with constant intensities manifest *spatially homogeneous scattering* of points on general state spaces.

Consider now a countable ensemble of independent Markov motions which take place in the state space $\mathcal{S}$ and whose Markov dynamics are characterized by the infinitesimal generator $\mathbf{G}$. Labeling the Markov motions by the index n, let $X_n(t)$ denote the position of the n^{th} motion at time t ($t \geqslant 0$), and let the ensemble $\mathcal{E}(t) = \{X_n(t)\}_n$ denote the motions' positions at time t ($t \geqslant 0$). The "displacement theorem" of the theory of Poisson processes asserts that, if the ensemble $\mathcal{E}(0)$ is a Poisson process, then so is the ensemble $\mathcal{E}(t)$ (for all $t > 0$). Namely, if the scattering of the motions' initial positions is Poissonian, then at any given time $t > 0$ the scattering of the motions' positions will also be Poissonian. More specifically, the "displacement theorem" asserts the following [9]: If the ensemble $\mathcal{E}(0)$ is a Poisson process with intensity $\Lambda(0,s)$ ($s \in \mathcal{S}$), then the ensemble $\mathcal{E}(t)$ is a Poisson process with intensity

$$\Lambda(t,s) = \int_{\mathcal{S}} \Lambda(0,x)P_x(t,s)m(dx), \tag{12}$$

where $P_x(t,s)$ ($t \geqslant 0, s,x \in \mathcal{S}$) is the solution of Eq. (3) with the "δ-function" initial condition $P(0,s) = \delta(s-x)$.

It is illuminating to examine Eq. (12) in the case of Brownian motion. To keep things simple, we consider one-dimensional Brownian motion. Starting from the initial position $X(0) = x$, the position $X(t)$ of Brownian motion at time t is Gauss distributed with mean $\mathbf{E}[X(t)] = x$ and variance $\mathbf{Var}[X(t)] = t$. Consequently, the Markov kernel $P_x(t,s)$ of Brownian motion is given by

$$P_x(t,s) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(s-x)^2}{2t}\right) \tag{13}$$

($t \geqslant 0, -\infty < s,x < \infty$). In Sec. III we have seen that the positive-valued solutions of the "steady-state equation" (8) are constant functions: $\psi(s) = c$ (where c is a positive constant). These constant functions are not integrable over the real line and, hence, are not admissible probability density functions. On the other hand, these constant functions are admissible Poissonian intensities on the real line. Moreover, a straightforward calculation implies that

$$\int_{-\infty}^{\infty} \psi(x)P_x(t,s)dx = \psi(s) \tag{14}$$

($t \geqslant 0, -\infty < s < \infty$). Combining together Eqs. (12) and (14), we obtain that if we set off from the initial Poissonian intensity $\Lambda(0,s) = \psi(s)$, then $\Lambda(t,s) = \psi(s)$ for all $t > 0$. This observation implies that the nonintegrable solutions $\psi(s) = c$ of the "steady-state equation" (8) are "stationary intensities" of Brownian motion, which is a *nonstationary* Markov motion. As noted above, the constant "stationary intensities" $\psi(s) = c$ represent Poisson processes which are *spatially homogeneous* over the real line, they manifest a *uniform* scattering of points across the real line, and they generalize the notion of *uniform* probability distributions.

Thus, the notion of steady state—which is inapplicable in the context of a single agent performing Brownian motion—is well applicable in the context of infinitely many independent agents performing Brownian motion. As we shall show in the next section, the notion of "stationary intensities" obtained in this section in the context of Brownian motion can be extended to general Markov dynamics.

V. POISSONIAN STEADY STATES

The goal of this paper is to address the case in which Eq. (4) has positive-valued [$\psi(s) > 0$] and nonintegrable [$\int_{\mathcal{S}} \psi(s)m(ds) = \infty$] solutions, an example of this scenario being Brownian motion. In the context of stationary densities, this scenario implies that the Markov motion considered is nonstationary and does not attain a steady state. In this section we will establish that in the context of *stationary intensities* this scenario implies that the Markov dynamics considered do attain *Poissonian steady states*.

Consider a Poissonian setting in which (i) there is a countable ensemble of independent Markov motions $\{X_n\}_n$ which take place in the state space $\mathcal{S}$ and whose Markov dynamics are characterized by the infinitesimal generator $\mathbf{G}$, (ii) the variable $X_n(t)$ denotes the position of the n^{th} motion at time t ($t \geqslant 0$) and the ensemble $\mathcal{E}(t) = \{X_n(t)\}_n$ denotes the motions' positions at time t ($t \geqslant 0$), and (iii) the scattering of the motions' initial positions is Poissonian and $\Lambda(t,s)$ denotes the intensity of the Poisson process $\mathcal{E}(t)$ ($t \geqslant 0, s \in \mathcal{S}$).

An analysis detailed in the appendix asserts that the *master equation* governs the dynamics of the ensembles' Poissonian intensities,

$$\frac{\partial}{\partial t}\Lambda = \mathbf{G}^*\Lambda, \tag{15}$$

where the initial condition $\Lambda(0,s)$ is an arbitrary Poissonian intensity function defined on the state space. A *stationary intensity* of the Markov dynamics is a Poissonian intensity $\Lambda_{\text{SI}}(s)$, defined on the state space $\mathcal{S}$, which satisfies the following stationarity condition: If $\Lambda(0,s) = \Lambda_{\text{SI}}(s)$, then $\Lambda(t,s) = \Lambda_{\text{SI}}(s)$ for all $t > 0$. The stationary intensities $\Lambda_{\text{SI}}(s)$ are the "Poissonian analogs" of the stationary densities $P_{\text{SD}}(s)$ discussed above. Substituting $\Lambda(t,s) = \Lambda_{\text{SI}}(s)$ into Eq. (15) implies that $\mathbf{G}^*\Lambda_{\text{SI}} = 0$. Thus, in order to compute the stationary intensities $\Lambda_{\text{SI}}(s)$ of the Markov dynamics one needs to solve the equation

$$\mathbf{G}^*\psi = 0. \tag{16}$$

The positive-valued solutions of Eq. (16) are the stationary intensities of the Markov dynamics characterized by the infinitesimal generator $\mathbf{G}$.

The "steady-state equations" (4) and (16) are identical. Positive-valued [$\psi(s) > 0$] and normalized [$\int_{\mathcal{S}} \psi(s)m(ds) = 1$] solutions of these equations are the stationary densities $P_{\text{SD}}(s)$ ($s \in \mathcal{S}$) of the Markov dynamics, whereas positive-valued [$\psi(s) > 0$] solutions of these equations are the stationary intensities $\Lambda_{\text{SI}}(s)$ ($s \in \mathcal{S}$) of the Markov dynamics. The stationary densities of given Markov dynamics are, thus, *a subset* of the stationary intensities of these dynamics.

In the case of a stationary density, a single agent performing the Markov dynamics is considered, and the notion of steady state regards the statistics of the single agent's motion: In steady state, the probability of observing the agent in the domain $D \subset \mathcal{S}$ is given by

$$P_{\text{SD}}(D) = \int_D P_{\text{SD}}(s)m(ds). \tag{17}$$

On the other hand, in the case of stationary intensities a countable ensemble of independent agents performing the Markov dynamics is considered, and the notion of steady state regards the statistics of the entire ensemble of agents. In steady state, the number of agents observed in the domain $D \subset \mathcal{S}$ is a Poisson-distributed random variable with mean

$$\Lambda_{\text{SI}}(D) = \int_D \Lambda_{\text{SI}}(s)m(ds). \tag{18}$$

In the scenario where the "steady-state equations" (4) and (16) yield positive-valued and integrable solutions, then each stationary density $P_{\text{SD}}(s)$ corresponds to a one-dimensional set of stationary intensities $\Lambda_{\text{SI}}(s)$ which coincide up to a multiplicative factor: $\Lambda_{\text{SI}}(s) = I(\Lambda_{\text{SI}})P_{\text{SD}}(s)$, where the multiplicative factor is given by the integral $I(\Lambda_{\text{SI}}) = \int_{\mathcal{S}} \Lambda_{\text{SI}}(s)m(ds)$. Moreover, in this scenario the Poisson process $\mathcal{E}(t)$ characterized by the stationary intensity $\Lambda_{\text{SI}}(s)$ is finite, and it admits the stochastic representation

$$\mathcal{E}(t) = \{X_1^*(t), \ldots, X_N^*(t)\}, \tag{19}$$

where (i) $\{X_n^*\}_{n=1}^{\infty}$ is a sequence of IID stationary Markov motions with stationary density $P_{\text{SD}}(s)$ and (ii) N is a Poisson-distributed random variable with mean $I(\Lambda_{\text{SI}})$, which is independent of the sequence of the IID stationary Markov motions.

On the other hand, in the scenario where the "steady-state equations" (4) and (16) yield positive-valued and nonintegrable solutions, the Markov dynamics considered have no stationary densities but they do have stationary intensities $\Lambda_{\text{SI}}(s)$ ($s \in \mathcal{S}$). In this scenario, the Poisson process $\mathcal{E}(t)$ is infinite, and it cannot be represented as a sequence of IID stationary Markov motions. An example of this scenario is Brownian motion whose stationary intensities are constant: $\Lambda_{\text{SI}}(s) = c$ ($s \in \mathcal{S}$; $c > 0$). Namely, the Poissonian steady states of Brownian motion are spatially homogeneous Poisson processes—the Poissonian generalization of uniform probability distributions.

In summary, we established that the notions of "Poissonian steady state" and "stationary intensity" are Poissonian generalizations of the standard Markov notions of "steady state" and "stationary density." These notions coincide in the scenario where the "steady-state equations" (4) and (16) yield positive-valued and integrable solutions. Indeed, in this scenario, the notion of steady state applies both to single agents and to ensembles of agents performing the Markov dynamics considered. On the other hand, in the scenario where the "steady-state equations" (4) and (16) yield positive-valued and nonintegrable solutions, the former Poissonian notions are still admissible, whereas the latter "regular" Markov notions are no longer admissible. In the latter scenario the notion of steady state applies only to infinite ensembles of agents performing the Markov dynamics considered.

VI. POISSONIAN FLUXES

In this section we re-establish the notion of stationary intensities from a different perspective, that of *Poissonian fluxes*. Throughout this section we consider the setting of Sec. V and use the shorthand notation $\Lambda(s) = \Lambda(0,s)$ $(s \in \mathcal{S})$ for the intensity of the Poisson process $\mathcal{E}(0)$ (the ensemble of the motions' positions at time 0). Also, we denote by $\mathbf{I}_D(s)$ the indicator function of the domain $D \subset \mathcal{S}$ [i.e., $\mathbf{I}_D(s) = 1$ for $s \in D$ and $\mathbf{I}_D(s) = 0$ for $s \notin D$].

The number of motions that were present in the domain $A \subset \mathcal{S}$ at time 0, and that are present in the domain $B \subset \mathcal{S}$ at time t $(t > 0)$, is given by

$$N_{A\mapsto B}(t) = \sum_n \mathbf{I}_A(X_n(0))\mathbf{I}_B(X_n(t)). \tag{20}$$

The random variable $N_{A\mapsto B}(t)$ represents the flow of motions from domain A (at time 0) to domain B (at time t). An analysis detailed in the appendix asserts that the flow $N_{A\mapsto B}(t)$ is a Poisson-distributed random variable with mean

$$\mathbf{E}[N_{A\mapsto B}(t)] = \int_A \left(\int_B P_x(t,s)m(ds) \right) \Lambda(x)m(dx), \tag{21}$$

where $P_x(t,s)$ $(t \geqslant 0, s,x \in \mathcal{S})$ is the solution of Eq. (3) with the "δ-function" initial condition $P(0,s) = \delta(s - x)$.

We define the *Poissonian flux* from domain A to domain B to be the derivative of the mean flow $\mathbf{E}[N_{A\mapsto B}(t)]$ at time $t = 0$. A computation detailed in the appendix asserts that the Poissonian flux is given by

$$\frac{d}{dt}\mathbf{E}[N_{A\mapsto B}(t)]|_{t=0} = \int_{\mathcal{S}} \mathbf{I}_A(s)[\mathbf{G}\mathbf{I}_B](s)\Lambda(s)m(ds). \tag{22}$$

Assume now that the entire ensemble of Markov motions is in "statistical equilibrium." Statistical equilibrium means that if we partition the state space $\mathcal{S}$ into two parts, then the Poissonian flux between the two parts should be equal. Namely, for any given domain D, the Poissonian flux from D to its complement $\bar{D}$ should equal the Poissonian flux from $\bar{D}$ back to D. Using Eq. (22), this equality of Poissonian fluxes reads out as follows:

$$\int_{\mathcal{S}} \mathbf{I}_D(s)[\mathbf{G}\mathbf{I}_{\bar{D}}](s)\Lambda(s)m(ds) = \int_{\mathcal{S}} \mathbf{I}_{\bar{D}}(s)[\mathbf{G}\mathbf{I}_D](s)\Lambda(s)m(ds). \tag{23}$$

Noting that $\mathbf{I}_{\bar{D}}(s) = \mathbf{I}_{\mathcal{S}}(s) - \mathbf{I}_D(s)$ and that $[\mathbf{G}\mathbf{I}_{\mathcal{S}}](s) = 0$ $(s \in \mathcal{S})$, Eq. (23) implies that

$$0 = \int_{\mathcal{S}} [\mathbf{G}\mathbf{I}_D](s)\Lambda(s)m(ds) = \int_D [\mathbf{G}^*\Lambda](s)m(ds) \tag{24}$$

[in the transition from the middle part to the right-hand side of Eq. (24) we applied Eq. (2)]. Since Eq. (24) holds for any domain $D \subset \mathcal{S}$ we conclude that $\mathbf{G}^*\Lambda = 0$. On the other hand, if $\mathbf{G}^*\Lambda = 0$ is satisfied, then reversing the arguments implies that the aforementioned equality of fluxes holds for any domain D (and its complement $\bar{D}$).

In summary, the statistical equilibrium condition of "balanced fluxes" was shown to be characterized by the equation $\mathbf{G}^*\psi = 0$, which is the very equation characterizing the stationary intensities defined in Sec. V. Consequently, the class of stationary intensities coincides with the class of Poissonian intensities yielding balanced fluxes. We have, thus, established a "flux perspective" to the notion of stationary intensities.

VII. POISSONIAN CORRELATIONS

In this section we explore the correlation structure of the ensemble of Markov motions $\mathcal{E}(t)$ $(t \geqslant 0)$. Throughout this section we consider the setting of Sec. V and denote by $\mathbf{I}_D(s)$ the indicator function of the domain $D \subset \mathcal{S}$ [i.e., $\mathbf{I}_D(s) = 1$ for $s \in D$ and $\mathbf{I}_D(s) = 0$ for $s \notin D$].

The occupation of the domain $D \subset \mathcal{S}$, at time t $(t \geqslant 0)$, is the number of motions present in the domain at this time epoch,

$$N_D(t) = \sum_n \mathbf{I}_D(X_n(t)). \tag{25}$$

Given two domains, $A \subset \mathcal{S}$ and $B \subset \mathcal{S}$, we focus on their occupations at two different time epochs. Since the motions are Markov, we set, with no loss of generality, the first time epoch to be 0 and the second time epoch to be t $(t > 0)$. The occupations are, thus, given by the random pair $(N_A(0), N_B(t))$. An analysis detailed in the appendix asserts that the probability-generating function of the random pair $(N_A(0), N_B(t))$ is given by

$$\begin{aligned}\mathbf{E}\left[z_1^{N_A(0)} z_2^{N_B(t)}\right] &= \exp\left((z_1 - 1)\mathbf{E}[N_A(0)]\right) \exp\left((z_2 - 1)\mathbf{E}[N_B(t)]\right) \\ &\quad \times \exp\left((z_1 - 1)(z_2 - 1)\mathbf{E}[N_{A\mapsto B}(t)]\right)\end{aligned} \tag{26}$$

$(z_1, z_2$ complex). Equation (26) has three implications, two that we already know and one new, as follows:

(a) The occupancy $N_A(0)$ is a Poisson-distributed random variable with mean

$$\mathbf{E}[N_A(0)] = \int_A \Lambda(0,s)m(ds). \tag{27}$$

This implication is obtained by setting $z_2 = 1$ in Eq. (26) and noting that, when doing so, the right-hand side of Eq. (26) reduces to the probability-generating function of a Poisson-distributed random variable with mean $\mathbf{E}[N_A(0)]$.

(b) The occupancy $N_B(t)$ is a Poisson-distributed random variable with mean

$$\mathbf{E}[N_B(t)] = \int_B \Lambda(t,s)m(ds). \tag{28}$$

This implication is obtained by setting $z_1 = 1$ in Eq. (26) and noting that, when doing so, the right-hand side of Eq. (26) reduces to the probability-generating function of a Poisson-distributed random variable with mean $\mathbf{E}[N_B(t)]$.

(c) The covariance between the occupancies $N_A(0)$ and $N_B(t)$ equals the mean flow from the domain A (at time 0) to the domain B (at time t) and is given by

$$\mathbf{Cov}[N_A(0),N_B(t)] = \mathbf{E}[N_{A\mapsto B}(t)] = \int_A \left(\int_B P_x(t,s)m(ds) \right) \Lambda(0,x)m(dx). \quad (29)$$

This implication is obtained by calculating the second-order derivative $\partial^2/\partial z_1 \partial z_2$ of Eq. (26) at the point $(z_1,z_2) = (1,1)$ (see the appendix for the details of this calculation).

The third implication implies that the correlation structure of the ensemble of Markov motions $\mathcal{E}(t)$ ($t \geqslant 0$) is governed by the mean flow $\mathbf{E}[N_{A\mapsto B}(t)]$ ($t \geqslant 0$). The term appearing in Eq. (29) is the *Poissonian correlation* of the ensemble of Markov motions $\mathcal{E}(t)$ ($t \geqslant 0$). We emphasize that the Poissonian correlation is always well-defined, even in cases where the underlying Markov motions have divergent variances and, thus, have no well-defined covariances. Last, we note that in a steady state which is quantified by the stationary intensity $\Lambda_{\mathrm{SI}}(s)$ ($s \in \mathcal{S}$), one has to set $\Lambda(0,s) = \Lambda_{\mathrm{SI}}(s)$ in Eq. (27), $\Lambda(t,s) = \Lambda_{\mathrm{SI}}(s)$ in Eq. (28), and $\Lambda(0,x) = \Lambda_{\mathrm{SI}}(x)$ in Eq. (29).

VIII. APPLICATIONS

We turn now to exemplify the application of Poissonian steady states. We present an array of examples including birth-death processes (Sec. VIII A), random walks (Sec. VIII B and VIII E), geometric random walks (Sec. VIII C and VIII F), renewal processes (Sec. VIII D), growth-collapse dynamics (Sec. VIII G), decay-surge dynamics (Sec. VIII H), Ito diffusions (Sec, VIII I), and Langevin dynamics (Sec. VIII J).

The method of Poissonian steady states facilitates the modeling of infinite ensembles of noninteracting agents governed by common Markov dynamics. In essence, the passage from finite to infinite such ensembles is analogous to the passage from random walks to Brownian motion. Let us explain this point before presenting the applications.

Random walks are finite objects in the sense that they carry a finite amount of information in any time window. On the other hand, Brownian motion—the universal scaling limit of random walks[3]—is an infinite object in the sense that it carries an infinite amount of information in any time window. Indeed, Brownian motion is a fractal object which fluctuates on all scales and, thus, it encapsulates infinite information in any time window (no matter how short).

In physical reality, there is always a lower bound cutoff, and, hence, random walks should be applied in relevant physical settings. Nonetheless, so very often the "mathematical" Brownian motion, rather than the "physical" random walks, is applied. Surprisingly, the mathematical model of Brownian motion, despite its physical shortcomings, is no less than an astounding success in applications [4,7,8].

It so turns out that in real-world modeling it is often that an infinite mathematical structure (which does not physically exist) approximates reality remarkably well. This holds in the case of calculus, which approximates deterministic processes that are essentially discrete on the atomic scale. This also holds in the case of Brownian motion (and stochastic calculus [28,29]), which approximates stochastic processes that are essentially discrete on the atomic scale.

The method of Poissonian steady states establishes an infinite mathematical model in the context of ensembles of noninteracting agents governed by common Markov dynamics. As in the cases of calculus and Brownian motion the infiniteness is an intrinsic and inherent feature of the model, which yields structures that are beyond the realm of analogous finite models.

A. Birth-death processes

Birth-death processes constitute a fundamental model of Markov evolutionary dynamics [1–3,30]. The state space of birth-death processes is the set of non-negative integers $\mathcal{S} = \{0,1,2,\ldots\}$, and the "natural measure" is the counting measure. The Markov dynamics of a birth-death process X are described as follows.

When at state $s \in \{1,2,\ldots\}$, the process can transit either to state $s+1$ or to state $s-1$. The transition to state $s+1$ occurs according to a state-dependent "birth rate" $B(s)$, and the transition to state $s-1$ occurs according to a state-dependent "death rate" $D(s)$. More specifically, as the process enters state $s \in \{1,2,\ldots\}$, two independent exponential timers are set: timer "B" with mean $1/B(s)$ and timer "D" with mean $1/D(s)$. On the expiration of either of the timers the process moves: If timer "B" expired first, then the process moves to state $s+1$, and, if timer "D" expired first, then the process moves to state $s-1$. As the process enters state $s=0$, an exponential timer with mean $1/B(0)$ is set and, on its expiration, the process moves to state $s=1$.

A birth-death process is characterized by its sequence of "birth rates" $\{B(s)\}_{s=0}^{\infty}$, and by its sequence of "death rates" $\{D(s)\}_{s=1}^{\infty}$. The master equation of birth-death processes is given by

$$\begin{aligned} \dot{P}(t,0) &= -B(0)P(t,0) + D(1)P(t,1) \quad (\text{if } s=0) \\ \dot{P}(t,s) &= B(s-1)P(t,s-1) - (B(s)+D(s))P(t,s) \\ &\quad + D(s+1)P(t,s+1) \quad (\text{if } s \neq 0) \end{aligned} \quad (30)$$

[3,30]. In turn, the corresponding "steady-state equation" (4) yields the following "balance equation":

$$\psi(s)B(s) = \psi(s+1)D(s+1) \quad (31)$$

[3,30]. In what follows, we set $\Pi(0) = 1$ and

$$\Pi(s) = \prod_{k=1}^{s} \frac{B(k-1)}{D(k)} \quad (32)$$

($s \neq 0$). The positive-valued solutions of the "balance equation" (31) are given by

$$\psi(s) = c\Pi(s), \quad (33)$$

where c is a positive constant.

Consider a given birth-death process, and set $\Sigma = \sum_{s=0}^{\infty} \Pi(s)$. The birth-death process attains a steady state if and only if the sum Σ is convergent ($\Sigma < \infty$). Indeed, if the sum Σ is convergent, then the stationary density of the birth-death process is given by $P_{\mathrm{SD}}(s) = \Pi(s)/\Sigma$. On the other

[3]More precisely, Brownian motion is the universal scaling limit of random walks with finite-variance jumps [27].

hand, if the sum Σ is divergent ($\Sigma = \infty$), then the birth-death process *explodes*, i.e., the birth-death process grows to infinity as time progresses: $\lim_{t\to\infty} \Pr(X(t) < \infty) = 0$. Namely, in the nonsummable case ($\Sigma = \infty$), the birth-death process is nonstationary and does not attain a steady state.

However, if we consider infinitely many independent birth-death processes (with common nonstationary birth-death dynamics), then we obtain a Poissonian steady state. Specifically, let $N(s)$ denote the number of birth-death processes whose size, at Poissonian steady state, is s. The Poissonian steady-state result then asserts that $\{N(s)\}_{s=0}^{\infty}$ are independent and Poisson-distributed random variables characterized by the means $\mathbf{E}[N(s)] = c\Pi(s)$. Thus, instead of tracking one birth-death evolution taking place in "one universe," we track infinitely many independent birth-death evolutions taking place in infinitely many "parallel universes." The birth-death evolution in each universe is nonstationary, yet the entire ensemble of infinitely many "parallel universes" attains Poissonian steady states.

1. The $M/M/1$ queueing system

As an illustrative example of birth-death processes, consider the "classic" $M/M/1$ queueing system [13,31]. In this queueing system, jobs arrive at a constant rate r_{arr} to a service station. The service station is staffed by one server which processes the jobs sequentially (one at a time) and at rate r_{ser}. The arriving jobs queue up in line according to their order of arrival and await service.[4] The $M/M/1$ motion is the random process X tracking the number of jobs in the system, $X(t)$ being the number of jobs in the system, either waiting in line for service or being served, at time t ($t \geqslant 0$). The $M/M/1$ motion is a birth-death process with constant birth rates and constant death rates: $B(s) = r_{\text{arr}}$ and $D(s) = r_{\text{ser}}$. Denoting by $\alpha = r_{\text{arr}}/r_{\text{ser}}$ the arrival-to-service ratio, Eqs. (32) and (33) imply that

$$\psi(s) = c\alpha^s. \tag{34}$$

Clearly, the $M/M/1$ queueing system is stable if and only if $\alpha < 1$, in which case its stationary density is geometric: $P_{\text{SD}}(s) = (1-\alpha)\alpha^s$. On the other hand, the $M/M/1$ queueing system is unstable if and only if $\alpha \geqslant 1$, in which case the server does not manage to effectively process the arriving jobs, and the queue of jobs awaiting service explodes. In the stable scenario the $M/M/1$ queueing system attains a "regular" steady state, and in the unstable scenario the $M/M/1$ queueing system attains Poissonian steady states.

2. The Galton-Watson branching process

As yet another illustrative example of birth-death processes, consider a continuous-time version of the Galton-Watson branching process [32,33]. In this version, particles flow into a system and thereafter branch and exit the system as follows: Particles flow into the system at a constant rate r_{in}; each particle produces new particles at a constant rate r_{new} and exits the system at a constant rate r_{ex}. The particles are independent of each other, and new particles are identical to incoming particles. The random motion X tracking the number of particles in the system, $X(t)$ being the number of particles in the system at time t ($t \geqslant 0$), is a Markov branching process [34–36]. More specifically, the branching process X is a birth-death process with affine birth rates and linear death rates: $B(s) = r_{\text{in}} + r_{\text{new}}s$ and $D(s) = r_{\text{ex}}s$. Denoting by $\alpha = r_{\text{new}}/r_{\text{ex}}$ the branching-to-exit ratio, and by $\beta = r_{\text{in}}/r_{\text{new}}$ the inflow-to-branching ratio, a straightforward calculation using Eqs. (32) and (33) implies that

$$\psi(s) = c\alpha^s \frac{\Gamma(s+\beta)}{\Gamma(s+1)} \tag{35}$$

(where $\Gamma(\cdot)$ denotes the Γ function). Note that at the parameter value $\beta = 1$ Eq. (35) yields Eq. (34). A further calculation involving Stirling's formula [37] yields the asymptotic approximation

$$\psi(s) \approx \frac{c}{\Gamma(\beta)} \alpha^s s^{\beta-1} \tag{36}$$

($s \gg 1$). From Eq. (36) it is evident that the branching process X is stable if and only if $\alpha < 1$, in which case its stationary density is a discrete analog of the continuous Gamma probability density function. On the other hand, the branching process X is unstable if and only if $\alpha \geqslant 1$, in which case it explodes. In the stable scenario, the branching process X attains a "regular" steady state, and in the unstable scenario the branching process X attains Poissonian steady states.

B. Random walks I

Random walks constitute the most elemental model of random motion in the sciences [38–40]. In the case of d-dimensional random walks, the state space is the d-dimensional Euclidean space $\mathcal{S} = \mathbb{R}^d$, and the "natural measure" is the Lebesgue measure $m(ds) = ds$. The structure of random walks is described as follows.

A d-dimensional random walk X is characterized by its jump rate r and by the distribution of its generic jump size J, an arbitrary d-dimensional random variable. Specifically, the positions of the random walk X are given by

$$X(t) = X(0) + \sum_{T_n \leqslant t} J_n \tag{37}$$

($t \geqslant 0$), where (i) the "jump times" $\{T_n\}_{n=1}^{\infty}$ are a standard Poisson process with intensity r,[5] (ii) the "jump sizes" $\{J_n\}_{n=1}^{\infty}$ are IID copies of the generic jump size J, and (iii) the "jump times" $\{T_n\}_{n=1}^{\infty}$ and the "jump sizes" $\{J_n\}_{n=1}^{\infty}$ are independent sequences. In what follows, $\rho(x)$ ($x \in \mathbb{R}^d$) denotes the probability density function of the generic jump size J.

Consider the random walk X along the infinitesimal time interval $(t, t+\delta)$ ($\delta \to 0$). During this time interval the random walk can either jump [with probability $r\delta + o(\delta)$] or stay

[4] In the nomenclature of queueing theory, the $M/M/1$ notation is an acronym for Markovian arrivals (quantified by the arrival rate r_{arr}), Markovian service (quantified by the service rate r_{ser}), and a single server.

[5] The precise definition of a standard Poisson process was given in Sec. IV.

still [with probability $1 - r\delta + o(\delta)$]. If it jumps, then it moves from position $X(t)$ to position $X(t+\delta) = X(t) + J$ (the equality being in law). Thus, a calculation of Eq. (1) implies that the random walk's infinitesimal generator is given by the integral operator

$$[\mathbf{G}\phi](s) = r \int_{\mathbb{R}^d} [\phi(s+x) - \phi(s)]\rho(x)dx. \tag{38}$$

In turn, a calculation of Eq. (2) implies that the random walk's adjoint operator is given by

$$[\mathbf{G}^*\psi](s) = r \int_{\mathbb{R}^d} [\psi(s-x) - \psi(s)]\rho(x)dx. \tag{39}$$

Consequently, we obtain that the corresponding "steady-state equation" (4) is given by

$$\psi(s) = \int_{\mathbb{R}^d} \psi(s-x)\rho(x)dx. \tag{40}$$

It is straightforward to observe that the solutions of the "steady-state equation" (40) are exponential functions. Specifically, the positive-valued solutions of the "steady-state equation" (40) are superpositions of constant and exponential functions,

$$\psi(s) = c_1 + c_2 \exp(-\langle v, s\rangle) \tag{41}$$

($s \in \mathbb{R}^d$), where c_1 and c_2 are positive constants and where v is a d-dimensional vector satisfying the Laplace condition

$$\mathbf{E}[\exp(\langle v, J\rangle)] = \int_{\mathbb{R}^d} \exp(\langle v, x\rangle)\rho(x)dx = 1. \tag{42}$$

Since neither constant functions nor exponential functions are integrable over the d-dimensional Euclidean space, the "steady-state equation" (40) has no solutions which are positive-valued and normalized. Consequently, we obtain that the random walk X is nonstationary and does not attain a steady state. However, the random walk X does attain Poissonian steady states.

In the case where the generic jump size J has a divergent Laplace transform, then the only admissible stationary intensities are constant functions: $\psi(s) = c_1$. In particular, we obtain that the stationary intensities of random walks with *Lévy jumps* are constant functions. Random walks with Lévy jumps play a major role in the transdisciplinary field of *anomalous diffusion* [41–43],[6] and they exhibit fluctuations which were classified by Mandelbrot as a "wild" form of randomness [47,48].

As noted in Sec. IV, the constant stationary intensities $\psi(s) = c_1$ represent Poisson processes which are *spatially homogeneous* over the Euclidean space, they manifest a *uniform* scattering of points across the Euclidean space, and they generalize the notion of *uniform* probability distributions. Moreover, note that the constant stationary intensities $\psi(s) = c_1$ are independent of the random walk's characteristics—the jump rate r and the generic jump size J. We, hence, conclude that spatially homogeneous Poisson processes are the "universal" Poissonian steady states of *all* random walks. Thus, in the context of random walks, the notions of "uniformity" and "universality" are in effect synonymous.

C. Geometric random walks I

Geometric random walks are the "geometric counterpart" of one-dimensional random walks. In random walks the evolution is additive, as evident from Eq. (37). However, there are many stochastic-dynamics settings, mainly prevalent in economics and finance [49,50], in which the evolution is multiplicative rather than additive [51–53]. Geometric random walks are a prototypical model of stochastic dynamics with multiplicative evolution.

In the case of geometric random walks the state space is the positive half-line $\mathcal{S} = (0,\infty)$, and the "natural measure" is the Lebesgue measure $m(ds) = ds$. A geometric random walk X is characterized by its jump rate r and by the distribution of its generic multiplicative jump J, an arbitrary positive-valued random variable. Specifically, a geometric random walk X is given by

$$X(t) = X(0) \prod_{T_n \leqslant t} J_n, \tag{43}$$

where (i) the "jump times" $\{T_n\}_{n=1}^{\infty}$ are a standard Poisson process with intensity r,[7] (ii) the "multiplicative jumps" $\{J_n\}_{n=1}^{\infty}$ are IID copies of the generic multiplicative jump J, and (iii) the "jump times" $\{T_n\}_{n=1}^{\infty}$ and the "multiplicative jumps" $\{J_n\}_{n=1}^{\infty}$ are independent sequences. In what follows, $\rho(x)$ $(x > 0)$ denotes the probability density function of the generic multiplicative jump J.

Consider the geometric random walk X along the infinitesimal time interval $(t, t+\delta)$ $(\delta \to 0)$. During this time interval, the geometric random walk can either jump [with probability $r\delta + o(\delta)$] or stay still [with probability $1 - r\delta + o(\delta)$]. If it jumps, then it moves from position $X(t)$ to position $X(t+\delta) = X(t)J$ (the equality being in law). Thus, a calculation of Eq. (1) implies that the geometric random walk's infinitesimal generator is given by the integral operator

$$[\mathbf{G}\phi](s) = r \int_0^\infty [\phi(sx) - \phi(s)]\rho(x)dx. \tag{44}$$

In turn, a calculation of Eq. (2) implies that the geometric random walk's adjoint operator is given by

$$[\mathbf{G}^*\psi](s) = r \int_0^\infty \left[\frac{1}{x}\psi\left(\frac{s}{x}\right) - \psi(s)\right]\rho(x)dx. \tag{45}$$

Consequently, we obtain that the corresponding "steady-state equation" (4) is given by

$$\psi(s) = \int_0^\infty \psi\left(\frac{s}{x}\right)\frac{\rho(x)}{x}dx. \tag{46}$$

It is straightforward to observe that the solutions of the "steady-state equation" (46) are power-law functions. Specifically, the positive-valued solutions of the "steady-state equation" (46) are superpositions of harmonic and power-law

[6] The pioneering works initiating the field of anomalous diffusion are due to Scher, Lax, Shlesinger, and Montroll [44–46].

[7] The precise definition of a standard Poisson process was given in Sec. IV.

functions,

$$\psi(s) = c_1 s^{-1} + c_2 s^{-\epsilon-1} \tag{47}$$

($s > 0$), where c_1 and c_2 are positive constants and where ϵ is a real-valued exponent satisfying the moment condition

$$\mathbf{E}[J^\epsilon] = \int_0^\infty x^\epsilon \rho(x) dx = 1. \tag{48}$$

Since neither harmonic functions nor power-law functions are integrable over the positive half-line, the "steady-state equation" (46) has no solutions which are positive-valued and normalized. Consequently, we obtain that the geometric random walk X is nonstationary and does not attain a steady state. However, the geometric random walk X does attain Poissonian steady states.

In case the generic multiplicative jump size J has divergent moments, then the only admissible stationary intensities are harmonic functions: $\psi(s) = c_1/s$. In particular, we obtain that the stationary intensities of geometric random walks with *log-Lévy multiplicative jumps* are harmonic functions. Geometric random walks with log-Lévy multiplicative jumps [54] exhibit fluctuations which were classified by Mandelbrot as an "extreme" form of randomness [47]. Note that the harmonic stationary intensities $\psi(s) = c_1/s$ are independent of the geometric-random-walk characteristics—the jump rate r and the generic multiplicative jump size J. Hence, we obtain that harmonic Poissonian intensities are the "universal" stationary intensities of geometric random walks. We emphasize that this "universality" holds for *all* geometric random walks.

D. Renewal processes

The standard Poisson process, defined in Sec. IV, underlies both additive and multiplicative random walks, which were discussed, respectively, in Secs. VIII B and VIII C. The standard Poisson process is, in effect, a special case of *renewal processes*, the fundamental model of regenerative stochastic phenomena [13,55,56]. Renewal processes underlie the continuous-time random walk (CTRW) model [41,57,58], a bedrock model of anomalous diffusion.

A series of events occurring at random time epochs $T_0 < T_1 < T_2 < \cdots$ form a renewal process if the sequence of consecutive interevent durations $\{T_k - T_{k-1}\}_{k=1}^\infty$ are IID copies of a generic interevent duration τ, an arbitrary positive-valued random variable. The standard Poisson process is a renewal process with exponentially distributed interevent time durations [9]. Henceforth, we denote by $\rho(t)$ ($t > 0$) the probability density function of the generic interevent duration τ and denote by $\rho_>(t)$ ($t > 0$) the corresponding tail probability

$$\rho_>(t) = \int_t^\infty \rho(u) du = \mathrm{Prob}(\tau > t). \tag{49}$$

In what follows, we shall use the *hazard rate function* $H(t)$ ($t > 0$) of the generic inter-event duration τ [56,59,60]. The hazard rate function is given by

$$H(t) = \lim_{\delta\to 0} \frac{1}{\delta} \mathrm{Prob}(\tau \leqslant t + \delta | \tau > t) = \frac{\rho(t)}{\rho_>(t)} \tag{50}$$

($t > 0$). Namely, $H(t)$ is the "rate of occurrence" of the interevent duration: the intensity at which the random variable τ realizes at time t, provided that it did not realize up to time t.

1. Renewal motions

The Markov motion X tracking a given renewal process—henceforth termed the *renewal motion*—is defined as follows: $X(t)$ is the time elapsing since the *last* renewal event that occurred up to time t. The state space of the renewal motion is the non-negative half-line $\mathcal{S} = [0,\infty)$, and the "natural measure" is the Lebesgue measure $m(ds) = ds$. The trajectories of the renewal motion are piecewise linear: Between the renewal events, the motion grows linearly, and, at the event epochs, the motion collapses to zero.

Consider the renewal motion X along the infinitesimal time interval $(t, t+\delta)$ ($\delta \to 0$). During this time interval, the renewal motion can either collapse to zero [with probability $H(X(t))\delta + o(\delta)$] or grow linearly [with probability $1 - H(X(t))\delta + o(\delta)$]. If it collapses, then it moves from position $X(t)$ to position $X(t+\delta) = 0$, and, if it grows, then it moves from position $X(t)$ to position $X(t+\delta) = X(t) + \delta$. Thus, a calculation of Eq. (1) implies that the renewal motion's infinitesimal generator is given by the differential operator

$$[\mathbf{G}\phi](s) = \phi'(s) - H(s)\phi(s) \tag{51}$$

[where $\phi(s)$ is an arbitrary test function that vanishes at the origin]. In turn, a calculation of Eq. (2) implies that the renewal motion's adjoint operator is given by

$$[\mathbf{G}^*\psi](s) = -\psi'(s) - H(s)\psi(s). \tag{52}$$

Consequently, we obtain that the corresponding "steady-state equation" (4) is given by

$$\psi'(s) = -H(s)\psi(s). \tag{53}$$

The positive-valued solutions of the "steady-state equation" (53) are multiples of the tail probability,

$$\psi(s) = c\rho_>(s), \tag{54}$$

where c is a positive constant.

Convergence to steady state is determined by the mean of the interevent time durations,

$$\mathbf{E}[\tau] = \int_0^\infty t\rho(t) dt = \int_0^\infty \rho_>(t) dt. \tag{55}$$

If the mean is finite ($\mathbf{E}[\tau] < \infty$), then the renewal motion attains a steady state which is quantified by the stationary density $P_{\mathrm{SD}}(s) = \rho_>(s)/\mathbf{E}[\tau]$. This stationary density is the probability density function of the "*residual lifetime*" of the renewal process [13,55,56]. On the other hand, if the mean is infinite ($\mathbf{E}[\tau] = \infty$), then the renewal process is nonstationary and *aging*: The occurrence of the events becomes rarer and rarer as time progresses. In the latter scenario ($\mathbf{E}[\tau] = \infty$), the renewal motion is nonstationary and does not attain a steady state. However, the renewal motion does attain Poissonian steady states.

2. Waiting times

So far, we have considered the time elapsing since the *last* renewal event. In practice, we are often interested in predicting when the *next* renewal event will take place. Tracking a renewal process, assume that we observe the corresponding renewal motion to be at state s. Let $W(s)$ denote the waiting time until the next renewal event: the time elapsing from the observation epoch until the first renewal event occurring after the observation epoch. It is straightforward to note that the probability density function of the waiting time $W(s)$ is given by

$$\rho(w;s) = \frac{\rho(w+s)}{\rho_>(s)} \tag{56}$$

($w > 0$).

Consider now a countable ensemble of independent renewal processes tracked by corresponding renewal motions, and observe the ensemble at an arbitrary time epoch. Each renewal process has its own waiting time till the next renewal event, and these waiting times are governed by the probability density function of Eq. (56). Specifically, let the ensemble $\{s_n\}_n$ denote the positions of the renewal motions at the observation epoch, and let the ensemble $\{W_n(s_n)\}_n$ denote the corresponding waiting times. The "displacement theorem" of the theory of Poisson processes asserts that [9]: if the ensemble $\{s_n\}_n$ is a Poisson process with intensity $\lambda(s)$ ($s > 0$), then the ensemble $\{W_n(s_n)\}_n$ of waiting times is a Poisson process with intensity

$$\tilde{\lambda}(w) = \int_0^\infty \lambda(s)\rho(w;s)ds \tag{57}$$

($w > 0$).

If the ensemble of renewal processes is observed at a Poissonian steady state, then $\lambda(s)$ is given by the stationary intensity of Eq. (54): $\lambda(s) = c\rho_>(s)$. Substituting this stationary intensity into Eq. (57), and using Eq. (56), a straightforward calculation implies that

$$\tilde{\lambda}(w) = c\rho_>(w) \tag{58}$$

($w > 0$). Remarkably so, we obtain that the stationary intensities of the renewal motion, $\Lambda_{SI}(s) = c\rho_>(s)$, are "fixed points" of the transformation given by Eq. (57). Namely, the input $\lambda(s) = \Lambda_{SI}(s)$ yields the identical output $\tilde{\lambda}(w) = \Lambda_{SI}(w)$.

As explained above, the stationarity of the single renewal processes is determined by the mean $\mathbf{E}[\tau]$ of the interevent time durations. If the mean is finite ($\mathbf{E}[\tau] < \infty$), then each renewal process attains a steady state, and, observing a renewal process at steady state, we obtain that (i) the time elapsing since the last renewal event before the observation epoch is equal, in law, to the waiting time till the next renewal event after the observation epoch and (ii) both the aforementioned elapsing and waiting times are random variables governed by the "*residual lifetime*" probability density function $P_{SD}(s) = \rho_>(s)/\mathbf{E}[\tau]$ ($s > 0$). On the other hand, if the mean is infinite ($\mathbf{E}[\tau] = \infty$), then each renewal process is nonstationary and aging, yet the entire ensemble of renewal processes attains Poissonian steady states. In the nonstationary case, observing an ensemble of renewal process at a Poissonian steady state, we obtain that (i) the random ensemble of the times elapsing since the last renewal events before the observation epoch ($\{s_n\}_n$) is equal, in law, to the random ensemble of waiting times till the next renewal events after the observation epoch ($\{W_n(s_n)\}_n$) and (ii) both the aforementioned ensembles of elapsing times and waiting times are Poisson processes governed by the stationary intensity $\Lambda_{SI}(s) = c\rho_>(s)$ ($s > 0$).

Thus, the Poissonian methodology established in this paper extends the probabilistic notion of "residual lifetime" from the case of stationary renewal processes (characterized by interevent time durations with finite mean) to the case of nonstationary renewal processes (characterized by interevent time durations with infinite mean). Indeed, the probabilistic "residual lifetime," given by the normalized probability density function $P_{SD}(s) = \rho_>(s)/\mathbf{E}[\tau]$ ($s > 0$), is replaced by a "Poissonian residual lifetime" given by the Poissonian intensities $\Lambda_{SI}(s) = c\rho_>(s)$ ($s > 0$).

The Poissonian residual lifetime provides a new perspective to the phenomenon of *ergodicity breaking* in the context of the CTRW model [61–63]. When tracking an aging CTRW ($\mathbf{E}[\tau] = \infty$, where τ is the CTRW's generic waiting-time), ergodicity happens to fail: Time averages of the CTRW's trajectories do not converge to their expected means; rather, each time-average measurement yields a different outcome, and the outcomes display a statistical pattern. Thus, single-trajectory measurements of aging CTRWs are seemingly useless, as characteristic means cannot be inferred from them. Nonetheless, if we simultaneously consider infinitely many IID CTRWs then the Poissonian residual lifetime captures the precise statistical structure underlying the observed ergodicity-breaking phenomenon. We emphasize that the Poissonian residual lifetime quantifies all forms of CTRWs displaying ergodicity breaking, the two most common scenarios being (i) subdiffusion $\rho_>(s) \approx 1/s^\epsilon$ (as $s \to \infty$), where ϵ is an exponent taking values in the unit interval $0 < \epsilon < 1$ and (ii) ultraslow diffusion $\rho_>(s) \approx 1/\ln(s)^\epsilon$ (as $s \to \infty$), where ϵ is a positive exponent.

E. Random walks II

In the random-walk model of Sec. VIII B the jump rate was constant. Namely, the jump rate was considered to be homogeneous across the d-dimensional Euclidean space. In this section we consider *spatially inhomogeneous* random walks with state-dependent jump rates. As in Sec. VIII B, the state space is the d-dimensional Euclidean space $\mathcal{S} = \mathbb{R}^d$, and the "natural measure" is the Lebesgue measure $m(ds) = ds$. The Markov dynamics of spatiallyinhomogeneous random walks are described as follows.

A d-dimensional random walk X is characterized by its state-dependent jump rate $R(s)$ ($s \in \mathbb{R}^d$) and by the distribution of its generic jump size J, an arbitrary d-dimensional random variable. The Markov dynamics of the random walk X, along the infinitesimal time interval $(t,t+\delta)$ ($\delta \to 0$), are given by

$$X(t+\delta) = \begin{cases} X(t) + J & \text{w.p.} \quad R(X(t))\delta + o(\delta), \\ X(t) & \text{w.p.} \quad 1 - R(X(t))\delta + o(\delta). \end{cases} \tag{59}$$

Namely, during the time interval $(t,t+\delta)$ the random walk can either jump [with state-dependent probability $R(X(t))\,\delta + o(\delta)$] or stay still [with state-dependent probability

$1 - R(X(t))\delta + o(\delta)$]. If it jumps then it moves from the position $X(t)$ to the position $X(t+\delta) = X(t) + J$ (the equality being in law).

As in Sec. VIII B we denote by $\rho(x)$ $(x \in \mathbb{R}^d)$ the probability density function of the generic jump size J. A calculation of Eq. (1), based on the Markov dynamics of Eq. (59), implies that the random walk's infinitesimal generator is given by the integral operator

$$[\mathbf{G}\phi](s) = R(s)\int_{\mathbb{R}^d}[\phi(s+x) - \phi(s)]\rho(x)dx. \quad (60)$$

In turn, a calculation of Eq. (2) implies that the random walk's adjoint operator is given by

$$[\mathbf{G}^*\psi](s) = \int_{\mathbb{R}^d}[R(s-x)\psi(s-x) - R(s)\psi(s)]\rho(x)dx. \quad (61)$$

Consequently, we obtain that the corresponding "steady-state equation" (4) is given by

$$R(s)\psi(s) = \int_{\mathbb{R}^d} R(s-x)\psi(s-x)\rho(x)dx. \quad (62)$$

Equation (62) is equivalent to Eq. (40). Indeed, setting $\tilde{\psi}(s) = R(s)\psi(s)$ in Eq. (62) yields Eq. (40). Thus, we obtain that the positive-valued solutions of the "steady-state equation" (62) are given by

$$\psi(s) = \frac{c_1 + c_2\exp(-\langle v,s\rangle)}{R(s)} \quad (63)$$

$(s \in \mathbb{R}^d)$, where c_1 and c_2 are positive constants and where v is a d-dimensional vector satisfying the Laplace condition of Eq. (42). Depending on the state-dependent jump rate $R(s)$, the stationary intensities of Eq. (63) can be either integrable or nonintegrable. In the former scenario the random walk X attains "regular" steady states. In the latter scenario the random walk X is nonstationary and does not attain a steady state but it does attain Poissonian steady states. We note that, in the case where the generic jump size J has a divergent Laplace transform, then the only admissible stationary intensities are multiples of the reciprocal of the state-dependent jump rate: $\psi(s) = c_1/R(s)$. Namely, the stationary intensities are inversely proportional to the state-dependent jump rate $R(s)$.

F. Geometric random walks II

In the geometric random-walk model of Sec. VIII C the jump rate was constant. Namely, the jump rate was considered to be homogeneous across the positive half-line. In this section, we consider *spatially inhomogeneous* geometric random walks with state-dependent jump rates. As in Sec. VIII C, the state space is the positive half-line $\mathcal{S} = (0,\infty)$, and the "natural measure" is the Lebesgue measure $m(ds) = ds$. The Markov dynamics of spatially inhomogeneous geometric random walks are described as follows.

A geometric random walk X is characterized by its state-dependent jump rate $R(s)$ $(s > 0)$ and by the distribution of its generic multiplicative jump size J, an arbitrary positive-valued random variable. The Markov dynamics of the inhomogeneous geometric random walk X, along the infinitesimal time interval $(t,t+\delta)$ $(\delta \to 0)$, are given by

$$X(t+\delta) = \begin{cases} X(t)\cdot J & \text{w.p.} \quad R(X(t))\delta + o(\delta), \\ X(t) & \text{w.p.} \quad 1 - R(X(t))\delta + o(\delta). \end{cases} \quad (64)$$

Namely, during the time interval $(t,t+\delta)$ the geometric random walk can either jump [with state-dependent probability $R(X(t))\delta + o(\delta)$] or stay still [with state-dependent probability $1 - R(X(t))\delta + o(\delta)$]. If it jumps, then it moves from position $X(t)$ to position $X(t+\delta) = X(t)\cdot J$ (the equality being in law).

As in Sec. VIII C, we denote by $\rho(x)$ $(x > 0)$ the probability density function of the generic multiplicative jump J. A calculation of Eq. (1), based on the Markov dynamics of Eq. (64), implies that the geometric random walk's infinitesimal generator is given by the integral operator

$$[\mathbf{G}\phi](s) = R(s)\int_0^\infty[\phi(sx) - \phi(s)]\rho(x)dx. \quad (65)$$

In turn, a calculation of Eq. (2) implies that the geometric random walk's adjoint operator is given by

$$[\mathbf{G}^*\psi](s) = \int_0^\infty\left[\frac{1}{x}R\left(\frac{s}{x}\right)\psi\left(\frac{s}{x}\right) - R(s)\psi(s)\right]\rho(x)dx. \quad (66)$$

Consequently, we obtain that the corresponding "steady-state equation" (4) is given by

$$R(s)\psi(s) = \int_0^\infty R\left(\frac{s}{x}\right)\psi\left(\frac{s}{x}\right)\frac{\rho(x)}{x}dx. \quad (67)$$

Equation (67) is equivalent to Eq. (46). Indeed, setting $\tilde{\psi}(s) = R(s)\psi(s)$ in Eq. (67) yields Eq. (46). Thus, we obtain that the positive-valued solutions of the "steady-state equation" (67) are given by

$$\psi(s) = \frac{c_1 + c_2 s^{-\epsilon}}{sR(s)} \quad (68)$$

$(s > 0)$, where c_1 and c_2 are positive constants and where ϵ is a real-valued exponent satisfying the moment condition of Eq. (48). Depending on the state-dependent jump rate $R(s)$, the stationary intensities of Eq. (68) can be either integrable or nonintegrable. In the former scenario the geometric random walk X attains "regular" steady states. In the latter scenario the geometric random walk X is nonstationary and does not attain a steady state, but it does attain Poissonian steady states. We note that in case the generic jump size J has divergent moments, then the only admissible stationary intensities are given by $\psi(s) = c_1/(sR(s))$.

G. Growth-collapse dynamics

The physical sciences are prevalent with processes whose dynamics exhibit growth-collapse evolutionary patterns: cycles of steady smooth and deterministic growth followed by a sudden discontinuous and random collapse. Examples of growth-collapse dynamics include renewal motions (described in Sec. VIII D), sand-pile models and systems in self-organized criticality [64], stick-slip models of interfacial friction [65], Burridge-Knopof–type models of earthquakes and continental drift [66], avalanche models [67], stochastic

Ornstein-Uhlenbeck capacitors [68], and geometric Langevin equations [53]. In queueing theory, growth-collapse dynamics emerge in the context of batch service [69–73]. Growth-collapse dynamics also emerge in the *asymmetric inclusion process* (ASIP), a bosonic counterpart of the fermionic *asymmetric exclusion process* (ASEP), which links together key models in statistical physics and queueing theory [74,75].

In the case of growth-collapse dynamics the state space is the positive half-line $\mathcal{S} = (0,\infty)$ and the "natural measure" is the Lebesgue measure $m(ds) = ds$. In what follows, we consider a general growth-collapse model which is characterized by the following triplet [53]: (i) a positive valued growth function $F(s)$ $(s > 0)$, which governs the smooth deterministic growth; (ii) a state-dependent collapse rate $R(s)$ $(s > 0)$; and (iii) a distribution of the generic collapse size J, an arbitrary random variable taking values in the unit interval $(0,1)$. The Markov dynamics, along the infinitesimal time interval $(t,t+\delta)$ $(\delta \to 0)$, of a random motion X governed by the aforementioned growth-collapse model are given by

$$X(t+\delta) = \begin{cases} X(t)J & \text{w.p.} \quad R(X(t))\delta + o(\delta), \\ X(t) + F(X(t))\delta & \text{w.p.} \quad 1 - R(X(t))\delta + o(\delta). \end{cases} \tag{69}$$

Namely, during the time interval $(t,t+\delta)$, the motion X can either collapse [with state-dependent probability $R(X(t))\delta + o(\delta)$] or grow smoothly [with state-dependent probability $1 - R(X(t))\delta + o(\delta)$]. If it collapses, then it moves from position $X(t)$ to position $X(t+\delta) = X(t)J$ (the equality being in law), and, if it grows, then it moves from position $X(t)$ to position $X(t+\delta) = X(t) + F(X(t))\delta$.

Note that in the aforementioned growth-collapse model the smooth and deterministic growth is additive, whereas the discontinuous and random collapse is multiplicative. In what follows, we denote by $\rho(x)$ $(0 < x < 1)$ the probability density function of the generic collapse size J. A calculation of Eq. (1), based on the Markov dynamics of Eq. (69), implies that the motion's infinitesimal generator is given by the integrodifferential operator

$$[\mathbf{G}\phi](s) = F(s)\phi'(s) + R(s)\int_0^1 [\phi(sx) - \phi(s)]\rho(x)dx. \tag{70}$$

In turn, a calculation of Eq. (2) implies that the motion's adjoint operator is given by

$$[\mathbf{G}^*\psi](s) = -[F(s)\psi(s)]' + \int_0^1 \left[\frac{1}{x}R\left(\frac{s}{x}\right)\psi\left(\frac{s}{x}\right) - R(s)\psi(s)\right]\rho(x)dx. \tag{71}$$

Consequently, we obtain that the corresponding "steady-state equation" (4) is given by

$$[F(s)\psi(s)]' + R(s)\psi(s) = \int_0^1 R\left(\frac{s}{x}\right)\psi\left(\frac{s}{x}\right)\frac{\rho(x)}{x}dx. \tag{72}$$

Setting $\tilde{\psi}(s) = R(s)\psi(s)$ and $\tilde{F}(s) = F(s)/R(s)$, Eq. (72) can be rewritten in the compact form

$$[\tilde{F}(s)\tilde{\psi}(s)]' + \tilde{\psi}(s) = \int_0^1 \tilde{\psi}\left(\frac{s}{x}\right)\frac{\rho(x)}{x}dx. \tag{73}$$

In general, Eq. (73) does not admit explicit closed-form solutions. However, if the ratio of the growth function $F(s)$ and the collapse-rate function $R(s)$ is a linear function, then explicit closed-form solutions are attainable. Indeed, assume that $\tilde{F}(s) = as$, where a is an arbitrary positive slope. It is then straightforward to observe that power-law functions solve Eq. (73) and yield the stationary intensities

$$\psi(s) = \frac{c_1 + c_2 s^{-\epsilon}}{sR(s)} \tag{74}$$

$(s > 0)$, where c_1 and c_2 are positive constants and where ϵ is a real-valued exponent satisfying the moment condition

$$\mathbf{E}[J^\epsilon] = \int_0^1 x^\epsilon \rho(x)dx = 1 - a\epsilon. \tag{75}$$

The stationary intensities of Eq. (74) are identical, in form, to the stationary intensities obtained in the context of spatially inhomogeneous geometric random walks [Eq. (68)].

H. Decay-surge dynamics

Decay-surge dynamics are the "mirror image" of growth-collapse dynamics. Decay-surge evolutionary patterns exhibit cycles of steady smooth and deterministic decay followed by a sudden discontinuous and random surge. Examples of decay-surge dynamics include shot noise [16,76], inverted stick-slip models of interfacial friction [77], and the running maxima of nonlinear shot noise processes [78,79]. In stochastic operations research decay-surge dynamics emerge in the context of workload in queueing systems [13,31] and in the context of water flow in dams [80].

In the case of decay-surge dynamics the state space is the positive half-line $\mathcal{S} = (0,\infty)$, and the "natural measure" is the Lebesgue measure $m(ds) = ds$. In what follows, we consider a general decay-surge model which is characterized by the following triplet [53]: (i) a positive valued decay function $F(s)$ $(s > 0)$ which governs the smooth deterministic decay; (ii) a state-dependent surge rate $R(s)$ $(s > 0)$; and (iii) a distribution of the generic surge size J, an arbitrary random variable taking values in the ray $(1,\infty)$. The Markov dynamics, along the infinitesimal time interval $(t,t+\delta)$ $(\delta \to 0)$, of a random motion X governed by the aforementioned decay-surge model are given by

$$X(t+\delta) = \begin{cases} X(t)J & \text{w.p.} \quad R(X(t))\delta + o(\delta), \\ X(t) - F(X(t))\delta & \text{w.p.} \quad 1 - R(X(t))\delta + o(\delta). \end{cases} \tag{76}$$

Namely, during the time interval $(t,t+\delta)$ the motion X can either surge [with state-dependent probability $R(X(t))\delta + o(\delta)$] or decay smoothly [with state-dependent probability $1 - R(X(t))\delta + o(\delta)$]. If it surges, then it moves from position $X(t)$ to position $X(t+\delta) = X(t)J$ (the equality being in law), and, if it decays, then it moves from position $X(t)$ to position $X(t+\delta) = X(t) - F(X(t))\delta$.

Note that in the aforementioned decay-surge model the smooth and deterministic decay is additive, whereas the discontinuous and random surge is multiplicative. In what follows, we denote by $\rho(x)$ ($x > 1$) the probability density function of the generic surge size J. A calculation of Eq. (1), based on the Markov dynamics of Eq. (76), implies that the motion's infinitesimal generator is given by the integrodifferential operator

$$[\mathbf{G}\phi](s) = -F(s)\phi'(s) + R(s)\int_1^\infty [\phi(sx) - \phi(s)]\rho(x)dx. \tag{77}$$

In turn, a calculation of Eq. (2) implies that the motion's adjoint operator is given by

$$[\mathbf{G}^*\psi](s) = [F(s)\psi(s)]' + \int_1^\infty \left[\frac{1}{x}R\left(\frac{s}{x}\right)\psi\left(\frac{s}{x}\right) - R(s)\psi(s)\right]\rho(x)dx. \tag{78}$$

Consequently, we obtain that the corresponding "steady-state equation" (4) is given by

$$R(s)\psi(s) - [F(s)\psi(s)]' = \int_1^\infty R\left(\frac{s}{x}\right)\psi\left(\frac{s}{x}\right)\frac{\rho(x)}{x}dx. \tag{79}$$

Setting $\tilde{\psi}(s) = R(s)\psi(s)$ and $\tilde{F}(s) = F(s)/R(s)$, Eq. (79) can be rewritten in the compact form

$$\tilde{\psi}(s) - [\tilde{F}(s)\tilde{\psi}(s)]' = \int_1^\infty \tilde{\psi}\left(\frac{s}{x}\right)\frac{\rho(x)}{x}dx. \tag{80}$$

In general, Eq. (80) does not admit explicit closed-form solutions. However, if the ratio of the decay function $F(s)$ and the surge-rate function $R(s)$ is a linear function, then explicit closed-form solutions are attainable. Indeed, assume that $\tilde{F}(s) = as$, where a is an arbitrary positive slope. It is then straightforward to observe that power-law functions solve Eq. (80) and yield the stationary intensities

$$\psi(s) = \frac{c_1 + c_2 s^{-\epsilon}}{sR(s)} \tag{81}$$

($s > 0$), where c_1 and c_2 are positive constants and where ϵ is a real-valued exponent satisfying the moment condition

$$\mathbf{E}[J^\epsilon] = \int_0^1 x^\epsilon \rho(x)dx = 1 + a\epsilon. \tag{82}$$

The stationary intensities of Eq. (81) are identical, in form, to the stationary intensities obtained in the context of spatially inhomogeneous geometric random walks [Eq. (68)].

I. Ito diffusions

Ito diffusions constitute the elemental scientific model of general state-dependent Markov diffusion processes [4,8,81,82]. In the case of one-dimensional Ito diffusions, the state space is a real domain $\mathcal{S} \subset (-\infty,\infty)$ and the "natural measure" is the Lebesgue measure $m(ds) = ds$. The Markov dynamics of a one-dimensional Ito diffusion X are governed by the Ito stochastic differential equation

$$\dot{X}(t) = F(X(t)) + \sqrt{D(X(t))}\dot{W}(t), \tag{83}$$

where (i) $F(s)$ is the underlying "force function"; (ii) $D(s)$ is the underlying "diffusion function"; and (iii) $\dot{W}(t)$ ($t \geqslant 0$) is the "white noise" driving the dynamics, the temporal derivative of an underling one-dimensional Brownian motion $W = (W(t))_{t\geqslant 0}$.

The infinitesimal generator of the Ito diffusion is given by the differential operator

$$[\mathbf{G}\phi](s) = F(s)\phi'(s) + \tfrac{1}{2}D(s)\phi''(s), \tag{84}$$

and the adjoint operator of the Ito diffusion is given by

$$[\mathbf{G}^*\psi](s) = -[F(s)\psi(s)]' + \tfrac{1}{2}[D(s)\psi(s)]'' \tag{85}$$

[8,26]. Substituting the adjoint operator of Eq. (85) into Eq. (3) yields the Fokker-Planck partial differential equation [4,83]. In turn, the corresponding "steady-state equation" (4) is given by

$$[F(s)\psi(s)]' = \tfrac{1}{2}[D(s)\psi(s)]''. \tag{86}$$

The positive-valued solutions of the "steady-state equation" (86) are given by

$$\psi(s) = \frac{\exp(U(s))}{D(s)}(c_1 V(s) + c_2), \tag{87}$$

where (i) c_1 and c_2 are positive constants, (ii) the function $U(s)$ satisfies $U'(s) = 2F(s)/D(s)$, and (iii) the function $V(s)$ satisfies $V'(s) = \exp(-U(s))$. The "equilibrium solutions" are the positive-valued solutions satisfying $F(s)\psi(s) = \frac{1}{2}[D(s)\psi(s)]'$ and are characterized by $c_1 = 0$.

1. Linear Brownian motion

Linear Brownian motion is an Ito diffusion whose dynamics are governed by the Ito stochastic differential equation

$$\dot{X}(t) = \mu + \sigma\dot{W}(t), \tag{88}$$

where μ is a real "drift" parameter and σ is a positive "volatility" parameter. The dynamics of linear Brownian motion take place on the real line $\mathcal{S} = (-\infty,\infty)$, and the Ito stochastic differential equation (88) admits the explicit solution

$$X(t) = X(0) + \mu t + \sigma W(t) \tag{89}$$

($t \geqslant 0$). Namely, linear Brownian motion is the superposition of Brownian motion and a deterministic linear motion. Linear Brownian motion is characterized by increments which are stationary, independent, and Gauss distributed. We note that linear Brownian motion is the only process, among the class of stochastic processes with stationary and independent increments, whose trajectories are continuous [84,85]. In the example of linear Brownian motion, Eq. (87) yields the stationary intensities

$$\psi(s) = c_1 + c_2\exp(\epsilon s) \tag{90}$$

($-\infty < s < \infty$), where ϵ is a real-valued exponent given by $\epsilon = 2\mu/\sigma^2$. The stationary intensities of Eq. (90) are the "diffusion analog" of the random-walk stationary intensities of Eq. (41). The corresponding equilibrium solutions of the steady-state equation (86) are given by the exponential stationary intensities $\psi(s) = c_2\exp(\epsilon s)$.

2. Geometric Brownian motion

Geometric Brownian motionis an Ito diffusion whose dynamics are governed by the Ito stochastic differential equation

$$\frac{\dot{X}(t)}{X(t)} = \mu + \sigma \dot{W}(t), \tag{91}$$

where μ is a real "drift" parameter, and where σ is a positive "volatility" parameter. The stochastic differential equation (91) is the "multiplicative counterpart" of the "additive" stochastic differential equation (88). The dynamics of geometric Brownian motion take place on the positive half-line $\mathcal{S} = (0,\infty)$, and the Ito stochastic differential equation (91) admits the explicit solution

$$X(t) = X(0)\exp(\bar{\mu}t + \sigma W(t)) \tag{92}$$

($t \geqslant 0$), where $\bar{\mu} = \mu - \sigma^2/2$. Namely, geometric Brownian motion (with drift μ and volatility σ) is the exponentiation of linear Brownian motion (with drift $\bar{\mu}$ and volatility σ). Geometric Brownian motion is the most commonly applied model of stock prices [86,87] and is the very bedrock of the Merton-Black-Scholes option pricing formula [88,89]. In the example of geometric Brownian motion, Eq. (87) yields the stationary intensities

$$\psi(s) = c_1 s^{-1} + c_2 s^{\epsilon-2} \tag{93}$$

($s > 0$), where ϵ is a real-valued exponent given by $\epsilon = 2\mu/\sigma^2$. The stationary intensities of Eq. (93) are the "diffusion analog" of the geometric random-walk stationary intensities of Eq. (47). The corresponding equilibrium solutions of the steady-state equation (86) are given by the power-law stationary intensities $\psi(s) = c_2 s^{\epsilon-2}$.

3. Bessel motion

Bessel motion is an Ito diffusion whose dynamics are governed by the Ito stochastic differential equation

$$\dot{X}(t) = \frac{p}{2X(t)} + \dot{W}(t), \tag{94}$$

where p is a positive parameter. The dynamics of Bessel motion take place on the positive half-line $\mathcal{S} = (0,\infty)$, and the Ito stochastic differential equation (94) does not attain an explicit solution. In the example of Bessel motion Eq. (87) yields the stationary intensities

$$\psi(s) = c_1 s + c_2 s^p \tag{95}$$

($s > 0$). The corresponding equilibrium solutions of the steady-state equation (86) are given by the power-law stationary intensities $\psi(s) = c_2 s^p$. Bessel motion with parameter $p = d - 1$ tracks the distance of a d-dimensional Brownian motion ($d = 2,3,\ldots$) from the origin [26,90]; in this case, the relevant stationary intensities are the equilibrium solutions $\psi(s) = c_2 s^{d-1}$.

4. Squared Bessel motion

Squared Bessel motionis an Ito diffusion whose dynamics are governed by the Ito stochastic differential equation

$$\dot{X}(t) = 2q + 2\sqrt{X(t)}\dot{W}(t), \tag{96}$$

where q is a positive parameter. The dynamics of squared Bessel motion take place on the positive half-line $\mathcal{S} = (0,\infty)$, and the Ito stochastic differential equation (96) does not attain an explicit solution. In the example of squared Bessel motion, Eq. (87) yields the stationary intensities

$$\psi(s) = c_1 + c_2 s^{q-1} \tag{97}$$

($s > 0$). The corresponding equilibrium solutions of the steady-state equation (86) are given by the power-law stationary intensities $\psi(s) = c_2 s^{q-1}$. Squared Bessel motion with parameter $q = d/2$ tracks the squared distance of a d-dimensional Brownian motion ($d = 2,3,\ldots$) from the origin [26,90]; in this case, the relevant stationary intensities are the equilibrium solutions $\psi(s) = c_2 s^{(d/2)-1}$.

J. Langevin dynamics

The Langevin equation describes the stochastic dynamics of a motion in a "potential well," which is perturbed by a random "noise." Introduced by Paul Langevin in 1908 [91], this equation is one of the most fundamental stochastic differential equations in the physical sciences [4,8,92]. In a d-dimensional setting the Langevin equation admits the form

$$\dot{X}(t) = F(X(t)) + \dot{\xi}(t), \tag{98}$$

where (i) $F(s) = [\nabla V](s)$ is the gradient of a general potential function $V(s)$ ($s \in \mathbb{R}^d$) and (ii) $\dot{\xi}(t)$ ($t \geqslant 0$) is a d-dimensional perturbing random noise. In this setting, the state space is the d-dimensional Euclidean space $\mathcal{S} = \mathbb{R}^d$, and the "natural measure" is the Lebesgue measure $m(ds) = ds$.

1. White noise

The common perturbing noise considered in the context of Langevin dynamics is "white noise," the temporal derivative of a d-dimensional Brownian motion. In this case, the Langevin motion's infinitesimal generator is given by the differential operator

$$[\mathbf{G}\phi](s) = \langle F(s),[\nabla\phi](s)\rangle + \tfrac{1}{2}[\Delta\phi](s), \tag{99}$$

and the adjoint operator of the Ito diffusion is given by

$$[\mathbf{G}^*\psi](s) = -\langle \nabla, \psi(s)F(s)\rangle + \tfrac{1}{2}[\Delta\psi](s). \tag{100}$$

[8,83,92]. In turn, the corresponding "steady-state equation" (4) is given by

$$\langle \nabla, \psi(s)F(s)\rangle = \tfrac{1}{2}[\Delta\psi](s). \tag{101}$$

Positive-valued solutions of the "steady-state equation" (101) are given by the exponentiation of the potential function,

$$\psi(s) = c\exp(2V(s)), \tag{102}$$

where c is a positive constant.

Depending on the structure of the potential function $V(s)$, the stationary intensities of Eq. (102) can be either integrable or nonintegrable. In the former scenario, the Langevin motion X attains a steady state, and in the latter scenario the Langevin motion X is nonstationary and does not attain a steady state, but it does attain Poissonian steady states.

2. *Lévy noise*

In recent years, there has been significant interest in Langevin dynamics driven by "Lévy noises" rather than by "white noise." Lévy-driven Langevin dynamics were comprehensively studied via different perspectives and approaches [93–100]. The modeling of Lévy-driven Langevin dynamics is accommodated by setting the random noise $\dot{\xi}(t)$ to be the temporal derivative of a random walk. A random-walk driving noise induces jumps in the trajectories of the Langevin motion X. In turn, "wild" Lévy jumps of the driving random-walk result in Lévy-driven Langevin dynamics.

Analogous to the inhomogeneous random-walk model presented in Sec. VIII E, we consider the jump mechanism of the Langevin motion X to be characterized by a state-dependent jump rate $R(s)$ ($s \in \mathbb{R}^d$) and by the distribution of its generic jump size J, an arbitrary d-dimensional random variable. Between the jumps the Langevin motion propagates according to the differential equation

$$\dot{X}(t) = F(X(t)). \tag{103}$$

Namely, between the jumps the Langevin motion "slides smoothly" along the gradient $F(s) = [\nabla V](s)$ of the potential function $V(s)$.

The Markov dynamics of the Langevin motion X, along the infinitesimal time interval $(t,t+\delta)$ $(\delta \to 0)$, are given by

$$X(t+\delta) = \begin{cases} X(t) + J & \text{w.p.} \quad R(X(t))\delta + o(\delta), \\ X(t) + F(X(t))\delta & \text{w.p.} \quad 1 - R(X(t))\delta + o(\delta), \end{cases} \tag{104}$$

Namely, during the time interval $(t,t+\delta)$, the Langevin motion can either jump [with state-dependent probability $R(X(t))\delta + o(\delta)$] or slide smoothly [with state-dependent probability $1 - R(X(t))\delta + o(\delta)$]. If it jumps, then it moves from position $X(t)$ to position $X(t+\delta) = X(t) + J$ (the equality being in law), and, if it slides smoothly, then it moves from the position $X(t)$ to the position $X(t+\delta) = X(t) + F(X(t))\delta$.

Let $\rho(x)$ ($x \in \mathbb{R}^d$) denote the probability density function of the generic jump size J. A calculation of Eq. (1), based on the Markov dynamics of Eq. (104), implies that the Langevin motion's infinitesimal generator is given by the integrodifferential operator

$$[\mathbf{G}\phi](s) = \langle F(s),[\nabla\phi](s)\rangle + R(s)\int_{\mathbb{R}^d} [\phi(s+x) - \phi(s)]\rho(x)dx. \tag{105}$$

In turn, a calculation of Eq. (2) implies that the Langevin motion's adjoint operator is given by

$$[\mathbf{G}^*\psi](s) = -\langle \nabla, F(s)\psi(s)\rangle + \int_{\mathbb{R}^d} [R(s-x)\psi(s-x) - R(s)\psi(s)]\rho(x)dx. \tag{106}$$

Consequently, we obtain that the corresponding "steady-state equation" (4) is given by

$$\langle \nabla, F(s)\psi(s)\rangle + R(s)\psi(s) = \int_{\mathbb{R}^d} R(s-x)\psi(s-x)\rho(x)dx. \tag{107}$$

Setting $\tilde{\psi}(s) = R(s)\psi(s)$ and $\tilde{F}(s) = F(s)/R(s)$, Eq. (107) can be rewritten in the compact form

$$\langle \nabla, \tilde{F}(s)\tilde{\psi}(s)\rangle + \tilde{\psi}(s) = \int_{\mathbb{R}^d} \tilde{\psi}(s-x)\rho(x)dx. \tag{108}$$

In general, Eq. (108) does not admit explicit closed-form solutions. However, if the ratio of the gradient function $F(s)$ and the jump-rate function $R(s)$ is spatially homogeneous, then explicit closed-form solutions are attainable. Indeed, assume that $\tilde{F}(s) = u$, where u is an arbitrary d-dimensional vector. In this case, we obtain that $\langle \nabla, \tilde{F}(s)\tilde{\psi}(s)\rangle = \langle u, \nabla\tilde{\psi}(s)\rangle$. In turn, it is straightforward to observe that exponential functions solve Eq. (73) and yield the stationary intensities

$$\psi(s) = \frac{c_1 + c_2 \exp(-\langle v,s\rangle)}{R(s)} \tag{109}$$

($s \in \mathbb{R}^d$), where c_1 and c_2 are positive constants and where v is a d-dimensional vector satisfying the Laplace condition

$$\mathbf{E}[\exp(\langle v,J\rangle)] = \int_{\mathbb{R}^d} \exp(\langle v,x\rangle)\rho(x)dx = 1 - \langle v,u\rangle. \tag{110}$$

The stationary intensities of Eq. (109) are identical, in form, to the stationary intensities obtained in the context of spatially inhomogeneous random walks [Eq. (63)].

IX. CONCLUSIONS

In this paper we explored the steady states of countable ensembles of independent motions governed by common Markov dynamics. Given general Markov dynamics taking place in a general state space, we initiated a countable ensemble of independent motions from initial positions which form a general Poisson process. Consequently, the ensemble's positions at all future times also form Poisson processes. This "Poissonian stability" led to the introduction of the notion of "stationary intensity." We defined a Poissonian intensity function to be a stationary intensity of the Markov dynamics if the following condition holds: If the initial Poisson process is governed by the given Poissonian intensity function, then so are the Poisson processes at all future times.

Analysis established that the aforementioned stationary intensities are steady-state solutions of the Markov dynamics' master equation. Thus, considering the positive-valued steady-state solutions of the master equation, we obtained that

(i) These solutions characterize the Poissonian steady states of the Markov dynamics.

(ii) If the solutions are integrable, then each motion attains a steady state, and the ensemble's Poissonian steady states are equivalent to the steady states of its composing motions.

(iii) If the solutions are nonintegrable, then each motion is nonstationary and does not attain a steady state but the entire ensemble does attain Poissonian steady states.

We have further shown that in the integrable scenario the ensemble consists of finitely many IID stationary motions, whereas in the nonintegrable scenario the ensemble consists of infinitely many nonstationary motions. In addition to the notion of Poissonian steady states, we also introduced the notions of Poissonian fluxes and Poissonian correlations and have shown how these three notions relate to each other. The results established were applied to a host of widely used stochastic models: Brownian motion, birth-death processes, random walks, geometric random walks, renewal processes, growth-collapse dynamics, decay-surge dynamics, Ito diffusions, and Langevin dynamics.

In this paper we set forth from the elemental Markov notions of "steady state" and "stationary density" and have generalized them to the notions of "Poissonian steady state" and "stationary intensity." These novel notions facilitate both the qualitative understanding and the quantitative description of the concept of statistical steady state in the context of general Markov dynamics. Moreover, these novel notions enable us to attain the oxymoronic goal of describing the stationary structures of nonstationary Markov dynamics. The applications of the results established in this paper apply to all fields of science in which nonstationary Markov dynamics arise.

APPENDIX

A key theorem from the theory of Poisson processes asserts that [9]: a random ensemble of points $\{p_n\}_n \subset \mathcal{S}$ is a Poisson process with intensity $\lambda(s)$ ($s \in \mathcal{S}$) if and only if

$$\mathbf{E}\left[\prod_n \phi(p_n)\right] = \exp\left\{\int_{\mathcal{S}} [\phi(s)-1]\lambda(s)m(ds)\right\} \tag{A1}$$

holds for all test functions $\phi(s)$ ($s \in \mathcal{S}$) for which the integral appearing on the right-hand side of Eq. (A1) converges. We shall use this theorem in the proofs given hereinafter.

1. Proof of Eq. (15)

In what follows we consider the setting and notation of Sec. V, fix an arbitrary test function $\phi(s)$ ($s \in \mathcal{S}$) and use the shorthand notation

$$\Phi(\delta,s) = \mathbf{E}[\phi(X_n(t+\delta))|X_n(t)=s] \tag{A2}$$

($t \geqslant 0, s \in \mathcal{S}$).

Applying Eq. (A1) to the Poisson process $\mathcal{E}(t+\delta)$ implies that

$$\mathbf{E}\left[\prod_n \phi(X_n(t+\delta))\right] = \exp\left\{\int_{\mathcal{S}} [\phi(s)-1]\Lambda(t+\delta,s)m(ds)\right\}. \tag{A3}$$

On the other hand, conditioning implies that

$$\mathbf{E}\left[\prod_n \phi(X_n(t+\delta))\right] = \mathbf{E}\left[\mathbf{E}\left[\prod_n \phi(X_n(t+\delta))|\mathcal{E}(t)\right]\right]. \tag{A4}$$

In turn, the independence of the Markov motions implies that

$$\mathbf{E}\left[\prod_n \phi(X_n(t+\delta))\right] = \mathbf{E}\left[\prod_n \mathbf{E}[\phi(X_n(t+\delta))|X_n(t)]\right]. \tag{A5}$$

Substituting Eq. (A2) into Eq. (A5) we obtain that

$$\mathbf{E}\left[\prod_n \phi(X_n(t+\delta))\right] = \mathbf{E}\left[\prod_n \Phi(\delta,X_n(t))\right]. \tag{A6}$$

Applying Eq. (A1) to the Poisson process $\mathcal{E}(t)$ we further obtain that

$$\mathbf{E}\left[\prod_n \phi(X_n(t+\delta))\right] = \exp\left\{\int_{\mathcal{S}} [\Phi(\delta,s)-1]\Lambda(t,s)m(ds)\right\}. \tag{A7}$$

Combining together Eqs. (A3) and (A7) we conclude that

$$\int_{\mathcal{S}} [\phi(s)-1]\Lambda(t+\delta,s)m(ds) = \int_{\mathcal{S}} [\Phi(\delta,s)-1]\Lambda(t,s)m(ds). \tag{A8}$$

Now, subtracting the term $\int_{\mathcal{S}}[\phi(s)-1]\Lambda(t,s)m(ds)$ from both sides of Eq. (A8) and thereafter dividing both sides of the equation by δ yields

$$\int_{\mathcal{S}} [\phi(s)-1]\frac{\Lambda(t+\delta,s)-\Lambda(t,s)}{\delta}m(ds) = \int_{\mathcal{S}} \frac{[\Phi(\delta,s)-1]-[\phi(s)-1]}{\delta}\Lambda(t,s)m(ds). \tag{A9}$$

Taking $\delta \to 0$ and using the definition of the infinitesimal generator $\mathbf{G}$ [Eq. (1)], Eq. (A9) implies that

$$\int_{\mathcal{S}} [\phi(s)-1]\frac{\partial\Lambda}{\partial t}(t,s)m(ds) = \int_{\mathcal{S}} [\mathbf{G}(\phi-1)](s)\Lambda(t,s)m(ds). \tag{A10}$$

Using the definition of the adjoint operator $\mathbf{G}^*$ [Eq. (2)], Eq. (A10) further implies that

$$\int_{\mathcal{S}} [\phi(s)-1]\frac{\partial\Lambda}{\partial t}(t,s)m(ds) = \int_{\mathcal{S}} [\phi(s)-1][\mathbf{G}^*\Lambda](t,s)m(ds). \tag{A11}$$

Equation (A11) holds for arbitrary test functions $\phi(s)$. Hence, we conclude that

$$\frac{\partial\Lambda}{\partial t}(t,s) = [\mathbf{G}^*\Lambda](t,s), \tag{A12}$$

where the initial condition $\Lambda(0,s)$ is an arbitrary Poissonian intensity function defined on the state space.

A more "compact" proof of Eq. (15) follows from Eq. (12)

$$\Lambda(t,s) = \int_{\mathcal{S}} \Lambda(0,x)P_x(t,s)m(dx), \tag{A13}$$

where $P_x(t,s)$ ($t \geqslant 0, s,x \in \mathcal{S}$) is the solution of Eq. (3) with the "δ-function" initial condition $P(0,s) = \boldsymbol{\delta}(s-x)$. Differentiating Eq. (A13) with respect to the time variable t yields

$$\frac{\partial\Lambda}{\partial t}(t,s) = \int_{\mathcal{S}} \Lambda(0,x)\left[\frac{\partial P_x}{\partial t}(t,s)\right]m(dx). \tag{A14}$$

In turn, using Eq. (3), we obtain that

$$\frac{\partial\Lambda}{\partial t}(t,s) = \int_{\mathcal{S}} \Lambda(0,x)[\mathbf{G}^* P_x](t,s)m(dx). \tag{A15}$$

Since the adjoint operator $\mathbf{G}^*$ is a linear operator that "acts" on the state-space variable s Eq. (A15) further implies that

$$\frac{\partial \Lambda}{\partial t}(t,s) = \mathbf{G}^* \left[\int_{\mathcal{S}} \Lambda(0,x) P_x(t,s) m(dx) \right] \quad \text{(A16)}$$

(with a slight abuse of notation). Finally, using Eq. (A13) we conclude that

$$\frac{\partial \Lambda}{\partial t}(t,s) = [\mathbf{G}^* \Lambda](t,s). \quad \text{(A17)}$$

2. Proofs of Eqs. (21) and (22)

In what follows, we consider the setting and notation of Sec. VI. The probability-generating function of the random variable $N_{A\mapsto B}(t)$—defined in Eq. (20)—is given by

$$\mathbf{E}[z^{N_{A\mapsto B}(t)}] = \mathbf{E}\left[\prod_n z^{\mathbf{I}_A(X_n(0))\mathbf{I}_B(X_n(t))} \right] \quad \text{(A18)}$$

(z complex). Conditioning implies that

$$\mathbf{E}[z^{N_{A\mapsto B}(t)}] = \mathbf{E}\left[\mathbf{E}\left[\prod_n z^{\mathbf{I}_A(X_n(0))\mathbf{I}_B(X_n(t))} | \mathcal{E}(0) \right] \right]. \quad \text{(A19)}$$

In turn, the independence of the Markov motions implies that

$$\mathbf{E}[z^{N_{A\mapsto B}(t)}] = \mathbf{E}\left[\prod_n \mathbf{E}[z^{\mathbf{I}_A(X_n(0))\mathbf{I}_B(X_n(t))} | X_n(0)] \right]. \quad \text{(A20)}$$

A straightforward calculation yields

$$\begin{aligned} &\mathbf{E}[z^{\mathbf{I}_A(X_n(0))\mathbf{I}_B(X_n(t))} | X_n(0)] \\ &= z^{\mathbf{I}_A(X_n(0))} \Pr(X_n(t) \in B | X_n(0)) \\ &\quad + [1 - \Pr(X_n(t) \in B | X_n(0))] \\ &= 1 + (z^{\mathbf{I}_A(X_n(0))} - 1) \Pr(X_n(t) \in B | X_n(0)) \\ &= 1 + (z-1)\mathbf{I}_A(X_n(0)) \Pr(X_n(t) \in B | X_n(0)) \\ &= 1 + (z-1)\mathbf{I}_A(X_n(0)) \int_B P_{X_n(0)}(t,s) m(ds). \end{aligned} \quad \text{(A21)}$$

Substituting Eq. (A21) into Eq. (A20) we obtain that

$$\mathbf{E}[z^{N_{A\mapsto B}(t)}] = \mathbf{E}\left[\prod_n \left[1 + (z-1)\mathbf{I}_A(X_n(0)) \times \int_B P_{X_n(0)}(t,s) m(ds) \right] \right]. \quad \text{(A22)}$$

Applying Eq. (A1) to the Poisson process $\mathcal{E}(0) = \{X_n(0)\}$ we further obtain that

$$\mathbf{E}[z^{N_{A\mapsto B}(t)}] = \exp\left((z-1) \int_{\mathcal{S}} \left[\mathbf{I}_A(x) \int_B P_x(t,s) m(ds) \right] \times \Lambda(x) dx \right). \quad \text{(A23)}$$

Equation (A23) implies that the random variable $N_{A\mapsto B}(t)$ is Poisson distributed with mean

$$\begin{aligned} \mathbf{E}[N_{A\mapsto B}(t)] &= \int_{\mathcal{S}} \left[\mathbf{I}_A(x) \int_B P_x(t,s) m(ds) \right] \Lambda(x) dx \\ &= \int_A \left(\int_B P_x(t,s) m(ds) \right) \Lambda(x) m(dx). \end{aligned} \quad \text{(A24)}$$

This proves Eq. (21).

Let X be a Markov motion whose dynamics are characterized by the infinitesimal generator $\mathbf{G}$. Using Eq. (A24) note that

$$\begin{aligned} \mathbf{E}[N_{A\mapsto B}(t)] &= \int_{\mathcal{S}} \left[\mathbf{I}_A(x) \int_B P_x(t,s) m(ds) \right] \Lambda(x) dx \\ &= \int_{\mathcal{S}} \mathbf{I}_A(x) \Pr(X(t) \in B | X(0) = x) \Lambda(x) m(dx) \\ &= \int_{\mathcal{S}} \mathbf{I}_A(x) \mathbf{E}[\mathbf{I}_B(X(t)) | X(0) = x] \Lambda(x) m(dx). \end{aligned} \quad \text{(A25)}$$

Consequently, we obtain that

$$\frac{d}{dt}\mathbf{E}[N_{A\mapsto B}(t)]|_{t=0} = \lim_{t\to 0} \frac{1}{t}(\mathbf{E}[N_{A\mapsto B}(t)] - \mathbf{E}[N_{A\mapsto B}(0)]). \quad \text{(A26)}$$

In turn, Eq. (A25) implies that

$$\begin{aligned} &\lim_{t\to 0} \frac{1}{t}(\mathbf{E}[N_{A\mapsto B}(t)] - \mathbf{E}[N_{A\mapsto B}(0)]) \\ &= \lim_{t\to 0} \frac{1}{t} \left(\int_{\mathcal{S}} \mathbf{I}_A(x) \mathbf{E}[\mathbf{I}_B(X(t)) | X(0) = x] \Lambda(x) m(dx) \right. \\ &\quad \left. - \int_{\mathcal{S}} \mathbf{I}_A(x) \mathbf{E}[\mathbf{I}_B(X(0)) | X(0) = x] \Lambda(x) m(dx) \right) \\ &= \int_{\mathcal{S}} \mathbf{I}_A(x) \left(\lim_{t\to 0} \frac{1}{t} \mathbf{E}[\mathbf{I}_B(X(t)) - \mathbf{I}_B(X(0)) | X(0) = x] \right) \\ &\quad \times \Lambda(x) m(dx). \end{aligned} \quad \text{(A27)}$$

Finally, combining together Eqs. (1), (A26), and (A27), we conclude that

$$\frac{d}{dt}\mathbf{E}[N_{A\mapsto B}(t)]|_{t=0} = \int_{\mathcal{S}} \mathbf{I}_A(x) [\mathbf{G}\mathbf{I}_B](x) \Lambda(x) m(dx). \quad \text{(A28)}$$

This proves Eq. (22).

3. Proofs of Eqs. (26) and (27)

First, note that conditioning implies that

$$\begin{aligned} \mathbf{E}\left[z_1^{N_A(0)} z_2^{N_B(t)}\right] &= \mathbf{E}\left[\mathbf{E}\left[z_1^{N_A(0)} z_2^{N_B(t)} | \mathcal{E}(0)\right]\right] \\ &= \mathbf{E}\left[z_1^{N_A(0)} \mathbf{E}\left[z_2^{N_B(t)} | \mathcal{E}(0)\right]\right] \end{aligned} \quad \text{(A29)}$$

(z_1, z_2 complex). Equation (25) further implies that

$$\begin{aligned} z_1^{N_A(0)} &= \prod_n z_1^{\mathbf{I}_A(X_n(0))} \\ &= \prod_n [z_1 \mathbf{I}_A(X_n(0)) + (1 - \mathbf{I}_A(X_n(0)))] \\ &= \prod_n [1 + (z_1 - 1)\mathbf{I}_A(X_n(0))], \end{aligned} \quad \text{(A30)}$$

and

$$\mathbf{E}\left[z_2^{N_B(t)} | \mathcal{E}(0)\right] = \mathbf{E}\left[\prod_n z_2^{\mathbf{I}_B(X_n(t))} | \mathcal{E}(0) \right]. \quad \text{(A31)}$$

In turn, the independence of the Markov motions implies that

$$\mathbf{E}\left[\prod_n z_2^{\mathbf{I}_B(X_n(t))}|\mathcal{E}(0)\right] = \prod_n \mathbf{E}\left[z_2^{\mathbf{I}_B(X_n(t))}|X_n(0)\right] = \prod_n [z_2 \Pr(X_n(t) \in B|X_n(0)) + (1-\Pr(X_n(t)\in B|X_n(0)))]$$
$$= \prod_n [1 + (z_2 - 1)\Pr(X_n(t) \in B|X_n(0))] = \prod_n \left[1 + (z_2 - 1)\int_B P_{X_n(0)}(t,s)m(ds)\right]. \quad \text{(A32)}$$

Using Eqs. (A30) and (A32), note that

$$lz_1^{N_A(0)}\mathbf{E}\left[z_2^{N_B(t)}|\mathcal{E}(0)\right] = \left(\prod_n [1 + (z_1 - 1)\mathbf{I}_A(X_n(0))]\right)\left(\prod_n \left[1 + (z_2 - 1)\int_B P_{X_n(0)}(t,s)m(ds)\right]\right)$$
$$= \prod_n [1 + (z_1 - 1)\mathbf{I}_A(X_n(0))]\left[1 + (z_2 - 1)\int_B P_{X_n(0)}(t,s)m(ds)\right]$$
$$= \prod_n \left[1 + (z_1 - 1)\mathbf{I}_A(X_n(0)) + (z_2 - 1)\left(\int_B P_{X_n(0)}(t,s)m(ds)\right)\right.$$
$$\left. + (z_1 - 1)(z_2 - 1)\mathbf{I}_A(X_n(0))\left(\int_B P_{X_n(0)}(t,s)m(ds)\right)\right]. \quad \text{(A33)}$$

Substituting Eq. (A33) into Eq. (A29), and applying Eq. (A1) to the Poisson process $\mathcal{E}(0) = \{X_n(0)\}$, we obtain that

$$\mathbf{E}\left[z_1^{N_A(0)}z_2^{N_B(t)}\right] = \exp\left(\int_{\mathcal{S}}\left[(z_1 - 1)\mathbf{I}_A(x) + (z_2 - 1)\left(\int_B P_x(t,s)m(ds)\right)\right.\right.$$
$$\left.\left. + (z_1 - 1)(z_2 - 1)\mathbf{I}_A(x)\left(\int_B P_x(t,s)m(ds)\right)\right]\Lambda(0,x)m(dx)\right). \quad \text{(A34)}$$

Now, note that
(i)

$$\int_{\mathcal{S}}(z_1 - 1)\mathbf{I}_A(x)\Lambda(0,x)m(ds) = (z_1 - 1)\int_A \Lambda(0,x)m(dx) = (z_1 - 1)\mathbf{E}[N_A(0)], \quad \text{(A35)}$$

(ii)

$$\int_{\mathcal{S}}(z_2 - 1)\left(\int_B P_x(t,s)m(ds)\right)\Lambda(0,x)m(dx) = (z_2 - 1)\int_B\left(\int_{\mathcal{S}} P_x(t,s)\Lambda(0,x)m(dx)\right)m(ds)$$
$$= (z_2 - 1)\int_B \Lambda(t,s)m(ds) = (z_2 - 1)\mathbf{E}[N_B(t)] \quad \text{(A36)}$$

(in the transition from the second to the third line we applied Eq. (12)),
(iii)

$$\int_{\mathcal{S}}(z_1 - 1)(z_2 - 1)\mathbf{I}_A(x)\left(\int_B P_x(t,s)m(ds)\right)\Lambda(0,x)m(dx) = (z_1 - 1)(z_2 - 1)\int_A\left(\int_B P_x(t,s)m(ds)\right)\Lambda(0,x)m(dx)$$
$$= (z_1 - 1)(z_2 - 1)\mathbf{E}[N_{A\mapsto B}(t)] \quad \text{(A37)}$$

[in the transition from the second to the third line we applied Eq. (21)]. Thus, substituting Eqs. (A35)–(A37) into Eq. (A34), we conclude that

$$\mathbf{E}\left[z_1^{N_A(0)}z_2^{N_B(t)}\right] = \exp((z_1 - 1)\mathbf{E}[N_A(0)])\exp((z_2 - 1)\mathbf{E}[N_B(t)])\exp((z_1 - 1)(z_2 - 1)\mathbf{E}[N_{A\mapsto B}(t)]) \quad \text{(A38)}$$

(z_1, z_2 complex). This proves Eq. (26).

Finally, differentiating Eq. (A38) twice—once with respect to the variable z_1 and once with respect to the variable z_2—we obtain that

$$\mathbf{E}\left[N_A(0)z_1^{N_A(0)-1}N_B(t)z_2^{N_B(t)-1}\right] = \mathbf{E}\left[z_1^{N_A(0)}z_2^{N_B(t)}\right]\mathbf{E}[N_{A\mapsto B}(t)] + \mathbf{E}\left[z_1^{N_A(0)}z_2^{N_B(t)}\right](\mathbf{E}[N_A(0)] + (z_1 - 1)\mathbf{E}[N_{A\mapsto B}(t)])(\mathbf{E}[N_B(t)]$$
$$+ (z_2 - 1)\mathbf{E}[N_{A\mapsto B}(t)]). \quad \text{(A39)}$$

Setting $z_1 = z_2 = 1$ in Eq. (A39) yields

$$\mathbf{E}[N_A(0)N_B(t)] = \mathbf{E}[N_{A\mapsto B}(t)] + \mathbf{E}[N_A(0)]\mathbf{E}[N_B(t)], \tag{A40}$$

which, in turn, implies that

$$\mathbf{Cov}[N_A(0), N_B(t)] = \mathbf{E}[N_{A\mapsto B}(t)]. \tag{A41}$$

This proves Eq. (27).

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