Energy transport in weakly nonlinear wave systems with narrow frequency band excitation

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(Received 22 June 2012; revised manuscript received 12 September 2012; published 17 October 2012)

A novel discrete model (D model) is presented describing nonlinear wave interactions in systems with small and moderate nonlinearity under narrow frequency band excitation. It integrates in a single theoretical frame two mechanisms of energy transport between modes, namely, intermittency and energy cascade, and gives the conditions under which each regime will take place. Conditions for the formation of a cascade, cascade direction, conditions for cascade termination, etc., are given and depend strongly on the choice of excitation parameters. The energy spectra of a cascade may be computed, yielding discrete and continuous energy spectra. The model does not require statistical assumptions, as all effects are derived from the interaction of distinct modes. In the example given—surface water waves with dispersion function $\omega^2 = gk$ and small nonlinearity—the D model predicts asymmetrical growth of side-bands for Benjamin-Feir instability, while the transition from discrete to continuous energy spectrum, excitation parameters properly chosen, yields the saturated Phillips' power spectrum $\sim g^2\omega^{-5}$. The D model can be applied to the experimental and theoretical study of numerous wave systems appearing in hydrodynamics, nonlinear optics, electrodynamics, plasma, convection theory, etc.

DOI: 10.1103/PhysRevE.86.041129 PACS number(s): 47.35.−i, 05.60.Cd, 89.75.Da

I. INTRODUCTION

A central topic in the theory of weakly nonlinear wave interactions is the mechanism of energy transport between modes. Considering what we can describe in theory and observe in experiment, there is good reason to believe that in any weakly nonlinear dispersive wave system there are two main types of energy transport: intermittency, which is a periodic or chaotic exchange of energy among a small number of modes; and energy cascade, which is a unidirectional flow of energy through scales in Fourier space.

In systems with a distributed initial state, energy transport is studied in the framework of *kinetic* wave turbulence theory (WTT) by means of the wave kinetic equation [1,2]. In this paper we explicitly study wave systems with narrow frequency band excitation.

Intermittency is based on finite-size effects in a resonator. The general properties of weakly nonlinear wave systems showing intermittency were first characterized through the solution of the kinematic resonance conditions [3], which reflect the geometry of the resonator. The general dynamical characteristics of this type of energy transport have been studied in the frame of *discrete* WTT [4] for systems with narrow frequency band excitation. The main mathematical object of the discrete WTT is a set of dynamical systems for the amplitudes of interacting waves; each dynamical system corresponds to a resonance cluster composed of a small number of resonant triads or quartets having joint modes [5].

Energy cascades in systems with narrow frequency band excitation have recently been described in Ref. [6] using the increment chain equation method (ICEM). An energy cascade is represented as a chain of modes with nonlinear frequencies triggered by modulation instability (MI) at each cascade step. The energy spectra $E(\omega)$ obtained by the ICEM have *exponential decay* and can be written as

$$E(\omega) \sim \sum_{i=1}^{i\geqslant 2} C_i \omega^{-\gamma_i}, \quad \gamma_i > 0, \tag{1}$$

where for a given linear dispersion function $\omega \sim k^\alpha$, C_i are known functions of excitation parameters and γ_i vary for different magnitudes of nonlinearity. For comparison, in systems with a distributed initial state, studied in the frame of *kinetic* wave turbulence theory (WTT), energy spectra decay according to a power law,

$$E(\omega) \sim \omega^{-\gamma}, \quad \gamma > 0, \tag{2}$$

with different γ for different wave systems [1,2].

In this paper we present, based on the resonance conditions, a common mathematical model, called the D model (D for discrete), incorporating both forms of energy transport, intermittency and cascades, and give the criteria regarding under what conditions to expect which behavior. In the D model, intermittency occurs for very small nonlinearity, $0 < \varepsilon < 0.1$, provided that the geometrical form of the resonator permits resonance. An energy cascade occurs at larger levels of nonlinearity, $\varepsilon \sim 0.1$–0.4, and its spectrum does not depend on the shape or finiteness of the interaction domain. The outcome of the model strongly depends on the excitation parameters.

The D model can explain the following phenomena observed in systems with narrow frequency band excitation.

(i) There is no cascade but recurrent wave patterns are observed (e.g., surface water waves) [7].

(ii) There is a cascade consisting of two distinct parts—discrete and continuous; the form of spectra does not follow a power law (e.g., a thin elastic steel plate [8] or gravity capillary waves in mercury [9]).

(iii) A discrete energy cascade develops a strongly nonlinear regime yielding breaking; a continuous part of the spectrum is not observed (e.g., surface water waves) [10].

(iv) The form of energy spectra depends on the parameters of excitation (e.g., gravity surface waves [11] and capillary water waves [12]).

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(v) Amplitudes of direct and inverse cascades are not symmetric (e.g., [13–17]).

(vi) Interactions of waves occur over several orders of magnitude (e.g., capillary waves in helium) [18].

The model is briefly described in Sec. II. To demonstrate how the D model works, we give an example determining the cascade direction and scenarios of cascade termination for surface water waves, depending on the excitation parameters, in Sec. III.

In Sec. IV we compare assumptions and predictions of the D model versus the kinetic WTT to give the experimentalist clues as to which model to apply in a given experimental setup. A short list of conclusions and open questions is given in Sec. V.

II. D MODEL

The time evolution of a wave field in a weakly nonlinear wave system is described by a weakly nonlinear partial differential equation of the form

$$L(\psi) = -\varepsilon N(\psi), \tag{3}$$

where N is a nonlinear operator, $0 < \varepsilon \ll 1$, and L is an arbitrary linear dispersive operator, i.e., $L(\varphi) = 0$ for Fourier harmonics $\varphi = A \exp i[\mathbf{k} \cdot \mathbf{x} - \omega(\mathbf{k})t]$, with constant A. Here $A, \mathbf{k}, \omega = \omega(\mathbf{k})$ denote the amplitude, wave vector, and dispersion function, respectively. The small parameter is usually introduced as the wave steepness $\varepsilon = Ak$, $k = |\mathbf{k}|$. If the nonlinearity is small enough, only resonant interactions have to be taken into account. The resonance conditions read as follows.

$$\text{For three waves:} \quad \begin{cases} \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3), \\ \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3. \end{cases} \tag{4}$$

$$\text{For four waves:} \quad \begin{cases} \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) + \omega(\mathbf{k}_4), \\ \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4. \end{cases} \tag{5}$$

Dynamical systems describing the time evolution of slowly changing amplitudes A_j of resonantly interacting modes can be obtained from (3) and (4) or from (3) and (5) using, e.g., a multiscale method. In a three-wave system $A_j = A_j(T)$, $T = t/\varepsilon$, and in a four-wave system $A_j = A_j(\widetilde{T})$, $\widetilde{T} = t/\varepsilon^2$. The corresponding dynamical systems (in canonic variables) are written as

$$i\dot{A}_1 = ZA_2^* A_3, \quad i\dot{A}_2 = ZA_1^* A_3, \quad i\dot{A}_3 = -ZA_1A_2; \tag{6}$$

$$\begin{aligned} i\dot{A}_1 &= VA_2^*A_3A_4 + (\tilde{\omega}_1 - \omega_1)A_1, \\ i\dot{A}_2 &= VA_1^*A_3A_4 + (\tilde{\omega}_2 - \omega_2)A_2, \\ i\dot{A}_3 &= V^*A_4^*A_1A_2 + (\tilde{\omega}_3 - \omega_3)A_3, \\ i\dot{A}_4 &= V^*A_3^*A_1A_2 + (\tilde{\omega}_4 - \omega_4)A_4, \\ \tilde{\omega}_j - \omega_j &= \sum_{i=1}^{4} (V_{ij}|A_j|^2 - \frac{1}{2} V_{jj}|A_i|^2), \end{aligned} \tag{7}$$

where the interaction coefficients $V_{ij} = V_{ji} \equiv V_{ij}^{ij}$ and $V = V_{34}^{12}$ are responsible for the nonlinear shifts of frequency and the energy exchange within a quartet, respectively, and $(\tilde{\omega}_j - \omega_j)$ are Stokes-corrected frequencies. For very small nonlinearity, dynamical system (7) can be regarded in a simplified form, with $\tilde{\omega}_j - \omega_j = 0$, i.e., without nonlinear correction of frequencies.

Three-wave interactions dominate in a weakly nonlinear wave system if resonance conditions (4) have solutions and the coupling coefficients $Z \neq 0$. Otherwise, the leading nonlinear processes are four-wave interactions.

The following results likewise hold for resonances and quasiresonances with small enough frequency mismatch.

A. Intermittency, $0 < \varepsilon < 0.1$

Excitation of a single mode in a three-wave system generates energy exchange within a resonance cluster only if this is the high-frequency mode $\omega(\mathbf{k}_3)$ from (4). In a four-wave system, excitation of a single mode generates energy exchange only if it is the high-frequency mode $\omega(\mathbf{k}_3)$ in a Phillips quartet,

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = 2\omega(\mathbf{k}_3), \quad \mathbf{k}_1 + \mathbf{k}_2 = 2\mathbf{k}_3, \tag{8}$$

which is a special case of (5) [19]. Solutions of resonance conditions (4) and (5) form a set of independent resonance clusters. The form of a cluster uniquely defines its dynamical system.

Solutions of dynamical systems (6) and (7) are known [20,21]; they describe periodic energy exchange within a resonant triad or quartet, respectively. Resonance clusters of a more complicated structure may have a dynamical system with periodic or chaotic evolution, [5].

In both three- and four-wave systems, resonant interactions are not local in k space; furthermore, in a four-wave system with dispersion function $\omega \sim k^\alpha$, modes with an arbitrary big difference in wavelengths can interact directly. In this case a parametric series of solutions of resonance conditions can be easily written as

$$\begin{aligned} &k_1^\alpha + k_2^\alpha = k_3^\alpha + k_4^\alpha, \quad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4, \\ &\Rightarrow \mathbf{k}_1 = (k_x, k_y), \quad \mathbf{k}_2 = (\mathbf{s}, -k_y), \\ &\quad \mathbf{k}_3 = (k_x, -k_y), \quad \mathbf{k}_4 = (\mathbf{s}, k_y), \end{aligned} \tag{9}$$

where $\mathbf{s}$ is an arbitrary real parameter (see Fig. 1).

In any given three-wave system, most of the modes are nonresonant. A nonresonant mode, being excited, does not change its energy at the slow time scale T. In the majority of four-wave systems, each mode satisfies (5). However, the excitation of a single mode does not generate resonance in the general case: the excited mode has to be the high-frequency mode in a Phillips quartet.

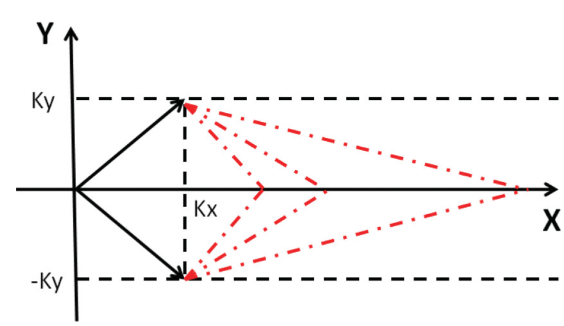

FIG. 1. (Color online) Nonlocal interactions in a four-wave system, $\omega \sim k^\alpha$. Each pair of dot-dashed (red) lines of equal length corresponds to a specific choice of the parameter $\mathbf{s}$.

B. D cascade, $\varepsilon \sim 0.1$–0.4

A D cascade is a cascade computed in the D model by the ICEM method first presented in [6]. In both three- and four-wave systems, D cascades are generated by MI. Accordingly, the ICEM method can be applied for all partial differential equations in which MI has been established: the focusing weakly nonlinear Schrödinger (NLS) equation [17], modified NLS [22,23], modified Korteweg–de Vries equation, [24,25], and Gardner equation [26].

In both three- and four-wave systems, D cascades are generated by MI which is described as a particular case of the Phillips quartet, (8), with $\omega_1 = \omega_0 + \Delta\omega$, $\omega_2 = \omega_0 - \Delta\omega$, $0 < \Delta\omega \ll 1$:

$$\omega_1 + \omega_2 = 2\omega_0, \quad \mathbf{k}_1 + \mathbf{k}_2 = 2\mathbf{k}_0. \tag{10}$$

The mode with frequency ω_0 is called the *carrier mode*. At each step in a discrete cascade, conditions (10) are satisfied, with a new carrier mode generated from the previous cascade step.

Time evolution of the quartet, (10), is studied in the frame of the NLS equation. The corresponding time scale $\tau = t/\varepsilon^2$ is called the Benjamin-Feir time scale and is shorter than the time scale of resonant interactions. To understand this, one has to take into account that the small parameter $\varepsilon_{\text{res}} < 0.1$ yielding resonance interactions is in fact substantially smaller than the $\varepsilon_{\text{MI}} \sim 0.1$–$0.4$ corresponding to MI: $\widetilde{T} = t/\varepsilon_{\text{res}}^2 > t/\varepsilon_{\text{MI}}^2 = \tau$. This fact is well established, e.g., in the theory of wind-generated oceanic waves [28].

The conditions under which MI occurs may be given as an instability interval for the initial real amplitude A and frequency ω of the carrier wave. For the NLS equation with dispersion relation $\omega^2 = gk$ and small nonlinearity $\varepsilon \sim 0.1$–0.25, the instability interval is described by

$$0 < \Delta\omega/Ak\omega \leqslant \sqrt{2}. \tag{11}$$

The most unstable mode in this interval satisfies the so-called maximum increment condition (in Benjamin-Feir form [17]):

$$\Delta\omega/\omega Ak = 1. \tag{12}$$

For moderate nonlinearity, $\varepsilon \sim 0.25$–0.4, the maximum increment condition reads (in Dysthe form [22])

$$\Delta\omega/\left(\omega Ak - \tfrac{3}{2}\omega^2 A^2 k^2\right) = 1. \tag{13}$$

Equations (12) and (13) each generate two chain equations (one for a direct D cascade and one for an inverse D cascade) describing the connection between the amplitudes of two neighboring modes in the D cascade, under the following assumptions [6].

(i) The fraction p of energy transported from one cascading mode to the next one depends only on the excitation parameters, and not on the step number of the cascade; p is called the cascade intensity.

(ii) Modes forming a D cascade have the maximum instability increment, i.e., a cascade is formed by the most unstable modes within the corresponding intervals of instability. This is a mathematical reformulation of the Phillips hypothesis that the spectral density is saturated at a level determined by wave breaking [27].

In particular, (12) generates chain equations connecting mode n to mode $n+1$,

$$\omega_{n+1} = \omega_n + \omega_n A(\omega_n) k_n, \tag{14}$$

$$\omega_{n+1} = \omega_n - \omega_n A(\omega_n) k_n, \tag{15}$$

for direct and inverse D cascades, respectively. This means that the D cascades are formed by *nonlinear frequencies* depending on the amplitudes. From the chain equations various properties of D cascades can be derived, including the form of the discrete and continuous energy spectra.

III. SURFACE WATER WAVES

To demonstrate the wide range of the predictions which are given by our model, we have chosen a classical example: surface water waves with dispersion function $\omega^2 = gk$ and small nonlinearity, $\varepsilon \sim 0.1$–0.25.

Before proceeding with our study we need to make an important remark on the terminology used below. Standard vocabulary for discussing wave resonant interactions is" a three-wave system" if (4) and (6) are satisfied and" a four-wave system" if (5) and (7) are satisfied. Regarding resonance conditions for a Phillips quartet, (8), one might formally conclude that this is a system of three waves with frequencies ω_1, ω_2, and $2\omega_3$. However, comparing the dynamical system for a three-wave system, (6), and the dynamical system for a Phillips quartet obtained from (7) by taking $A_3 = A_4$, we can see immediately that these systems are different. Accordingly, a Phillips quartet may be referred to in the literature as a four-wave system.

In the text below we call system (10) a four-wave system, although in the original papers whose results are interpreted using the D model, this system is often called a three-wave system. Our terminology also allows us to avoid confusion when discussing cascade termination due to intermittency in Sec. III C3.

A. Discrete and continuous energy spectra

For determining D-cascade direction and scenarios of D-cascade termination we first need to compute the form of the discrete energy spectrum. Detailed computation of D spectra for various wave systems is given in [6]. For the reader's convenience, below we outline this computation for surface water waves with small nonlinearity.

All computations below are performed with chain equation (14) and yield energy spectra for direct cascade. Computations for inverse cascade should be conducted similarly but with chain equation (15); they are omitted here.

Assumptions a and b (see Sec. II B) mean that $E_n = pE_{n-1}$ at any cascade step n, $E_n \sim A_n^2$ being the energy of the mode with amplitude A_n. As the dispersion function in this case has the form $\omega^2 = gk$, this allows us to rewrite (14) as

$$\sqrt{p} A(\omega_n) = A\left(\omega_n + \omega_n^3 A_n/g\right) = \sum_{s=0}^{\infty} \frac{A_n^{(s)}}{s!} \left(\omega_n^3 A_n/g\right)^s \tag{16}$$

[here the notation $A_n = A(\omega_n)$ is used].

Restricting ourselves to the first two terms in the Taylor expansion for the left-hand side of (16), we can obtain an ordinary differential equation and solve it analytically:

$$\omega_n^3 A_n^{'} A_n/g + (1-\sqrt{p})A_n = 0 \tag{17}$$

$$\Rightarrow A_n = g\frac{(1-\sqrt{p})}{2}\omega_n^{-2} + C, \quad C = A_0 - g\frac{(1-\sqrt{p})}{2}\omega_0^{-2}. \tag{18}$$

Accordingly, *the discrete energy spectrum* for the direct cascade reads

$$E_n = E(\omega_n) \sim A_n^2 = g^2\left[\frac{(1-\sqrt{p})}{2}\omega_n^{-2} + C\right]^2, \tag{19}$$

where ω_0 and A_0 are the excitation parameters and $p = p(\omega_0, A_0)$.

The corresponding *continuous energy spectrum* $E(\omega)$ is computed as $\lim_{n\to\infty} |E_{n+1} - E_n|/|\omega_{n+1} - \omega_n|$, yielding

$$E(\omega) \sim 2g^2\,[(1-\sqrt{p})\omega^{-5} - C\omega^{-3}]. \tag{20}$$

In particular, the special choice of excitation parameters $C = 0$ yields

$$E(\omega) \sim g^2\omega^{-5}, \tag{21}$$

which is the saturated Phillips' spectrum, [27]; this is also in accordance with the JONSWAP spectrum (an empirical relationship based on experimental oceanic data). Kinetic WTT predicts $\sim\omega^{-4}$ in this case [1,2].

B. Cascade direction

Combining the chain equation and expression for the amplitudes of the cascading modes we can study how the cascade direction depends on the choice of excitation parameters. For instance, for direct cascade $\omega_{n+1} - \omega_n > 0$ with $C \neq 0$, the use of (14), (17), and (18) yields

$$0 < \omega_{n+1} - \omega_n = \omega_n^3 A(\omega_n)/g \tag{22}$$

$$= \omega_n^3\left[g\frac{(1-\sqrt{p})}{2}\omega_n^{-2} + C\right]/g \tag{23}$$

$$= \frac{(1-\sqrt{p})}{2}\omega_n + \left[A_0 - g\frac{(1-\sqrt{p})}{2}\omega_0^{-2}\right]\omega_n^3/g \tag{24}$$

$$= \frac{(1-\sqrt{p})}{2} + \left[A_0 - g\frac{(1-\sqrt{p})}{2}\omega_0^{-2}\right]\omega_n^2/g \tag{25}$$

$$\Rightarrow g(1-\sqrt{p}) + \left[2A_0 - g\,(1-\sqrt{p})\omega_0^{-2}\right]\omega_n^2 > 0. \tag{26}$$

As $(1-\sqrt{p}) > 0$, the range of frequencies forming a direct cascade depends only on the sign of the expression $2A_0 - g\,(1-\sqrt{p})\omega_0^{-2}$.

An easy examination of (23) and (26) shows how to choose excitation parameters A_0 and ω_0 in order to observe the direct cascade.

$$\text{If} \quad 2A_0 \geqslant g\,(1-\sqrt{p})\omega_0^{-2}, \tag{27}$$

the only restriction on the range of frequencies forming the direct cascade is trivial, $\omega_n > \omega_0$, and accordingly, only direct cascade will occur;

$$\text{if} \quad 2A_0 < g(1-\sqrt{p})\omega_0^{-2}, \tag{28}$$

direct cascade will be observed for the range of frequencies $\omega_0 < \omega_n \leqslant \omega_{n_{st}}$, where

$$\omega_{n_{st}} = \sqrt{\frac{g(1-\sqrt{p})}{g(1-\sqrt{p})\omega_0^{-2} - 2A_0}}. \tag{29}$$

To simplify further formulas we introduce here a small parameter, $\varepsilon_0 = A_0 k_0 = A_0\omega_0^2/g$, and rewrite (29) as

$$\omega_{n_{st}} = \omega_0\sqrt{\frac{(1-\sqrt{p})}{(1-\sqrt{p}) - 2\varepsilon_0}}. \tag{30}$$

The physical meaning of the frequency $\omega_{n_{st}}$ is explained in Sec. III C.

Similar computations can be performed for an inverse cascade, and also the case where both direct and inverse cascades are possible can be studied in this way. In particular, for some choice of excitation parameters both direct and inverse cascade can be initiated simultaneously. This scenario is supported by a wide range of experimental studies (e.g., [13–15]).

All formulas, (17), (18), and (30), are given in terms of excitation parameters A_0 and ω_0 and cascade intensity p. This means that we should also compute p as a function of A_0, ω_0, and $p = p(A_o, \omega_0)$. This tedious computation will be given elsewhere. However, in the next section we give an example of the computation for a particular form of the solution, (17). Note that for studying predictions of the D model in experimental data one can just measure $\sqrt{p}$ as the ratio of amplitudes of two consecutive cascading modes, $\sqrt{p} = A_{n+1}/A_n$, and apply formulas afterwards.

C. Cascade termination

1. *Breaking*

It was first shown in [29] that the amplitude of the carrier wave may become so high that its steepness locally exceeds the maximum steepness of gravity waves, yielding the onset of wave breaking. In order to demonstrate that this effect can be reproduced in the D model, let us regard a particular solution of (17) with $C = 0$:

$$A_n = g\,\frac{(1-\sqrt{p})}{2}\omega_n^{-2}. \tag{31}$$

As for this solution

$$A_0 = g\frac{(1-\sqrt{p})}{2}\omega_0^{-2} \tag{32}$$

$$\Rightarrow \begin{cases} p = (1-2\varepsilon_0)^2, \\ A_n = p^{n/2}A_0 = (1-2\varepsilon_0)^n A_0, \end{cases} \tag{33}$$

any choice of ε_0 and A_0 uniquely defines a cascade intensity p and the amplitude of the nth cascading mode.

It follows from (31) and (32) that in this case all cascading modes have the same steepness $\varepsilon_n = \varepsilon_0, \forall n$:

$$\varepsilon_n = A_n k_n = A_n\omega_n^2/g = \frac{(1-\sqrt{p})}{2} = \varepsilon_0. \tag{34}$$

This allows us to compute the steepness ε of the total wave packet at step n (before breaking) as

$$\varepsilon \approx \sum_n \varepsilon_n \approx (n+1)\varepsilon_0. \tag{35}$$

Accordingly, though the amplitudes of the cascading modes are decreasing, the steepness of the total packet is growing with an increasing number of cascade steps.

For instance, direct computations demonstrate that if the initial steepness $\varepsilon_0 = 0.1$, then after three cascade steps $A_0 \times 100\%/A_3 \approx 0.5\%$. However, the total steepness of the wave packet is $\varepsilon = 4 \times 0.1 = 0.4$, and according to the Stokes criterion for the limiting steepness being about 0.44, we conclude that mode A_3 is about to break. A different choice of the initial steepness, say $\varepsilon_0 = 0.05$, yields the same total steepness, $\varepsilon = 8 \times 0.05 = 0.4$, at step $n = 7$ and cascading mode A_7 contains about 23% of the excitation energy while $A_0 \times 100\%/A_7 \approx 48\%$. Thus, by varying the excitation parameters one can predict the occurrence of breaking at the different cascade steps.

Denoting the limiting steepness of the wave package before breaking as $\varepsilon_{\rm br}$, we conclude that the cascade terminates due to breaking if $(n_{\rm br}+1)\varepsilon_0 = \varepsilon_{\rm br} \approx 0.44$, i.e., at the finite step $n_{\rm br}$,

$$n_{\rm br} \approx 0.44/\varepsilon_0 - 1. \tag{36}$$

At the end of this section we point out again that all results given by (32)–(36) are obtained for a specific form of solution of (17), namely, for $C = 0$. In the general case $C \neq 0$ some results might be qualitatively different: for instance, breaking may occur at infinity rather than at some finite step. In this section we did not aim to present all possible formulas in their most general form but rather to demonstrate that growth of nonlinearity followed by breaking—an experimentally well-established phenomenon [10,13–15]—can be reproduced by the D model.

2. Stabilization

If at some cascade step $n_{\rm st}$ the mode with frequency $\omega_{n_{\rm st}}$ is stable, then condition (11) is not fulfilled, no additional mode can be generated, and the D cascade stops due to stabilization at some frequency $\omega_{n_{\rm st}}$.

From (11), (17), and (18) it may be concluded that

$$\omega_{n_{\rm st}} = \omega_{n_{\rm st}+1} \Rightarrow 0 = \omega_{n_{\rm st}} - \omega_{n_{\rm st}+1} \tag{37}$$

$$= A_{n_{\rm st}}\omega_{n_{\rm st}}k_{n_{\rm st}} = \left[g\frac{(1-\sqrt{p})}{2}\omega_{n_{\rm st}}^{-2} + C\right]\omega_{n_{\rm st}}^3 \Big/ g \tag{38}$$

$$\Rightarrow 0 = \frac{(1-\sqrt{p})}{2}\omega_{n_{\rm st}} + C\omega_{n_{\rm st}}^3/g \tag{39}$$

$$\Rightarrow \omega_{n_{\rm st}}^2 = \frac{(1-\sqrt{p})}{2}\Big/C = \frac{g(1-\sqrt{p})}{g(1-\sqrt{p})\omega_0^{-2} - 2A_0}, \tag{40}$$

and for direct cascade, stabilization occurs if

$$\omega_n > \omega_{n_{\rm st}} = \omega_0\sqrt{\frac{(1-\sqrt{p})}{(1-\sqrt{p})-2\varepsilon_0}}, \tag{41}$$

which is in accordance with (30).

It follows from (41) that

(i) direct cascade stabilizes at the finite step $\omega \leqslant \omega_{n_{\rm st}} < \infty$ if $1-\sqrt{p} > 2\varepsilon_0$;

(ii) direct cascade stabilizes at infinity if $1-\sqrt{p} = 2\varepsilon_0$, and then $C = 0$ in Eq. (18) and the corresponding continuous energy spectrum is Phillips spectrum $\sim\omega^{-5}$ (see Sec. III C1); and

(iii) stabilization does not occur if $1-\sqrt{p} < 2\varepsilon_0$ while the expression on the right-hand side of (41) becomes complex and has no physical meaning, i.e., stabilization conditions can never be fulfilled.

Similar computations can be performed for inverse cascade. Though formally the termination conditions may allow the inverse cascade to be terminated at a negative frequency, this is physically irrelevant. This means that in a real physical system an inverse cascade terminates in somewhat the vicinity of the zero frequency mode, which might yield a substantial concentration of energy near the zero frequency mode, also observed experimentally, e.g., in Ref. [12].

3. Fermi-Pasta-Ulam (FPU)-like recurrence

The fact that the long-time evolution of nonlinear wave trains of surface water waves may evolve in a recurrent fashion (FPU-like recurrence), where the wave form returns periodically to its previous form, has been discovered experimentally and described in the pioneering paper of Lake *et al.* [16]. The next milestone step in the study of this effect was performed by Tulin and Waseda in Ref. [13], where the authors refined the experimental technique so that not only the excitation frequency but also the initial side bands and the amplitude strength can be chosen. More experimental results can be found in Refs. [14,15] and references therein.

In the D model, the formation of a recurrent phenomenon (intermittency) is due to the formation of a cluster of resonant quartets—in the simplest case, an isolated Phillips quartet, (8). Its occurrence depends strongly on the form of the experimental tank.

For some aspect ratio of the tank side lengths, intermittency cannot occur, as kinematic resonance conditions cannot be satisfied. If, for a given aspect ratio, solutions of (5) exist, the interaction coefficient $V \neq 0$, and initially excited resonant mode(s) are modulationally stable, then a recurrence may be observed.

Below we give a short list of experimental observations with their respective explanations.

(i) *No cascade is observed;rather, recurrent patterns on the water surface are observed* [7]. The initial steepness is too small to initiate MI.

(ii) *No intermittency is observed; rather, a discrete cascade terminated by wave breaking* [10]. The initial steepness is big enough to cause MI and $\omega_{\rm br} < \omega_{\rm st}$ or stabilization is generally not possible for the chosen excitation parameters.

(iii) *No intermittency is observed in the nonbreaking regime* [30]. The initial steepness is big enough to cause MI, cascade terminates due to stabilization, i.e., $\omega_{\rm st} < \omega_{\rm br}$, and the mode with frequency $\omega_{\rm st}$ is not a resonant mode in a resonant cluster possible for the chosen experimental tank.

(iv) *Intermittency is observed in the nonbreaking regime* [14,15]. The cascade stabilizes at the frequency $\omega_{\rm st}$, and the

ω_{st} mode is a resonant mode and may excite a resonant cluster with another cascading mode. In particular, if the ω_{st} mode and ω_0 mode form a resonance, complete FPU-like recurrence will be observed [13–15]. If the ω_{st} mode forms a resonance with a cascading mode with frequency $\widetilde{\omega} \neq \omega_0$, then partial recurrence will occur, with the spectral peak being downshifted to the frequency $\widetilde{\omega}$ [30].

(v) *Intermittency is observed at the postbreaking stage* [13–15]. As the essential part of the energy is lost due to breaking, the amplitudes of newly excited modes may become modulationally stable and form a resonance with some of the previously excited cascading modes. This is only a qualitative explanation; quantified prediction is an important separate topic which lies outside the scope of this paper. A possible theoretical scenario of the energy redistribution at the postbreaking stage is developed in [13].

In this section we have shown how to use the chain equation to determine, depending on the excitation parameters, the direction of the energy cascade and how the cascade will terminate. It should be noted that also the asymmetry of direct and inverse cascades as known from experiments (e.g. [13–16]) may be deduced from the chain equation [31].

IV. D MODEL VERSUS KINETIC WTT

For almost 50 years, kinetic WTT, which requires a distributed initial state, was used to describe experiments using narrow frequency band excitation. This was considered legal, as the assumption was, and still is, that from the excitation frequency as a starting point, a distributed state will quickly be established. The discrete part of the spectrum which was well observed in experiments was ignored in theoretical discussion, focusing on the continuous part of the spectrum.

That this approach is not without problems was acknowledged within the community. As Newell noted recently, seems to agree with the theory but experiments [do] not" [32] (see also the recent review in [33]). Indeed, a distributed initial state as needed for applicability of kinetic WTT is easy to create in numerical simulations but not in laboratory experiments.

Though the D model and kinetic WTT differ greatly in their assumptions and consequent range of applicability, sometimes the predicted form of the continuous energy spectrum is very close. To gain more understanding of which approach to apply in a given experimental setup, here we provide a comparison of the assumptions and predictions of the D model and kinetic WTT (a short list is given in Table I).

The crucial difference in descriptions of energy cascades between the D model and kinetic WTT is the physical mechanism generating a cascade: MI in an arbitrary s-wave system versus s-wave interactions, $s = 3, 4, \ldots$. This means, in particular, that a D cascade is generated by a mechanism which provides locality of interactions automatically. In kinetic WTT the locality has to be assumed, and no mechanism is suggested, which allows us to choose local interactions in wave systems where nonlocal interactions are also possible, as shown in Sec. II A and Eqs. (9) and also observed experimentally [18]. The assumption of locality—only interactions among waves with close wavelengths are allowed—is basic in kinetic WTT; without locality energy exchange among different scales k is possible and the energy spectrum cannot be regarded as a function of only k.

Another important point is that the influence of the excitation parameters on the form of the continuous energy spectrum observed experimentally (e.g., in [12] and [34–36]) generally cannot be included in kinetic WTT but is reproduced in the D model. One more considerable difference between the D model and kinetic WTT is the origin of cascade termination. In kinetic WTT this is always dissipation, while in the D model various scenarios can be reproduced, depending on the excitation parameters and direction of the cascade. D cascades can terminate, e.g., due to breaking, stabilization, or formation of the FPU-like recurrent phenomenon; all these effects are observed experimentally [13–15].

Assumption a of the D model (see Sec. II B), i.e., a constant cascade intensity, $p = \text{const}$, is absent in kinetic WTT. This assumption is not substantial for the D model and can easily be removed. Indeed, if the cascade intensity at step n is $p_n \neq \text{const}$, chain equations (14) and (15) do not change, while the ordinary differential equation, (17), and its solutions can be trivially rewritten by changing p to p_n. The only nontrivial change would be the construction of the transition from discrete to continuous energy spectra. Of course, the estimates for determining cascade direction, termination, etc., should be recalculated and might get a more complicated form, though not necessarily. For instance, all estimates made for the particular solution of (17) with $C = 0$ remain valid, while for so chosen excitation parameters A_0 and ω_0 the cascade intensity is a constant defined by A_0 and ω_0:

$$C = 0 \;\Rightarrow\; A_0 - g\frac{(1-\sqrt{p_n})}{2}\omega_0^{-2} = 0 \qquad (42)$$

$$\Rightarrow\; p_n = \sqrt{1 - 2A_0\omega_0^2/g} \equiv \text{const}\,. \qquad (43)$$

Accordingly, the transition from a discrete to a continuous spectrum can be performed as above, producing a saturated Phillips spectrum.

A wide range of experimental data shows that $p = \text{const}$ in various wave systems and accordingly the discrete energy spectrum has an exponential form (e.g., [37] and references therein); this was our motivation for choosing a constant cascade intensity in this presentation.

Last but not least. It was shown in a recent experimental study of capillary waves that "from the measured wavenumber-frequency spectrum it appears that the [linear] dispersion relation is only satisfied approximately.... This disagrees with weak WTT where exact satisfaction of the dispersion relation is pivotal. We find approximate algebraic frequency and wavenumber spectra but with exponents that are different from those predicted by weak wave turbulence theory" [38].

On the other hand, D cascades are formed by modes with nonlinear frequencies, and not by modes with linear frequencies, as assumed in kinetic WTT. This is a manifestation of the very important difference between the D-model and kinetic WTT. Cascades in kinetic WTT are due to resonant interactions and therefore are possible at the time scale T or $\tilde{T}$ with a very small nonlinearity $0 < \varepsilon < 0.1$. In the D model, only intermittency is formed at these time scales, while a D cascade occurs at the faster time scale τ and for the larger nonlinearity $\varepsilon \sim 0.1$–0.4.

TABLE I. Assumptions and predictions used in the D model and kinetic WTT.

	Property	D model	Kinetic WTT
Assumption			
1	Cascade origin in an S-wave system	Modulation instability, no dependence on S	S-wave kinetic equation, depends on S
2	Initial state	Narrow frequency band	Distributed state
3	Locality of interactions	No assumptions	Necessary
4	Existence of inertial interval	No assumptions	Necessary
5	Origin of cascade termination	No assumptions	Dissipation
6	Range of wave steepness	$0 < \varepsilon \sim 0.1$–0.4	$0 < \varepsilon < 0.1$
7	Cascade intensity	Is constant	No assumptions
8	Energy flux	No assumptions	Is constant
Prediction			
1	Cascade is formed by	Nonlinear frequencies	Linear frequencies
2	Spectrum form:		
	(a)	Discrete and continuous	Continuous
	(b)	Depends on the excitation	Does not depend on the excitation
3	Transition from discrete to continuous spectrum	Included	Not included
4	Direction of cascade	Included	Included
5	Intermittency	Included	Not included
6	Origin of cascade termination	Various scenarios: stabilization, breaking, FPU-like recurrence	(See assumptions)

V. CONCLUSIONS AND OPEN QUESTIONS

In this paper we have presented a D model which describes nonlinear wave systems with narrow frequency band excitation. It allows us to reproduce in a single theoretical frame various nonlinear wave phenomena, in particular, finite-size effects in resonators and formation of energy cascades. The cascades do not depend on the shape or finiteness of the interaction domain, as they are triggered by the local mechanism of MI.

The main predictions of the D model can be stated as follows.

(i) Intermittency is formed by a set of distinct modes with *linear frequencies*; intermittency may occur in systems with very small nonlinearity, $0 < \varepsilon < 0.1$, at the slow time scale T or $\tilde{T}$; and the underlying physical mechanism is resonant wave interaction.

(ii) An energy cascade is formed by a chain of distinct modes with *nonlinear frequencies*; a cascade may occur in systems with small to moderate nonlinearity, $\varepsilon \sim 0.1$–0.4, at the Benjamin-Feir time scale τ; and the underlying physical mechanism is MI.

(iii) The discrete and continuous energy spectra of a cascade can be computed by the ICEM [6]; the form of spectra, cascade direction, and scenario of cascade termination depend on the excitation parameters.

(iv) Various scenarios of energy cascade termination, known from laboratory experiments—stabilization, breaking, and the appearance of FPU-like recurrence—can be reproduced in the D model.

As discussed in Sec. IV, all these predictions are quite different from those of kinetic WTT developed for wave systems with a distributed initial state. In the latter case an energy cascade occurs at the slow time scale of resonant interactions, is formed by linear frequencies, terminates (by assumption) always due to dissipation, etc.

The D model explains known physical phenomena, as well as the results of individual laboratory experiments. In addition, the D model makes predictions which may be easily verified in experiments, e.g., increasing the amplitude of excitation increases the distance between cascading modes in k space (a direct consequence of the chain equation).

It should be mentioned that within the wide range of excitation-dependent spectra predicted by the D model, the saturated Phillips spectrum ω^{-5} has two special properties. First, as shown in Sec. III C2, of all the possible spectra, only the Phillips spectrum does not stabilize after a finite number of cascade steps but at infinity (in k space). Second, for the Phillips spectrum it is easy to prove that the cascade intensity is constant [see Eq. (43)]; for other spectra it is not known. What this means physically is presently under study.

The D model may be refined in many ways; two examples are given here.

(i) In Eq. (17) just two terms of the Taylor expansion are taken to compute the energy spectrum; instead, one might regard the hierarchy of finite-order ordinary differential equations obtained by cutting off the Taylor expansion at three terms, four terms, and so on.

(ii) Dissipation (depending on the frequency) can be taken into account in the following way: the cascade intensity p, which describes the fraction of energy going from mode n to mode $n+1$, may be considered as a function increasing with frequency, $p = p(\omega) \neq \mathbf{const}$. So stabilization of the cascade will occur earlier, and also the form of the energy spectrum will change.

Many more problems can be studied in the framework of the D model than have been mentioned in this paper. For instance, (i) Is it possible to use the D model to describe real-life phenomena where excitation parameters are not known *a priori*? Most naturally one might study the probability of various initial states in a given situation and choose as input for the model either the most probable state or an average state—for instance, the known prevailing direction of the wind blowing over the ocean during a season.

(ii) MI plays a central role in the formation of extreme waves (e.g., [39–41]). Is it possible to use the D model to predict freak waves in the ocean? The Benjamin-Feir index, which is the ratio of the parameter of nonlinearity ε to the relative spectral width, characterizes the evolution of a unidirectional wave field with a narrow spectrum. As either the frequency range or the directional spreading widens, the probability of the appearance of extremely steep waves decreases [40,41]. Using a chain equation, one may, e.g., compute an upper estimate for the Benjamin-Feir index at each cascade step as a function of the excitation parameters and study the characteristic behavior of this function.

(iii) In the special case of surface water waves the Zakharov equation is the model of choice. So it would be of great interest to compare the predictions of the D model with the predictions of the Zakharov equation. Some results are already known, for instance, side-band asymmetry of Benjamin-Feir instability has been established in numerical simulations with the Zakharov equation [42]. Moreover, it was recently shown by Onorato [43] that a D cascade has a direct correspondence in the Zakharov equation: the frequencies of cascading modes as determined in the D model form exact four-wave resonances in the Zakharov equation with nonlinear Stokes corrected frequencies.

The latter result is of the upmost importance, as it opens a broad avenue for further studies of nonlinear wave systems with a higher degree of nonlinearity. The question is, Is it possible to compute energy cascades in nonlinear wave systems with a distributed initial state using a new type of wave kinetic equation based on resonances of nonlinear Stokes corrected frequencies with greater nonlinearity than is possible for the applicability of kinetic WTT?

ACKNOWLEDGMENTS

Author acknowledges K. Dysthe, A. Maurel, A. Newell, M. Onorato, E. Pelinovsky, I. Procaccia, M. Shats, I. Shugan and H. Tobisch for valuable discussions and anonymous Referees for useful remarks and recommendations. This research has been supported by the Austrian Science Foundation (FWF) under project P22943-N18, and in part–by the Project of Knowledge Innovation Program (PKIP) of Chinese Academy of Sciences, Grant No. KJCX2.YW.W10.

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