Effective-average-action-based approach to correlation functions at finite momenta

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We present a truncation scheme of the effective-average-action approach of the nonperturbative renormalization group that allows for an accurate description of the critical regime as well as of correlation functions at finite momenta. The truncation is a natural modification of the standard derivative expansion that includes both all local correlations and two-point and four-point irreducible correlations to all orders in the derivatives. We discuss schemes for both the symmetric and the symmetry broken phase of the O(N) model and present results for $D = 3$. All approximations are done directly in the effective average action rather than in the flow equations of irreducible vertices. The approach is numerically relatively easy to implement and yields good results for all N both for the critical exponents and for the momentum dependence of the two-point function.

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I. INTRODUCTION

The nonperturbative renormalization group technique is based on an exact flow equation of the effective average action (or generating functional of irreducible vertices) [1,2] and has been applied to a large variety of systems (see, e.g., Refs. [3–6] for reviews). It proved especially useful when applied to critical phenomena where often even relatively simple truncation schemes yield an accurate description of the critical region [3,4]. While the exact flow equation of the effective average action can almost never be solved, it allows for novel nonperturbative approximation techniques. One successful approximation strategy is the derivative expansion, where the effective average action is expanded consistently to a given order in spatial derivatives, but no truncation is made in the power of the fields. The derivative expansion has been applied with success to O(N) models [3,7,8].

The derivative expansion allows strict control over the symmetries of the studied models since all approximations are done in the effective action that is expanded in invariants of the model. This approach automatically yields flow equations that both are closed and further obey the symmetry of the original model. In contrast, if a direct field expansion of the effective action is employed and approximations are done at the level of the flow equations of irreducible vertices, the invariance of the action is not guaranteed. In general, approximations at the level of flow equations of vertices are therefore more difficult to control. However, the derivative expansion can only access the asymptotic small momentum regime of the theory and it cannot be applied to calculate correlation functions at finite momenta.

Recently, several approximation schemes that allow one to calculate correlation functions at finite momenta were developed [9–12]. The most sophisticated of these is the scheme presented in Refs. [10,11] [the Blaizot-Méndez-Wschebor (BMW) scheme], which is based on an approximate solution of the exact flow equation of the two-point vertex in the presence of a background field. All of these approaches, however, rely on approximations at the level of flow equations for irreducible vertices. Instead, here we want to develop a scheme in which all approximations are done directly at the level of the effective average action. This allows for a transparent calculation of the momentum dependence where the full symmetry of the model is always obeyed by the flow equations. The scheme we discuss below is a natural modification of the usual derivative expansion and is based on a local potential that is supplemented by a momentum-dependent potential that accounts for nonlocal correlations up to the four-point vertex. Similar schemes were previously used to calculate the one-particle spectral function of Bose condensates [13,14] (see also Refs. [15,16]) and also the thermal fluctuations of crystalline membranes such as graphene [17] (see also Ref. [18]). Here we extend the scheme to include, besides the nonlocal terms, the full local potential and test it on the O(N) model (see, e.g., Ref. [19] for a recent summary of results on the O(N) model). We develop two schemes, a nonlocal potential approximation (NLPA) for the ordered state and a NLPA for the symmetric state. Both schemes allow for an investigation of the critical region. We first introduce in Sec. II the NLPA approach for both the ordered state and the symmetric state. We present results from a numerical solution of the flow equations in Sec. III, with results for the critical exponents presented in Sec. III A. In Sec. III B we assess the quality of the approach in the finite momentum regime. We conclude in Sec. IV.

II. NONLOCAL POTENTIAL APPROXIMATION

We begin with an approximation scheme for the ordered state, where we explicitly incorporate a finite order parameter into the invariant effective action.

A. The NLPA for the ordered state

The starting point of the NLPA approach is an effective average action that consists of both a nonlocal potential term, which is restricted to second order in the invariant densities and is characterized by the coupling function $u_\Lambda(k)$, and a local potential term $U_\Lambda(\rho - \rho_\Lambda^0)$, which may be an arbitrary function of $\rho - \rho_\Lambda^0$, where ρ_Λ^0 is the (cutoff-dependent) order parameter density and $\rho = \boldsymbol{\varphi}^2/2$, where $\boldsymbol{\varphi}$ is a field with N components. To avoid double counting of correlations, we define $u_\Lambda(k)$ to be completely nonlocal with $u_\Lambda(0) = 0$. We furthermore keep the full momentum dependence $\sigma_\Lambda(k)$ of the quadratic term in the action, and thus approximate the effective average action, after subtraction of the noninteracting

contribution $(1/2)\int_k G_{0,\Lambda}^{-1}(k)\boldsymbol{\varphi}_{\boldsymbol{k}}\cdot\boldsymbol{\varphi}_{-\boldsymbol{k}}$, as

$$\Gamma_\Lambda[\varphi] = \frac{1}{2}\int_k [\sigma_\Lambda(k)\boldsymbol{\varphi}_{\boldsymbol{k}}\cdot\boldsymbol{\varphi}_{-\boldsymbol{k}} + u_\Lambda(k)\Delta\rho_{\boldsymbol{k}}\Delta\rho_{-\boldsymbol{k}}] + \int_x U_\Lambda\big(\rho-\rho_\Lambda^0\big), \tag{1}$$

where $\Delta\rho_{\boldsymbol{k}}$ is the Fourier transform of $\rho(\boldsymbol{x})-\rho_\Lambda^0$. We use the notation $\int_k = \int d^Dk/(2\pi)^D$ and $\int_x = \int d^Dx$ for integrals over momenta and integrals over coordinate space, respectively. Here $U_\Lambda(\tau)$ can, for finite cutoff Λ, be expanded in τ,

$$U_\Lambda(\tau) = \sum_n \frac{1}{n!}U_\Lambda^{(n)}\tau^n, \tag{2}$$

with $\tau=\rho-\rho_\Lambda^0$ or, for the symmetric scheme discussed in Sec. II B, $\tau=\rho$. Note that the effective action (1) does not contain all terms of a complete derivative approximation to order q^2, in which the derivative term of the action would also include an expansion to all powers of $\rho-\rho_\Lambda^0$. In the present scheme, one could easily improve upon the action (1) by including, for example, also additional terms that can be parametrized by only one momentum. One obvious extension would be to include a term $\int_{x,y}[\rho(\boldsymbol{x})-\rho_\Lambda^0]^2[\rho(\boldsymbol{y})-\rho_\Lambda^0]\kappa(\boldsymbol{x}-\boldsymbol{y})$ with some function $\kappa(\boldsymbol{x})$, which would be a generalization of the $(\partial_\mu\rho)^2(\rho-\rho_\Lambda^0)$ term encountered in a derivative expansion. Such an extension is both straightforward and numerically feasible.

At the same time, the present approach goes well beyond the derivative expansion in that it includes the full momentum dependence in the first two terms of Eq. (1). As in the derivative expansion, the effective average action obeys the full O(N) invariance throughout the entire flow. To determine the flow of U_Λ, we can use the standard technique [1] and evaluate the flow of $\Gamma_\Lambda[\bar{\boldsymbol{\varphi}}]$ for a homogeneous (x-independent) field $\bar{\boldsymbol{\varphi}}$ such that $V^{-1}\Gamma_\Lambda[\bar{\boldsymbol{\varphi}}] = U_\Lambda(\bar\rho-\rho_\Lambda^0)$ with $\bar\rho=\bar{\boldsymbol{\varphi}}^2/2$ and where V is the volume. We shall now assume that $N\geqslant 2$, so there is at least one gapless transverse mode. The flow of the local potential is then given by [1]

$$\partial_\Lambda U_\Lambda\big(\bar\rho-\rho_\Lambda^0\big) = \frac{1}{2}\int_k \partial_\Lambda R_\Lambda(k)\{\bar G_{\Lambda,\parallel}(k,\bar\rho) + (N-1)\bar G_{\Lambda,\perp}(k,\bar\rho)\}, \tag{3}$$

where

$$\bar G_{\Lambda,\perp}^{-1}(k,\bar\rho) = \sigma_\Lambda(k) + U_\Lambda'\big(\bar\rho-\rho_\Lambda^0\big) + G_{0,\Lambda}^{-1}(k), \tag{4a}$$

$$\bar G_{\Lambda,\parallel}^{-1}(k,\bar\rho) = \sigma_\Lambda(k) + 2\bar\rho\big[u_\Lambda(k) + U_\Lambda''\big(\bar\rho-\rho_\Lambda^0\big)\big] + U_\Lambda'\big(\bar\rho-\rho_\Lambda^0\big) + G_{0,\Lambda}^{-1}(k). \tag{4b}$$

Here the cutoff regulated noninteracting Green's function is

$$G_{0,\Lambda}^{-1}(k) = k^2 + R_\Lambda(k) \tag{5}$$

and $R_\Lambda(k)$ is a regulator for small momenta with $k\lesssim\Lambda$. The only difference in Eq. (3) from the standard form used in a derivative expansion of Γ_Λ is the presence of the full functions $u_\Lambda(k)$ and $\sigma_\Lambda(k)$ in Eqs. (4a) and (4b) rather than just their leading terms of a k expansion. To determine the flows of $\sigma_\Lambda(k)$ and $u_\Lambda(k)$ we invoke a field expansion of $\Gamma_\Lambda[\varphi]$ in terms of $\Delta\varphi_k^a = \varphi_k^a - \varphi_\Lambda^0\delta_{a1}\delta_{k,0}$ with $\rho_\Lambda^0 = (\varphi_\Lambda^0)^2/2$. Here we have assumed, without loss of generality, an order parameter field φ_Λ^0 that is directed in the $a=1$ direction of the internal space. To determine the flows of σ_Λ and u_Λ, we need the lowest-order irreducible vertices (up to the four-point vertex), which have the form

$$\Gamma_{\Lambda,ab}^{(2)}(\boldsymbol{k},-\boldsymbol{k}) = \delta_{ab}\sigma_\Lambda(k) + 2\delta_{a1}\delta_{b1}\rho_\Lambda^0\tilde u_\Lambda(k), \tag{6a}$$

$$\Gamma_{\Lambda,abc}^{(3)}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = \varphi_\Lambda^0[\delta_{a1}\delta_{bc}\tilde u_\Lambda(k_1) + \delta_{b1}\delta_{ac}\tilde u_\Lambda(k_2) + \delta_{c1}\delta_{ab}\tilde u_\Lambda(k_3)] + \big(\varphi_\Lambda^0\big)^3U_\Lambda^{(3)}\delta_{a1}\delta_{b1}\delta_{c1}, \tag{6b}$$

$$\Gamma_{\Lambda,abcd}^{(4)}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) = \delta_{ab}\delta_{cd}\tilde u_\Lambda(k_{12}) + \delta_{ac}\delta_{bd}\tilde u_\Lambda(k_{13}) + \delta_{ad}\delta_{bc}\tilde u_\Lambda(k_{14}) + 2\rho_\Lambda^0U_\Lambda^{(3)} \times[\delta_{ab}\delta_{cd}(\delta_{a1}+\delta_{c1}) + \delta_{ac}\delta_{bd}(\delta_{a1}+\delta_{b1}) + \delta_{ad}\delta_{bc}(\delta_{a1}+\delta_{b1})] + 4\big(\rho_\Lambda^0\big)^2U_\Lambda^{(4)}\delta_{a1}\delta_{b1}\delta_{c1}\delta_{d1}, \tag{6c}$$

where we defined $\tilde u_\Lambda(k) = u_\Lambda(k) + U_\Lambda^{(2)}$ and $k_{ij} = |\boldsymbol{k}_i+\boldsymbol{k}_j|$. The flow of the order parameter follows from the requirement that $\partial_\Lambda\Gamma_\Lambda^{(1)}=0$. This yields [20]

$$\partial_\Lambda\rho_\Lambda^0 = \frac{-1}{2\tilde u_\Lambda(0)}\int_q \big\{\big[\tilde u_\Lambda(0) + 2\tilde u_\Lambda(q) + 2\rho_\Lambda^0U_\Lambda^{(3)}\big]\dot G_{\Lambda,\parallel}(q) + (N-1)\tilde u_\Lambda(0)\dot G_{\Lambda,\perp}(q)\big\}, \tag{7}$$

where $G_{\Lambda,\alpha}(k) = \bar G_{\Lambda,\alpha}(k,\rho_\Lambda^0)$ for $\alpha=\perp,\parallel$ and $\dot G_{\Lambda,\alpha}(k) = -G_{\Lambda,\alpha}^2(k)\partial_\Lambda R_\Lambda(k)$. The flow of $\sigma_\Lambda(k)$ follows from the flow of $\Gamma_{\Lambda,\perp}^{(2)}(k) = \Gamma_{\Lambda,aa}^{(2)}(k,-k)$, where $a\neq 1$ is a direction transverse to the order parameter field,

$$\partial_\Lambda\sigma_\Lambda(k) = \int_q\{\dot G_\perp(q)\tilde u_\Lambda(q') - \dot G_\parallel(q)\tilde u_\Lambda(q)\} - 2\rho_\Lambda^0\int_q\big\{\dot G_\parallel(q')\times G_\perp(q)\tilde u_\Lambda^2(q') + \dot G_\perp(q')G_\parallel(q)\tilde u_\Lambda^2(q)\big\}, \tag{8}$$

and we defined $q'=|\boldsymbol{k}+\boldsymbol{q}|$. The flow equation of $\tilde u_\Lambda(k)$ can be obtained from the flow of $\Gamma_{\Lambda,\parallel}^{(2)}(k) = \Gamma_{\Lambda,11}^{(2)}(k,-k)$, which reads

$$\begin{aligned}\partial_\Lambda\Gamma_{\Lambda,\parallel}^{(2)}(k) &= \frac{1}{2}\int_q\big\{(N-1)\dot G_{\Lambda,\perp}(q)\big[\tilde u_\Lambda(0) + 2\rho_\Lambda^0U_\Lambda^{(3)}\big] + \dot G_{\Lambda,\parallel}(q)\\ &\quad\times\big[\tilde u_\Lambda(0) + 2\tilde u_\Lambda(q') + 12\rho_\Lambda^0U_\Lambda^{(3)} + 4\big(\rho_\Lambda^0\big)^2U_\Lambda^{(4)}\big]\big\}\\ &\quad - 2\rho_\Lambda^0\int_q\big\{(N-1)\dot G_{\Lambda,\perp}(q')G_{\Lambda,\perp}(q)\tilde u_\Lambda^2(k)\\ &\quad + \dot G_{\Lambda,\parallel}(q')G_{\Lambda,\parallel}(q)\big[\tilde u_\Lambda(q) + \tilde u_\Lambda(q') + \tilde u_\Lambda(k)\\ &\quad + 2\rho_\Lambda^0U_\Lambda^{(3)}\big]^2\big\} + \big[\tilde u_\Lambda(0) + 2\tilde u_\Lambda(k) + 2\rho_\Lambda^0U_\Lambda^{(3)}\big]\partial_\Lambda\rho_\Lambda^0.\end{aligned} \tag{9}$$

Combining Eqs. (7)–(9) and keeping in mind that $\Gamma^{(2)}_{\Lambda,\parallel}(k) = \sigma_\Lambda(k) + 2\rho^0_\Lambda \tilde{u}_\Lambda(k)$, one finds

$$\begin{aligned}\partial_\Lambda \tilde{u}_\Lambda(k) &= \frac{1}{2\rho^0_\Lambda}\int_q [\dot{G}_{\Lambda,\parallel}(q) - \dot{G}_{\Lambda,\perp}(q)]\tilde{u}_\Lambda(q') + \int_q \dot{G}_{\Lambda,\parallel}(q) \\ &\times \left\{2U^{(3)}_\Lambda + \rho^0_\Lambda U^{(4)}_\Lambda - U^{(3)}_\Lambda\left[\tilde{u}_\Lambda(q) + \rho^0_\Lambda U^{(3)}_\Lambda\right]/\tilde{u}_\Lambda(0)\right\} \\ &- \int_q \left\{(N-1)\dot{G}_{\Lambda,\perp}(q')G_{\Lambda,\perp}(q)\tilde{u}^2_\Lambda(k) + \dot{G}_{\Lambda,\parallel}(q') \right. \\ &\times \left. G_{\Lambda,\parallel}(q)\left[\tilde{u}_\Lambda(q) + \tilde{u}_\Lambda(q') + \tilde{u}_\Lambda(k) + 2\rho^0_\Lambda U^{(3)}_\Lambda\right]^2\right\} \\ &+ \int_q \left\{\dot{G}_\parallel(q')G_\perp(q)\tilde{u}^2_\Lambda(q') + \dot{G}_\perp(q')G_\parallel(q)\tilde{u}^2_\Lambda(q)\right\}. \quad (10)\end{aligned}$$

This completes the derivation of the flow equations, which are uniquely determined by the effective action (1). The flow equations (3), (7), (8), and (10) form a closed set that can be used to calculate the full momentum dependence of the self-energies in a controlled manner and the only approximation is the form of the effective action as stated in Eq. (1). By construction, the approach reproduces exactly the correct structure of the leading-order perturbation theory, which is dominant at large momenta. Also by construction it reproduces the leading terms in a derivative expansion of both $u_\Lambda(k)$ and $\sigma_\Lambda(k)$ to lowest order in the fields, which dominate the behavior in the infrared. The same is true also for the symmetric scheme, which we discuss below.

B. The NLPA for the symmetric state

We now derive flow equations that are valid for the symmetric phase, which are even simpler. In the NLPA for the symmetric state the distance to the critical point is controlled by a mass term r_Λ in the propagator that vanishes at criticality in the limit $\Lambda \to 0$. We write the ansatz for Γ_Λ in the NLPA as

$$\begin{aligned}\Gamma_\Lambda[\varphi] = \frac{1}{2}\int_k \{[\sigma_\Lambda(k) + r_\Lambda]\boldsymbol{\varphi}_{\boldsymbol{k}}\cdot\boldsymbol{\varphi}_{-\boldsymbol{k}} \\ + u_\Lambda(k)\rho_{\boldsymbol{k}}\rho_{-\boldsymbol{k}}\} + \int_x U_\Lambda(\rho), \quad (11)\end{aligned}$$

where ρ_k is the Fourier transform of $\rho(\boldsymbol{x}) = \boldsymbol{\varphi}^2(\boldsymbol{x})/2$ and we set $U^{(1)}_\Lambda = 0$ to avoid double counting of the mass term, which is already accounted for by r_Λ. The action (11) yields again unique flow equations for r_Λ and the functions $U_\Lambda(\rho)$, $\sigma_\Lambda(k)$, and $u_\Lambda(k)$, which can be easily derived. We define the vertices now as expansion coefficients of Γ_Λ around $\boldsymbol{\varphi} = 0$. The flow for the two-point vertex $\Gamma^{(2)}_{\Lambda,ab}(k,-k) = \delta_{ab}\Sigma_\Lambda(k)$ is then

$$\partial_\Lambda \Sigma_\Lambda(k) = \frac{1}{2}\int_q \dot{G}_\Lambda(q)[2\tilde{u}_\Lambda(q') + N\tilde{u}_\Lambda(0)], \quad (12)$$

where $\Sigma_\Lambda(k) = r_\Lambda + \sigma_\Lambda(k)$ with $\sigma_\Lambda(0) = 0$ and $\dot{G}_\Lambda(q) = -G^2_\Lambda(q)\partial_\Lambda R_\Lambda(q)$ with $G^{-1}_\Lambda = G^{-1}_{0,\Lambda} + \Sigma_\Lambda(k)$. The flow of the two-point vertex is in the symmetric phase not sufficient to extract also the flow of $\tilde{u}_\Lambda(k) = u_\Lambda(k) + U^{(2)}_\Lambda$ and we must extract its flow from the four-point vertex. This yields

$$\begin{aligned}\partial_\Lambda \tilde{u}_\Lambda(k) = \frac{4+N}{2}\int_q \dot{G}_\Lambda(q)U^{(3)}_\Lambda - \int_q \dot{G}_\Lambda(q)G_\Lambda(q') \\ \times \{(N-1)\tilde{u}_\Lambda(k)^2 + [\tilde{u}_\Lambda(k) + \tilde{u}_\Lambda(q') + \tilde{u}_\Lambda(q)]^2\} \\ - \int_q \dot{G}_\Lambda(q)G_\Lambda(q)\{[\tilde{u}_\Lambda(q') - \tilde{u}_\Lambda(q)] \\ \times [\tilde{u}_\Lambda(0) + 2\tilde{u}_\Lambda(q)]\}, \quad (13)\end{aligned}$$

with $q' = |\boldsymbol{k} + \boldsymbol{q}|$. The flow of the local potential $U_\Lambda(\rho)$ is given by Eq. (3) with $\rho^0_\Lambda = 0$ and with $\sigma_\Lambda(k)$ replaced by $\sigma_\Lambda(k) + r_\Lambda$ in Eqs. (4a) and (4b).

III. RESULTS

We have solved the flow equations in both the symmetric phase and the symmetry broken phase numerically for $D = 3$ and different values of N. For $D = 3$ the field expansion of the local potential actually converges relatively fast [7], so one can work with a finite-order approximation of the local potential. We have used an expansion of $U_\Lambda(\rho)$ up to eighth order in ρ in both the symmetric and the symmetry broken schemes and have checked that the values of the anomalous dimension η are already converged at this level of truncation. The convergence can clearly be seen in Fig. 1, where we show the values of η for different maximal powers of ρ. All results presented below were calculated with all terms up to order ρ^8.

For numerical stability we choose an exponential cutoff

$$R_\Lambda(q^2) = \alpha Z_\Lambda \frac{q^2}{\exp(q^2/\Lambda^2) - 1}, \quad (14)$$

where $Z_\Lambda = 1 + \partial_{k^2}\sigma_\Lambda(k)|_{k=0}$ is the wave function renormalization. Usually the prefactor α is tuned in such a way as to extremize the critical exponents, e.g., the anomalous dimension

$$\eta = -\Lambda\partial_\Lambda \ln Z_\Lambda. \quad (15)$$

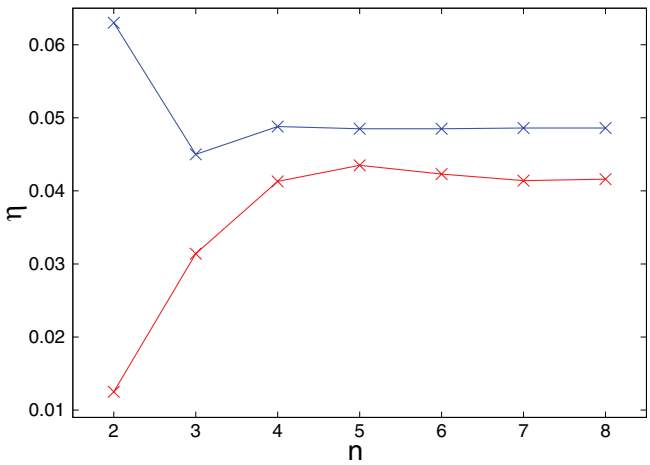

FIG. 1. (Color online) Dependence of the anomalous dimension η on the order n of the polynomial approximation of the local potential $U_\Lambda(\tau) = \sum_{j=0}^n U^{(j)}_\Lambda \tau^j/j!$ for the symmetry broken phase (top curve) and the symmetric scheme (bottom curve) in which the fixed point is approached from within the symmetric phase. Values shown are for $N = 2$ and $D = 3$.

TABLE I. Values for the anomalous dimension η for various N and $D = 3$ from different approaches. The columns correspond to the symmetric NLPA (SNLPA), the NLPA for the ordered phase (ONLPA), results from the background field scheme (BMW) [11], the first-order derivative expansion (DE), field theory (FT), variational perturbation theory (VPT) [21], and Monte Carlo simulations (MC).

N	SNLPA	ONLPA	BMW	DE	FT	VPT	MC
0	0.042		0.034	0.039[a]	0.0272(3)[b]	0.031(1)	0.0303(3)[c]
1	0.042		0.039	0.0443[d]	0.0318(3)[b]	0.034(7)	0.03627(1)[e]
2	0.041(5)	0.049	0.041	0.049[a]	0.0334(2)[b]	0.035(6)	0.0381(2)[f]
3	0.040	0.046	0.040	0.049[a]	0.0333(3)[b]	0.035(0)	0.0375(5)[g]
4	0.038	0.042	0.038	0.047[a]	0.0350(45)[h]	0.031	0.0365(10)[i]
10	0.026	0.024(5)	0.022	0.028[a]	0.024[j]	0.0216	

[a]Reference [22].
[b]Reference [23].
[c]Reference [24].
[d]Reference [7].
[e]Reference [25].
[f]Reference [26].
[g]Reference [27].
[h]Reference [28].
[i]Reference [29].
[j]Reference [30].

This ensures a minimal sensitivity of the results to small variations in α [7]. In the present scheme we do not observe an extremal value of η as a function of α. Instead, we observe a steady decrease of η when α is increased and a minimum that is only reached asymptotically for large α. For the symmetric scheme the dependence of η on α is already essentially flat for $\alpha \geqslant 5$ and we choose $\alpha = 5$ for our analysis below. Similarly, in the symmetry broken phase only a small decrease of η is detected on increasing α from 1 to 2 and η is then essentially unchanged up to $\alpha = 3$. We set $\alpha = 2$ for the analysis below.

A. Critical exponents η and ν

The value of η can be easily determined from the flow of the quantity $\sigma_\Lambda(k)$ and its low momentum structure via Eq. (15). To determine the thermal exponent ν, we use in the symmetric phase the value of the fully renormalized mass term $r_* = \lim_{\Lambda\to 0} r_\Lambda$, which scales as $r_* \simeq (r_{\Lambda_0} - r_c)^{2\nu}$, where r_c is the critical value of the mass term at the initial cutoff scale Λ_0. Similarly, in the symmetry broken phase we analyze the scaling of the order parameter $\rho_* = \lim_{\Lambda\to 0} \rho_\Lambda$, which scales as $\rho_* \simeq (\rho_{\Lambda_0} - \rho_c)^{2\beta}$, where β is the critical exponent of the order parameter and ρ_c is the critical value of ρ_Λ at the initial scale Λ_0. From β and η we can extract ν via the hyperscaling relation $\nu = 2\beta/(D - 2 + \eta)$.

In Tables I and II we show our results for the critical exponents η and ν and compare them with results from various other approaches. Somewhat surprisingly, in contrast to what is observed in a standard derivative expansion, the results for the critical exponents are generally better in the scheme where one

TABLE II. Values for the anomalous dimension ν for various N and $D = 3$ from different approaches. The columns correspond to the symmetric NLPA (SNLPA), the NLPA for the ordered state (ONLPA), results from the background field scheme (BMW) [11], the first-order derivative expansion (DE), field theory (FT), variational perturbation theory (VPT) [21], and Monte Carlo simulations (MC).

N	SNLPA	ONLPA	BMW	DE	FT	VPT	MC
0	0.58		0.589	0.590[a]	0.5886(3)[b]	0.5883	0.5872(5)[c]
1	0.62		0.632	0.6307[d]	0.6306(5)[b]	0.6305	0.63002(10)[e]
2	0.66	0.68	0.674	0.666[a]	0.6700(6)[b]	0.6710	0.6717(1)[f]
3	0.70	0.72	0.715	0.704[a]	0.7060(7)[b]	0.7075	0.7112(5)[g]
4	0.74	0.76	0.754	0.739[a]	0.741(6)[h]	0.737	0.749(2)[i]
10	0.89	0.89	0.889	0.859[a]	0.859[j]	0.866	

[a]Reference [22].
[b]Reference [23].
[c]Reference [31].
[d]Reference [7].
[e]Reference [25].
[f]Reference [26].
[g]Reference [27].
[h]Reference [28].
[i]Reference [29].
[j]Reference [30].

TABLE III. Values for the quantity c defined in Eq. (16), from both the symmetric NLPA (SNLPA) and the NLPA for the ordered state (ONLPA) as well as from a perturbative FRG approach (PFRG), the background field scheme (BMW), variational perturbation theory (VPT), and Monte Carlo simulations (MC).

N	SNLPA	ONLPA	PFRG[a]	BMW[b]	VPT[c]	MC
1	1.38			1.15	1.07(10)	1.09(9)[d]
2	1.49	1.60	1.23	1.37	1.27(10)	1.29(5)[e]
						1.32(2)[f]
3	1.59	1.72		1.50	1.43(11)	
4	1.68	1.82		1.63	1.54(11)	1.60(10)[d]
10	2.02	2.11		2.02		

[a]Reference [9].
[b]Reference [11].
[c]Reference [37].
[d]Reference [34].
[e]Reference [35].
[f]Reference [36].

approaches the critical point from the symmetric side, where a field expansion around $\rho = 0$ rather than around a finite value ρ^0 is employed. As can be seen from Table I, the values for η in the symmetric scheme are, except for $N = 0$, quite close to those obtained within the BMW scheme of Ref. [11]. For large N, it is known that η behaves as $\eta = 0.27/N$ [32] and the result from the symmetric scheme for $N = 10$ is already close to this value.

The results for the approach from the symmetry broken phase are similar to those of the leading-order derivative expansion [where terms up to $O(q^2)$ are kept] (see Table I). A possible reason for the inferior accuracy of the scheme for the symmetry broken phase compared with the accuracy of the symmetric scheme is that all nonlocal correlations are determined already from the two-point function, whereas in the symmetric scheme the nonlocal potential flow is determined from the four-point function. Including further terms in the effective action is expected to improve also the results of the symmetry broken NLPA.

The results for the thermal exponent ν are similar in both schemes and generally close to the most accurate Monte Carlo (MC) results with deviations never more than about 3%. Our values are also close to values from other approaches.

B. Beyond the universal regime

Both the schemes for the symmetric and the symmetry broken phase reproduce the logarithmic behavior of the self-energy at large momenta $\Sigma(k) \simeq u_{\Lambda_0}^2 \ln(k/u_{\Lambda_0})$, which can be derived from perturbation theory [33]. To assess the accuracy of the calculated self-energy over the whole momentum regime a useful quantity is the small-u_{Λ_0} limit of the one-dimensional integral [$\zeta(z)$ is the Riemann zeta function]

$$c = \frac{128}{3\pi u_{\Lambda_0}} \zeta(3/2)^{-4/3} \int_0^\infty dq \frac{\Sigma(q)}{q^2 + \Sigma(q)}, \qquad (16)$$

where $\Sigma(q)$ is the full self-energy at criticality, $\Gamma^{(2)}_{\Lambda,ab}(k,-k) = \delta_{ab}\Sigma_\Lambda(k)$, and $\Sigma(k) = \lim_{\Lambda\to 0}\Sigma_\Lambda(k)$. The quantity c is finite in the limit $u_{\Lambda_0}/\Lambda_0 \to 0$ and has physical significance for $N = 2$, where it relates to the suppression of the critical temperature of the weakly interacting Bose gas in $D = 3$ dimensions [33]. The integral in Eq. (16) is dominated by contributions from the crossover regime $k \simeq u_{\Lambda_0}$, where the momentum dependence of the self-energy changes from the perturbative $\ln(k)$ behavior at large momenta to the anomalous $k^{2-\eta}$ scaling at small momenta. The value of c has been estimated for different N from Monte Carlo simulations [34–36] and has been used to quantify the accuracy of various approaches [9,11,37]. To calculate c we used a small initial value of u_{Λ_0}, $u_{\Lambda_0}/\Lambda_0 = 0.001$. Again we find generally better values for the symmetric scheme. For $N = 2$ the value is about 15% too high when compared to MC results and for $N = 1$ the difference is slightly larger. For $N = 4$ the difference is less than 5%. In comparison with the BMW scheme, the differences are 8% for $N = 2$ and rapidly decrease for larger N (see Table III).

IV. CONCLUSION

We have presented a straightforward nonlocal potential approximation that allows access to finite momentum properties of correlation functions and also allows for an accurate calculation of critical exponents. In the NLPA all truncations are done at the level of the effective action, a property it shares with the derivative expansion. This allows for a strict control of the symmetries of the underlying model and also allows for extensions of the approach. While the present approach includes both terms of arbitrary powers in the fields (in the local term), the nonlocal terms are restricted up to fourth order in the fields. In contrast, in the BMW scheme [11] all vertices have a momentum dependence that is, however, only approximately taken into account. The present scheme can easily be extended by including, for example, terms of the type $\rho(\boldsymbol{x})\rho(\boldsymbol{y})^2\kappa(\boldsymbol{x}-\boldsymbol{y})$, which would result in momentum-dependent vertices with up to six legs, and similar terms of higher order in the densities can of course also easily be constructed. In $D = 3$ it might suffice to limit such terms only to a small maximal power in ρ to get converged values for the critical exponents. For each such additional term a new coupling function must be introduced, so the nonlocality in the scheme will always be restricted to a finite order in the fields. The computational cost of such an extension is relatively modest since one would still deal with the flow of a small

number of one-parameter functions. In contrast, in the BMW scheme the flow must be analyzed for a two-point function that is defined on a two-dimensional grid, one dimension each for the dependence on fields and momenta, which is numerically more difficult.

Even in the the simplest NLPA truncation analyzed here, the results for both the critical exponents and for the momentum dependence of the two-point function are already surprisingly good and the present scheme offers direct access to both universal and nonuniversal quantities. The present scheme is also certainly useful for more complex models where even the local terms are restricted to a finite order in the fields [13–18].

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