

**Bright solitons in defocusing media with spatial modulation of the quintic nonlinearity**Jianhua Zeng<sup>1</sup> and Boris A. Malomed<sup>2</sup><sup>1</sup>*State Key Laboratory of Low Dimensional Quantum Physics, Department of Physics, Tsinghua University, Beijing 100084, China*<sup>2</sup>*Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering, Tel Aviv University, Tel Aviv 69978, Israel*

(Received 5 April 2012; published 24 September 2012)

It has been recently demonstrated that self-defocusing (SDF) media with cubic nonlinearity, whose local coefficient grows from the center to the periphery fast enough, support stable bright solitons without the use of any linear potential. Our objective is to test the genericity of this mechanism for other nonlinearities, by applying it to one- and two-dimensional (1D and 2D) quintic SDF media. The models may be implemented in optics (in particular, in colloidal suspensions of nanoparticles), and the 1D model may be applied to the description of the Tonks-Girardeau gas of ultracold bosons. In 1D, the nonlinearity-modulation function is taken as  $g_0 + \sinh^2(\beta x)$ . This model admits a subfamily of exact solutions for fundamental solitons. Generic soliton solutions are constructed in a numerical form and also by means of the Thomas-Fermi and variational approximations (TFA and VA). In particular, a new ansatz for the VA is proposed, in the form of “raised sech,” which provides for an essentially better accuracy than the usual Gaussian ansatz. The stability of all the fundamental (nodeless) 1D solitons is established through the computation of the corresponding eigenvalues for small perturbations and also verified by direct simulations. Higher-order 1D solitons with two nodes have a limited stability region, all the modes with more than two nodes being unstable. It is concluded that the recently proposed inverted Vakhitov-Kolokolov stability criterion for fundamental bright solitons in systems with SDF nonlinearities holds here too. Particular exact solutions for 2D solitons are produced as well.

DOI: [10.1103/PhysRevE.86.036607](https://doi.org/10.1103/PhysRevE.86.036607)

PACS number(s): 05.45.Yv, 03.75.Kk, 03.75.Lm, 42.65.Tg

**I. INTRODUCTION**

The guiding of matter waves in Bose-Einstein condensates (BECs) [1] and light in optical waveguides [2] by means of effective potentials which are induced by optical lattices or photonic crystals, has drawn a great deal of attention in recent years. Such settings play a crucial role in the creation and stabilization of ordinary and gap solitons [1,3,4], supported by the balance of the diffraction, the lattice-induced potentials, and the self-focusing or defocusing nonlinearity, respectively.

In addition to linear lattices, many works have been dealing with their nonlinear counterparts (the spatial modulation of the local nonlinearity coefficient) [4–10] and combined linear-nonlinear [11–14] lattices. The nonlinear lattices may be realized, severally in optics and BECs, by means of properly designed photonic-crystal structures or nonlinearity landscapes induced by the Feshbach resonance controlled by nonuniform external fields. A recent noteworthy result is the existence of stable bright solitons in self-defocusing (SDF)  $D$ -dimensional media with the strength of the cubic nonlinearity growing toward the periphery, as a function of radius  $r$ , at any rate faster than  $r^D$  [15,16].

Along with the cubic nonlinearity, quintic terms appear in various physical settings too. In optics, the quintic nonlinearity has been theoretically predicted [17,18] and experimentally observed in fluids [19,20] and glasses [21–25]. In the context of BEC, the quintic nonlinearity represents three-body interactions in a dense condensate, provided that collision-induced losses may be neglected [26]. The self-focusing quintic nonlinearity is critical in the one-dimensional (1D) case, similar to its cubic counterpart in the two-dimensional (2D) case [27]; i.e., it gives rise to a degenerate family of *Townes solitons* [28], with a single value of the norm for the entire family. In the free space, the Townes solitons are unstable, but they can be readily stabilized by linear potentials,

as demonstrated in 1D [29–31] and 2D [3,32] settings alike. The use of nonlinear potentials for the same purpose turns out to be tricky: in the 2D space with the cubic self-focusing, only nonlinearity-modulation profiles with *sharp edges* are able to stabilize the Townes solitons in a limited parameter region [4,9]. In the 1D system with the quintic nonlinearity, nonlinear lattices with a *smooth* (sinusoidal) modulation profile stabilize the respective Townes solitons against the critical collapse, but only in a narrow region [8,10].

On the other hand, in Ref. [33] it has been demonstrated that the 1D Tonks-Girardeau (TG) gas of bosons with the hard-core repulsion, emulating a degenerate Fermi gas, obeys the 1D nonlinear Schrödinger (NLS) equation with the quintic SDF term (an additional nonlocal cubic interaction appears if the bosons carry dipole moments [34]). While this equation is inappropriate for the description of dynamical effects caused by interference of bosonic wave packets [35], it correctly predicts stationary patterns in the trapped gas [33–35]. Experimentally, the TG gas has been realized in an ultracold gas of <sup>87</sup>Rb loaded into a tightly confined potential pipe [36]. In fact, the “quantum Newton’s cradle,” realized in a trapped chain of <sup>87</sup>Rb atoms [37], is another example of the TG gas.

In this work, we address the possibilities to support stable bright solitons in 1D and 2D media with the SDF *quintic* term whose local strength grows fast enough at  $r \rightarrow \infty$  (in fact, the solitons exist for the growth rates faster than  $r^{2D}$ ). These results generalize those recently reported in Refs. [15] and [16] for SDF media with the modulated cubic nonlinearity and thus demonstrate the genericity of the method for creating robust bright solitons using solely the SDF nonlinearity. We focus on 1D and 2D models with modulation profiles in the form of hyperbolic functions, see Eqs. (5) and (27) below, for which some soliton solutions can be obtained in an exact analytical form and generic ones are constructed numerically. The family of the fundamental

solitons is completely stable, similar to the situation reported in Refs. [15] and [16] in the case of the spatially modulated cubic SDF nonlinearity. We give an additional argument, based on energy estimates, in favor of the conjecture that the fundamental solitons may realize the system’s ground state. In terms of the dependence between the soliton’s propagation constant and norm,  $k(N)$ , the bright solitons in the SDF medium obey the *inverted* Vakhitov-Kolokolov (alias *anti-VK* [11]) criterion,  $dk/dN < 0$ , contrary to the usual VK criterion for fundamental solitons in self-focusing media,  $dk/dN > 0$  [27,38]. It appears that, as conjectured in Ref. [11], the anti-VK criterion is a necessary but, in the general case, not sufficient condition for the stability of bright solitons supported by SDF nonlinearity. Higher-order 1D solitons, whose shapes feature different numbers of nodes ( $n$ ), are found too. A part of the soliton family with  $n = 2$  is stable, while all the modes with  $n \geq 3$  are found to be unstable.

It is relevant to mention that the model considered here, as well as its previously studied counterpart with cubic nonlinearity [15,16], is not integrable. Indeed, it was proven (see, e.g., Ref. [39]) that the NLS equation in 1D with any nonlinearity different from pure cubic is nonintegrable; hence our quintic model is definitely nonintegrable too. For this reason, the localized modes that we study in this paper are not “solitons” in the rigorous mathematical meaning of the word, but rather “solitary waves.” Nevertheless, we call the modes “solitons,” following the usage commonly adopted in the current literature. In this work, we do not study interactions between the solitons because multisoliton configurations trapped in the nonlinear potential well are less relevant than single-soliton states.

The rest of the paper is organized as follows. In Sec. II, we introduce the model, report particular exact solutions, and develop the variational approximation (VA) for generic 1D solitons, based on Gaussian and sech *ansätze* (the latter one is taken in a generalized form, based on “raised sech,” which is an essential technical novelty). Numerical results for the 1D fundamental and higher-order solitons, including the stability analysis, are reported in Sec. III. Two-dimensional solitons are briefly considered in Sec. IV, and the paper is concluded by Sec. V.

## II. THE QUINTIC MODEL AND ANALYTICAL APPROXIMATIONS

### A. The model and exact solutions

The NLS equation with the quintic SDF nonlinearity for field amplitude  $u(\mathbf{r}, z)$  is

$$iu_z = -\frac{1}{2}\nabla^2 u + [g_0 + g(\mathbf{r})]|u|^4 u, \quad (1)$$

where  $g_0 \geq 0$  is the strength of the uniform quintic term; Laplacian  $\nabla^2 = \partial_x^2 + \partial_y^2$  or  $\nabla^2 = \partial_x^2$  acts on transverse coordinates  $\mathbf{r} = (x, y)$  or  $x$  in 2D and 1D settings, respectively;  $z$  is the propagation coordinate in the optical medium (in the Gross-Pitaevskii equation for matter waves,  $z$  is replaced by time  $t$ ); and positive function  $g(r)$  accounts for the nonlinearity strength growing at  $r \rightarrow \infty$ . In optics, the 1D version of this modulated nonlinearity may be implemented in a planar waveguide of a variable transverse width, filled with a colloidal suspension of metallic nanoparticles, as proposed in Ref. [10].

The same 1D model applies to the TG gas trapped in a potential pipe whose confinement strength varies along the axial coordinate, similar to how it was recently proposed to induce a modulated cubic nonlinearity in the effectively 1D BEC with attractive interatomic interactions [40].

Stationary solutions to Eq. (1) with propagation constant  $k$  are sought for as  $u(\mathbf{r}, z) = U(r) \exp(ikz)$ , where the real function  $U(r)$  obeys the equation

$$kU - \frac{1}{2}\nabla^2 U + [g_0 + g(\mathbf{r})]U^5 = 0, \quad (2)$$

which can be derived from the Lagrangian density

$$2\mathcal{L} = kU^2 + \frac{1}{2}(\nabla U)^2 + \frac{1}{3}[g_0 + g(\mathbf{r})]U^6. \quad (3)$$

In the 1D case, Eq. (2) taken at the inflexion point,  $U'' = 0$ , demonstrates that all solitons may exist only with  $k < 0$ . A similar argument readily predicts  $k < 0$  for 2D solitons. With regard to this, the Thomas-Fermi approximation (TFA), which neglects the diffraction term,  $\nabla^2 U$ , yields the solution

$$U_{\text{TFA}}^2(\mathbf{r}) \approx \sqrt{-k/g(\mathbf{r})} \quad (4)$$

at  $r \rightarrow \infty$ ; hence, as mentioned above, the modes with a finite norm,  $N_{2D} = \iint U^2(\mathbf{r}) dx dy$  or  $N_{1D} = \int_{-\infty}^{+\infty} U^2(x) dx$ , exist if  $g(\mathbf{r})$  grows faster than  $r^{2D}$  at  $r \rightarrow \infty$ .

In the present work, the 1D modulation function is taken as

$$g(x) = \sinh^2(\beta x), \quad (5)$$

the coefficient in front of which is scaled to be 1, while  $g_0$  is kept as a free parameter in Eq. (1). In fact, the unlimited exponential growth of the nonlinearity strength at  $|x| \rightarrow \infty$ , which may be difficult to realize in real physical settings, is not necessary for the creation of the solitons supported by the spatially modulated SDF nonlinearity. Indeed, the strong localization of the solitons predicted by expressions (6) and (11) (see below) implies that the modulation profile must actually represent a deep nonlinear-potential well of a finite extension, rather than the unlimitedly growing structure, as a particular form of  $g(x)$  at  $|x| \gg \beta^{-1}$  does not affect the solution.

In the case of  $g_0 < 1$ , Eq. (2) with  $g(x)$  taken as per Eq. (5) admits an exact soliton at a particular value of  $k$ :

$$U^2(x) = \sqrt{\frac{3\beta^2}{8(1-g_0)}} \text{sech}(\beta x), \quad (6)$$

$$k = -\frac{\beta^2}{8} \frac{2+g_0}{1-g_0}. \quad (7)$$

It is worthy to mention that a particular exact solution can be also found in the 1D model with a more general nonlinearity-modulation function:

$$iu_z = -\frac{1}{2}u_{xx} + [\cosh(\beta x)]^\mu [g_0 + \sinh^2(\beta x)]|u|^4 u, \quad (8)$$

where the real power takes values  $\mu > -2$ . In fact, most interesting are the negative values,  $-2 < \mu < 0$ , because in that case the growth of the nonlinearity at  $|x| \rightarrow \infty$  is *slower*.

The exact soliton solution found in this model is

$$u = e^{ikz} A[\operatorname{sech}(\beta x)]^{(2+\mu)/4}, \quad (9)$$

$$A^4 = \frac{\beta^2(2+\mu)(6+\mu)}{32(1-g_0)},$$

$$k = -\frac{\beta^2(2+\mu)}{32} \left[ \frac{6+\mu}{1-g_0} - (2+\mu) \right]. \quad (10)$$

In the limit of  $\mu \rightarrow -2$ , when the nonlinearity modulation in Eq. (8) ceases to be growing at  $|x| \rightarrow \infty$ , expression (10) for the amplitude degenerates into  $A = 0$ . On the other hand, in the limit of  $2+\mu \approx C(1-g_0) \rightarrow 0$  with fixed  $C > 0$ , Eq. (8) degenerates into the NLS equation with the constant coefficient of the SDF quintic nonlinearity. In this case, the soliton goes over into the continuous-wave (cw) solution with the constant amplitude,  $A_{\text{cw}}^4 = \beta^2 C/8$ . It is obvious that the latter solution is modulationally stable [27]; hence the continuity suggests that the fundamental soliton solutions of Eq. (8) may be stable too, which is confirmed below (for  $\mu = 0$ ).

Coming back to general soliton solutions to the 1D version of Eq. (1) with modulation function (5), it is easy to find the asymptotic form of the soliton solutions at  $|x| \rightarrow \infty$ , which accounts for its strong localization:

$$U(x) \approx \left(\frac{1}{2}\beta^2 - 4k\right)^{1/4} \exp\left(-\frac{1}{2}\beta|x|\right) \quad (11)$$

(note that, unlike the usual solitons, the localization length determined by this expression,  $\sim\beta^{-1}$ , does not depend on  $k$ ). Further, the TFA applies to the entire 1D solution in the case of  $-8k \gg \beta^2$ , predicting the solution in the form of Eq. (4) at all  $x$ , the respective norm being

$$N_{\text{TFA}} \approx (2I/\beta)\sqrt{-k}, \quad (12)$$

with constant  $I \equiv \int_0^\infty (2g_0 - 1 + \cosh y)^{-1/2} dy$ . It is obvious that the fundamental (nodeless) solitons are stable within the framework of the TFA.

It may be conjectured that the fundamental solitons realize the ground state in the systems of the present type [15]. A “naive” counterargument against this assumption is a proposal to spread out the given norm,  $N$ , into a layer of an indefinitely large length,  $L$ , with a vanishingly small squared amplitude,  $N/L$ , so that the energy of such a state would fall to zero, along with its amplitude, while the total energy of the fundamental solitons would, obviously, be positive. However, a straightforward estimate of the energy of the stretching layer [the energy density is actually the same as the Lagrangian density (3), except for the first term in it] yields an estimate,  $E(L) \approx (N^3/12\beta L^3) \exp(2\beta L)$ , which *diverges*, rather than vanishes, in the limit of  $L \rightarrow \infty$ .

A curious fact is that expression (11) gives an exact solution for the *entire family* of fundamental solitons in the model with  $g(x) = (1/4) \exp(2\beta|x|)$ , which emulates the asymptotic form of function (5), if, in addition to the effective nonlinear potential, an attractive linear  $\delta$ -functional potential is placed at  $x = 0$ . The respective stationary equation is

$$kU - \frac{1}{2}U'' + \frac{1}{4}e^{2\beta|x|}U^5 - \frac{\beta}{2}\delta(x)U = 0. \quad (13)$$

The linear potential  $-(\beta/2)\delta(x)$  in Eq. (13) is necessary to balance the peakon singularity in expression (11), if it is

considered as the exact solution. Obviously, the norm of this solution is  $N(k) = \beta^{-1}\sqrt{2(\beta^2 - 8k)}$ , with the propagation constant taking values  $k \leq \beta^2/8$  (note that this  $k$  may be positive too).

Furthermore, the spatially modulated SDF nonlinearity can trap a soliton against the action of the linear *repulsive* potential. To demonstrate this possibility, one can take the following modification of Eqs. (2) and (5),

$$kU - \frac{1}{2}\nabla^2 U + [g_0 + \sinh^2(\beta x)]U^5 + W\operatorname{sech}^2(\beta x)U = 0, \quad (14)$$

where  $W > 0$  is the strength of the repulsive linear potential. An exact solution to Eq. (14) for the trapped mode can be found in the following form, which generalizes the above solution given by Eqs. (6) and (7):

$$U^2(x) = \sqrt{\frac{1}{1-g_0} \left( \frac{3\beta^2}{8} + W \right)} \operatorname{sech}(\beta x), \quad (15)$$

$$k = -\frac{1}{1-g_0} \left[ \frac{\beta^2}{8}(2+g_0) + W \right]. \quad (16)$$

Numerical results [not shown here in detail, but essentially the same as presented below, i.e., based on the analysis of small perturbations—see Eq. (24)—and direct simulations] demonstrate that this solution is stable.

## B. The variational approximation

### 1. The Gaussian ansatz

To search for soliton solutions of Eq. (2) by means of the VA, we start with the simplest Gaussian ansatz,

$$U^2(x) = \frac{N}{\sqrt{\pi}W} \exp\left(-\frac{x^2}{2W^2}\right), \quad (17)$$

where  $W$  and  $N$  are the width and norm of the soliton. The substitution of this ansatz into the Lagrangian density (3) and subsequent integration yields the total Lagrangian,  $L = \int_{-\infty}^{+\infty} \mathcal{L} dx$ , from which it is straightforward to derive the variational equations,  $\partial L_{\text{eff}}/\partial N = \partial L_{\text{eff}}/\partial W = 0$ :

$$\frac{1}{4W^2} + \frac{N^2}{2\sqrt{3}\pi W^2} \left[ 2g_0 + 1 - e^{-\frac{(\beta W)^2}{3}} \right] = -k, \quad (18)$$

$$\frac{1}{2W^2} + \frac{N^2}{3\sqrt{3}\pi} \left[ \frac{(2g_0+1)}{W^2} + \left( \frac{\beta^2}{3} - \frac{1}{W^2} \right) e^{\frac{(\beta W)^2}{3}} \right] = 0. \quad (19)$$

### 2. The raised-sech ansatz

The availability of the particular exact solution (6), expressed in terms of *sech*, suggests to use this function as an alternative variational ansatz, as it is natural to have one which is able to reproduce a particular exact solution, suggesting a better accuracy in the general case too (which turns out to be correct, see below). Usually, the ansatz based on *sech* is introduced with arbitrary amplitude and width [41]. However, in the present model the functional form of the ansatz must be fixed as  $\operatorname{sech}(\beta x)$ , with the same  $\beta$  as in Eq. (5), as otherwise the integration of the term in the Lagrangian density (3) containing  $g(x)$  is impossible in an analytical form. The arbitrary width can be accommodated differently, adopting the

following *raised-sech* ansatz (sech raised to arbitrary power  $\nu > 0$ ):

$$U(x) = A[\text{sech}(\beta x)]^\nu, \quad (20)$$

where  $A$  and  $\nu$  are to be treated as variational parameters. In particular, in the case of  $\nu \ll 1$ , the standard full width at half maximum width of ansatz (20) is  $(\ln 2)/(\beta\nu)$ , which demonstrates how  $\nu$  controls the width.

To the best of our knowledge, the VA based on ansatz (20), with power  $\nu$  dealt with as the variational parameter, is a technical novelty, which may be helpful too in studies of other

models with the width of the shape-defining function fixed by the form of the given equation(s). As shown below, this ansatz leads to rather complex variational equations which, nevertheless, can be solved, leading to a good agreement with directly found numerical results. It is relevant to note that some still more sophisticated trial functions were recently proposed for the description of solitons, such as the so-called  $q$ -Gaussian ansatz [42], with an extra parameter ( $q$ ) controlling a transition between the limit cases of the Gaussian and the TFA ansätze.

The effective Lagrangian, produced by the integration of density (3) with ansatz (20) substituted into it is

$$\frac{2}{\sqrt{\pi}\beta}L = \left[ k + \frac{\nu^2\beta^2}{2(2\nu+1)} \right] \frac{A^2\Gamma(\nu)}{\Gamma(\nu+\frac{1}{2})} + \frac{A^6}{3} \frac{g_0\Gamma(3\nu)}{\Gamma(3\nu+\frac{1}{2})} + \frac{A^6}{3(6\nu-1)} \frac{\Gamma(3\nu-1)}{\Gamma(3\nu-\frac{1}{2})}, \quad (21)$$

where  $\Gamma$  is the Gamma function (below,  $\Gamma'$  stands for its derivative). The corresponding variational equations,  $\partial L/\partial(A^2) = 0$  and  $\partial L/\partial\nu = 0$ , take the following forms:

$$\left[ k + \frac{\nu^2\beta^2}{2(2\nu+1)} \right] \frac{\Gamma(\nu)}{\Gamma(\nu+\frac{1}{2})} + \frac{g_0A^4\Gamma(3\nu)}{\Gamma(3\nu+\frac{1}{2})} + \frac{A^4}{6\nu-1} \frac{\Gamma(3\nu-1)}{\Gamma(3\nu-\frac{1}{2})} = 0, \quad (22)$$

$$\begin{aligned} & \left[ k + \frac{\nu^2\beta^2}{2(2\nu+1)} \right] \left[ \frac{\Gamma'(\nu)}{\Gamma(\nu+\frac{1}{2})} - \frac{\Gamma(\nu)\Gamma'(\nu+\frac{1}{2})}{\Gamma^2(\nu+\frac{1}{2})} \right] + \frac{\beta^2\nu(\nu+1)}{(2\nu+1)^2} \frac{\Gamma(\nu)}{\Gamma(\nu+\frac{1}{2})} + \frac{g_0A^4}{3} \left[ \frac{\Gamma'(3\nu)}{\Gamma(3\nu+\frac{1}{2})} - \frac{\Gamma(3\nu)\Gamma'(3\nu+\frac{1}{2})}{\Gamma^2(3\nu+\frac{1}{2})} \right] \\ & + \frac{A^4}{3(6\nu-1)} \left[ \frac{\Gamma'(3\nu-1)}{\Gamma(3\nu-\frac{1}{2})} - \frac{\Gamma(3\nu-1)\Gamma'(3\nu-\frac{1}{2})}{\Gamma^2(3\nu-\frac{1}{2})} \right] = 0. \end{aligned} \quad (23)$$

Despite the relative complexity of these equations, it is possible to check that, setting  $\nu = 1/2$  and  $g_0 = 0$ , they reproduce the corresponding exact solution (6) with  $k$  given by Eq. (7).

The comparison between both versions of the VA, produced by numerical solutions of Eqs. (18) and (19) and Eqs. (22) and (23), respectively, and the results obtained from a numerical solution of Eq. (2) are presented in Fig. 1. It is observed that ansatz (20) provides for a substantial improvement of the accuracy of the VA [expect for the limit case of small  $N$  in Fig. 1(b)]. It is interesting to note that, according to the variational and numerical results alike, the entire family

of the fundamental solitons satisfies the above-mentioned *anti-VK* criterion,  $dk/dN < 0$ , which was recently proposed, at a semiempiric level, as a stability criterion for fundamental bright solitons in media with repulsive nonlinearities [11,14]. Indeed, it is demonstrated below that the fundamental solitons are fully stable in the present model.

It is also worth stressing that, as shown by the  $k(N)$  curves (the numerical and VA-predicted ones alike) in Fig. 1, the family of the fundamental solitons features no existence threshold; i.e., the solutions persist up to the limit of  $N \rightarrow 0$ . An analytical consideration of Eqs. (2) and (5) suggests that, in this limit, the soliton acquires a shape of a quasiflat shelf of width  $L \simeq (2/\beta) \ln(\beta/N)$ , with the squared amplitude  $A^2 \simeq N/L$ .

### III. NUMERICAL RESULTS

#### A. The linear stability analysis

It is crucially important to test stability of the solitons. To this end, perturbed solutions are taken as

$$u(x,z) = \exp(ikz)[U(x) + V(x)\exp(\lambda z) + W^*(x)\exp(\lambda^*z)], \quad (24)$$

where  $V$  and  $W$  are components of a perturbation mode with growth rate  $\lambda$ , while  $U(x)$  is the stationary solution to Eq. (1) with propagation constant  $k$  (the asterisk stands for the

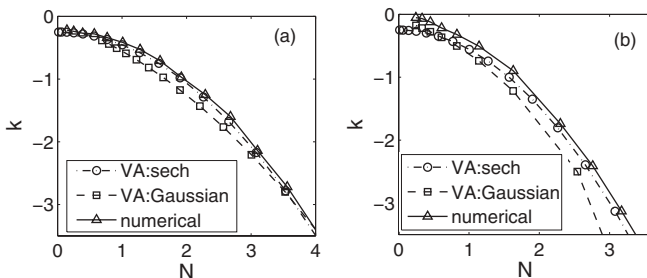


FIG. 1. The comparison of predictions of the VA for fundamental solitons, based on the Gaussian and raised-sech ansätze, with numerically found dependencies  $k(N)$ , for  $\beta = 2$  and  $g_0 = 0$  (a) and  $g_0 = 0.5$  (b).

complex conjugate). The substitution of expression (24) into Eq. (1) and subsequent linearization gives rise to the eigenvalue problem for  $\lambda$ :

$$i\lambda V = -(1/2)V_{xx} + kV + [g_0 + g(x)]U^4(3V + 2W), \quad (25)$$

$$i\lambda W = +(1/2)W_{xx} - kW - [g_0 + g(x)]U^4(3W + 2V). \quad (26)$$

Obviously, the stationary solution,  $U(x)$ , is stable only if  $\text{Re}\{\lambda\} = 0$  for all the eigenvalues.

The eigenvalue problem can be solved numerically by means of a finite-difference scheme. Specifically, we first took the Gaussian ansatz as the initial guess to construct solutions  $U(x)$  of Eq. (2) and then tested the stability of the so found solutions by solving the eigenvalue problem based on Eqs. (25) and (26). Finally, the predicted (in)stability was tested in direct simulations of Eq. (1). The numerical computations were performed in the domain of size  $-10 \leq x \leq 10$  on a grid of 128 points, with the usual absorbing boundary conditions. This method is reliable, as it has been applied extensively to soliton-related problems. For further details, see Ref. [43], which contains the relevant code. Results produced by the linear-stability analysis are presented below.

### B. Fundamental and higher-order solitons

Examples of fundamental (nodeless) and higher-order solitons found in the numerical form are displayed in Fig. 2(a) and Figs. 2(b) and 2(c), respectively. Note that the raised-sech ansatz predicts the profile of the fundamental soliton which is virtually identical to its numerically found counterpart. Both

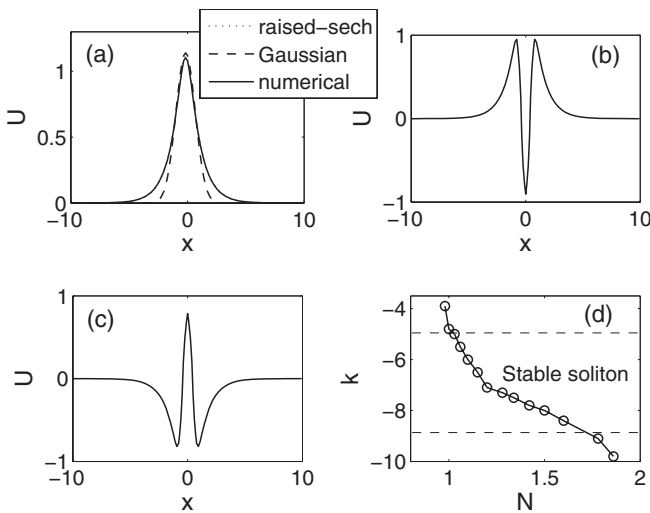


FIG. 2. (a) An example of a stable 1D fundamental soliton and the comparison with the Gaussian and raised-sech ansätze. The latter one is indistinguishable from the numerically found profile. (b,c) Examples of stable higher-order solitons with two nodes. These examples are displayed for  $\beta = 2$  and  $g_0 = 0$ , other parameters being (a)  $k = -1$ ,  $N = 1.91$ , (b)  $k = -8.65$ ,  $N = 1.6$ , and (c)  $k = -7.1$ ,  $N = 1.2$ . (d) The propagation constant vs the norm for the higher-order solitons with two nodes at  $\beta = 2$  and  $g_0 = 0$ . The solitons are stable between the dashed lines.

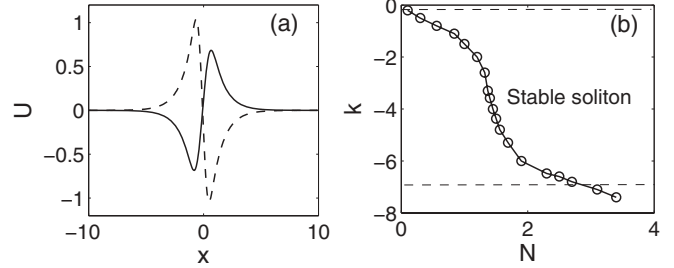


FIG. 3. (a) Examples of stable dipole solitons for  $\beta = 2$  and  $g_0 = 0$ . Other parameters are  $k = -2.6$ ,  $N = 1.32$  and  $k = -4.8$ ,  $N = 1.56$  (the dashed and solid profiles, respectively). (b) Curve  $k(N)$  for the dipole solitons at  $\beta = 2$  and  $g_0 = 0$ . The solitons are stable between the dashed lines.

the computation of the eigenvalues for small perturbations and the direct simulations demonstrate that the entire family of the fundamental solitons is stable.

Higher-order solitons are characterized by the number of nodes (zeros of the field). In particular, the examples shown in Figs. 2(b) and 2(c) feature two zero crossings, and examples of dipole solitons, with the single node, are presented in Fig. 3.

The solitons with two nodes, as well as the dipole solitons, are stable in a part of their existence regions, as shown in Figs. 2(d) and 3(b), respectively. Note that the entire  $k(N)$  curves in these figures satisfy the anti-VK criterion,  $dk/kN < 0$ , like they did for the fundamental solitons (cf. Fig. 1). The fact that only a part of the families of the higher-order solitons is stable complies with the general fact that the usual VK criterion, in models with self-focusing nonlinearities, is necessary but not sufficient for the full stability of bright solitons [27,38]. Further, the conclusion that the fundamental solitons supported by the SDF nonlinearity with the local coefficient growing at  $r \rightarrow \infty$  are completely stable, while a part of higher-order families are unstable, agrees with the results recently reported in Refs. [15] and [16] for the cubic nonlinearity with similar modulation profiles. In fact, it was found that, in the case of the steep (anti-Gaussian) modulation, the solitons with one and two nodes are completely stable, instability regions appearing for modes with three zeros [15], while, under the action of a milder algebraic modulation, with  $g(x) \sim |x|^\alpha$  ( $\alpha > 1$ ), the solitons with one and two zeros may be unstable too, only the fundamental family remaining completely stable. The exponential modulation function adopted in the present model, see Eq. (5), is intermediate between its steep anti-Gaussian and mild algebraic counterparts.

We have found that all the higher-order modes with  $\geq 3$  zeros are unstable in the present model, as illustrated by Figs. 4 and 5. The instability transforms the unstable solitons into chaotically oscillating localized modes (see examples in Fig. 5).

### IV. SOLITONS IN THE 2D MODEL

An issue of obvious interest is to extend the model and its analysis to the 2D geometry (cf. Refs. [15] and [16]). Although the TG model is irrelevant in 2D, the abovementioned optical realization, in terms of the colloidal suspensions, applies to the 2D case too. To produce an example of exact 2D solutions,

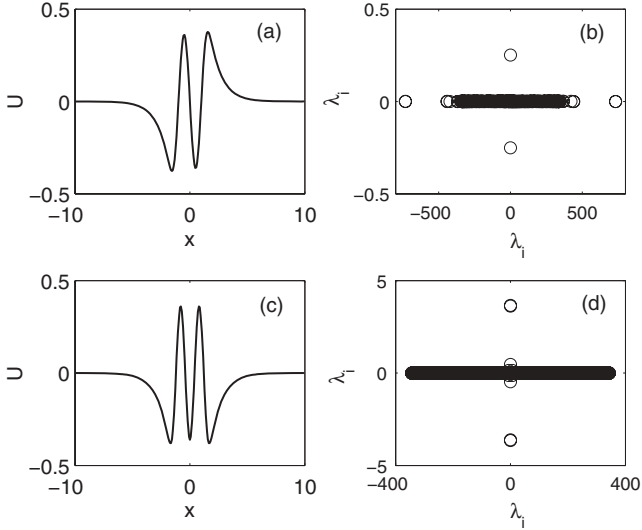


FIG. 4. The shapes (left) and their perturbation spectra (right) of unstable 1D modes with three (a,b) and four (c,d) zeros, at  $\beta = 2$  and  $g_0 = 0$ . (a,b)  $k = -5$ ,  $N = 1.03$ ; (c,d)  $k = -7.6$ ,  $N = 1.07$ .

we set  $g_0 = 0$  in Eq. (1) and take

$$g(x,y) = g_0 + [\beta^{-2} \sinh^2(\beta x) \cosh^2(\alpha y) + \alpha^{-2} \sinh^2(\alpha y) \cosh^2(\beta x)] \quad (27)$$

[cf. Eq. (5) in 1D]. In this case, the following exact fundamental soliton can be found for  $g_0 = 0$ :  $u(x,y,t) = U(x,y) \exp(ikz)$ , with  $k = -(\alpha^2 + \beta^2)/4$  and

$$U^2(x,y) = \sqrt{3/8} \alpha \beta \operatorname{sech}(\beta x) \operatorname{sech}(\alpha y). \quad (28)$$

Examples of isotropic ( $\alpha = \beta = 1$ ) and anisotropic ( $\alpha = 1$  and  $\beta = 1/2$ ) 2D solitons, found in the numerical form for modulation function (27) with  $g_0 = 0.1$ , when exact solutions

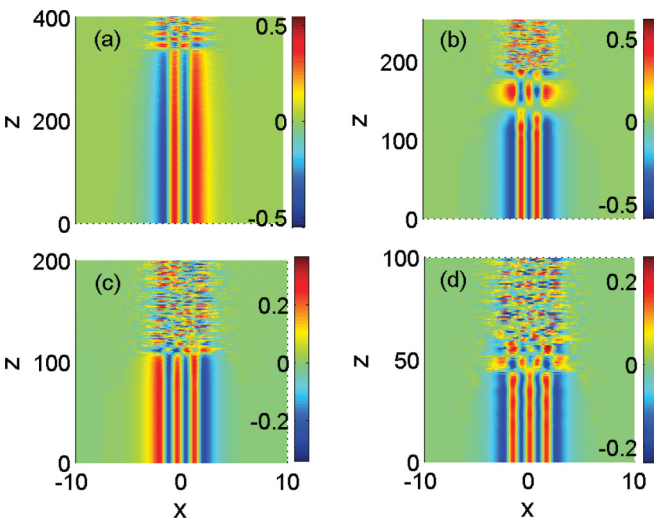


FIG. 5. (Color online) The evolution, shown in terms of  $\operatorname{Re}\{u(x,z)\}$ , of unstable 1D higher-order solitons with the number of nodes  $n = 3, 4, 5$ , and  $6$  at  $\beta = 2$  and  $g_0 = 0$ . (a)  $k = -5.3$ ,  $N = 1.05$ ; (b)  $k = -8.9$ ,  $N = 1.35$ ; (c)  $k = -7.6$ ,  $N = 1.07$ ; (d)  $k = -7.7$ ,  $N = 1.2$ .

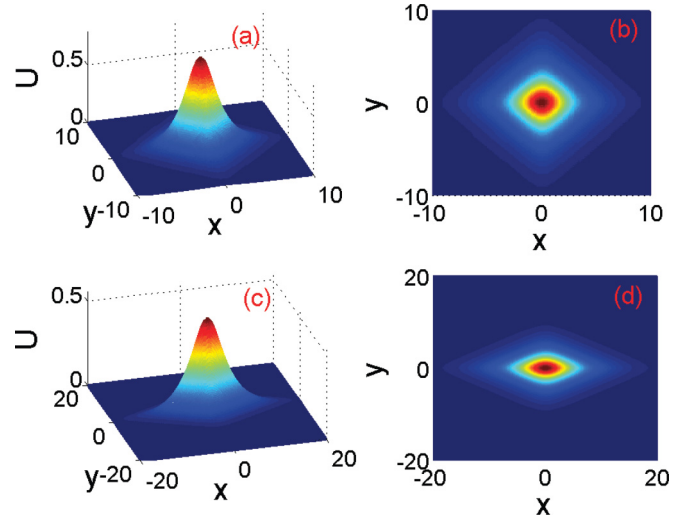


FIG. 6. (Color online) Shapes of stable isotropic and anisotropic 2D fundamental solitons found from Eqs. (2) and (27) with  $g_0 = 0.1$ . (a, b)  $\alpha = \beta = 1$  and  $k = -0.5$ ,  $N = 4$ ; (c, d)  $\alpha = 2\beta = 1$  and  $k = -0.3125$ ,  $N = 3.5$ .

are not available, are shown in Fig. 6. The stability of these solitons was verified by direct simulations of the perturbed evolution. Systematic results for 2D solitons will be reported elsewhere.

## V. CONCLUSION

We have introduced the 1D and 2D models with the SDF quintic nonlinearity growing at  $r \rightarrow \infty$ . The model may be realized in optical media, and its 1D version can be also implemented in the TG gas. Recently, stable bright solitons were found in similar models with the cubic nonlinearity. The objective of the present work was to test the genericity of the mechanism creating bright solitons in cubic SDF media with the spatially modulated nonlinearity coefficient, by testing it with the other (quintic) nonlinearity. We found particular exact solutions for fundamental solitons in the 1D model with the modulation function defined as per Eqs. (2) and (5). General solutions have been found in the numerical form and also analytically in the framework of the TFA and the VA. In particular, a new ansatz for the VA, based on the raised sech, was developed. It yields an accuracy essentially better than that of the usual Gaussian ansatz. All the fundamental solitons are stable, while higher-order ones have a finite stability region for modes with one and two nodes, all the solitons with  $\geq 3$  nodes being unstable. It has been found that the recently proposed anti-VK stability criterion for bright solitons in SDF media is valid, as a necessary stability condition, in the present model too. The 2D model was also considered, in a brief form. Particular exact solutions for 2D solitons were produced, and examples of numerically found stable solitons in 2D were reported, both isotropic and anisotropic ones. The analysis of the 2D model calls for an extension; in particular, a challenging problem is to construct solutions for vortex solitons.

## ACKNOWLEDGMENTS

J.Z. acknowledges support from the Natural Science Foundation of China (Project No. 11204151). The work of B.A.M.

was supported, in a part, by the German-Israel Foundation (Grant No. I-1024-2.7/2009).

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