Motion-dependent levels of order in a relativistic universe

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Consider a generally closed system of continuous three-space coordinates $\mathbf{x}$ with a differentiable amplitude function $\psi(\mathbf{x})$. What is its level of order $\mathbb{R}$? Define $\mathbb{R}$ by the property that it decreases (or stays constant) after the system is coarse grained. Then $\mathbb{R}$ turns out to obey $\mathbb{R} = 8^{-1}L^2 I$,where quantity $I = 4\int d\mathbf{x}\nabla\psi^* \cdot \nabla\psi$ is the classical Fisher information in the system and L is the longest chord that can connect two points on the system surface. In general, order $\mathbb{R}$ is (i) unitless, and (ii) invariant to uniform stretch or compression of the system. On this basis, the order $\mathbb{R}$ in the Universe was previously found to be invariant in time despite its Hubble expansion, and with value $\mathbb{R} = 26.0 \times 10^{60}$ for flat space. By comparison, here we model the Universe as a string-based "holostar," with amplitude function $\psi(\mathbf{x}) \propto 1/r$ over radial interval $r = (r_0, r_H)$. Here r_0 is of order the Planck length and r_H is the radial extension of the holostar, estimated as the known value of the Hubble radius. Curvature of space and relative motion of the observer must now be taken into account. It results that a stationary observer observes a level of order $\mathbb{R} = (8/9)(r_H/r_0)^{3/2} = 0.42 \times 10^{90}$; while for a free-falling observer $\mathbb{R} = 2^{-1}(r_H/r_0)^2 = 0.85 \times 10^{120}$. Both order values greatly exceed the above flat-space value. Interestingly, they are purely geometric measures, depending solely upon ratio r_H/r_0. Remarkably, the free-fall value $\sim 10^{120}$ of $\mathbb{R}$ approximates the negentropy of a universe modeled as discrete. This might mean that the Universe contains about equal amounts of continuous and discrete structure.

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I. WHAT IS ORDER?

Consider a closed system of *continuous* coordinates $\mathbf{x} = (x_1, \ldots, x_K)$, K arbitrarily obeying a probability density function (PDF) $p = p(\mathbf{x})$ for some property such as mass.

A. Candidate measures of order

The order of, by comparison, a *discrete* system, characterized by a probability law P_i, $i = 1, \ldots, N$ of discrete states i, is long known to be its "negentropy,"

$$\mathbb{R}_H \equiv +\sum_{i=1}^{N} P_i \ln P_i. \quad (1)$$

See, e.g., Ref. [1]. This measure satisfies our defining criterion that it decrease under coarse graining (see below). More generally, consider the Kullback-Leibler or "cross entropy" measure $H_{\rm KL} \equiv \sum_i P_i \ln(P_i/Q_i)$. Probability law Q_i is some chosen "reference" law, and $H_{\rm KL}$ is a measure of the "distance between" the two laws. It was shown [2] that $H_{\rm KL}$ also satisfies the coarse-graining criterion.

But our system $p(\mathbf{x})$ *is continuous*, not discrete. And the measure (1) *cannot* be converted into continuous form: its continuous limit $P_i \to p(\mathbf{x})d\mathbf{x}$ is not well defined, since then $\mathbb{R}_H \to \sum_i p(\mathbf{x}_i)d\mathbf{x} \ln p(\mathbf{x}_i)d\mathbf{x}$, which contains the unbounded term $\sum_i p(\mathbf{x}_i)d\mathbf{x} \ln d\mathbf{x} \to 1 \times \ln d\mathbf{x} = \sum_i \ln dx_i \to -\infty$.

However, the measure $H_{\rm KL}$ *does* hold in the continuous limit. Then the cross entropy $H_{\rm KL}$ remains a candidate measure of order for our continuous system.

B. Fisher order measure

In recent work [3,4] the order in this *continuous case* was found to be

$$\mathbb{R} = 8^{-1}L^2 I, \quad \text{where } I = 4\int d\mathbf{x}\nabla\psi^* \cdot \nabla\psi,$$
$$\psi = \psi(\mathbf{x}), \quad p \equiv |\psi|^2, \quad (2)$$

again using the coarse-graining requirement. Here I is Fisher's information measure [5,6]. Amplitude function $\psi(\mathbf{x})$ is generally complex, and the asterisk denotes a complex conjugate. In a three-dimensional system, parameter L is the maximum chord length connecting two system surface points [4]. That the order $\mathbb{R}$ in measure (2) should increase with system size L is intuitive. For example, consider the order in a typical apartment building of height L. The structural details of each story is ordinarily about the same. This suggests that the total order should increase with the number of stories L.

Still, the problem remains that with both measures $\mathbb{R}$ and $H_{\rm KL}$ satisfying the coarse-graining requirement, which should be used? In this regard note that $H_{\rm KL}$ contains an arbitrary reference law Q_i. However, to apply the mathematical measure consistently $H_{\rm KL}$ to all systems requires deciding upon *one definite* law Q_i. A possibility is $Q_i = \text{const.}$, making $H_{\rm KL}$ effectively the negentropy $\mathbb{R}_H$. However this choice is disqualified since (as found) it becomes unbounded for our continuous coordinates.

Next, consider the choice $Q_i = p(\mathbf{x}_i - \Delta\mathbf{x})$, $\Delta x \to 0$. This reference function is simply the infinitesimally shifted system PDF. This causes [7] the proportionality $H_{\rm KL} \propto I$. Then by Eq. (2) $H_{\rm KL} \propto \mathbb{R}$, and *the two measures converge*to the common measure (2), which we use.

II. COARSE GRAINING

A coarse-graining operation on a system merges its finest microstates into larger or coarser macrostates [2]. Thus, a coarse-grained description of a system is limited to its grosser subcomponents.

The measure $\mathbb{R}$ obeying form (2) derives [3,4] uniquely from a single physical requirement: *After a continuous system* $p(x)$ *is coarse grained during a tiny time interval* Δt, $\mathbb{R}$ *must either decrease or stay the same*,

$$\Delta\mathbb{R} \leqslant 0 \quad \text{for} \quad \Delta t > 0. \tag{3}$$

Uses of $\mathbb{R}_H$ as an alternative order measure are limited to systems that are very close to thermodynamic equilibrium. But of course most natural systems are *not* near equilibrium. It is therefore apt that in the derivation [3,4] of Eq. (2), $p(\mathbf{x})$ is by definition an *arbitrary* system state.

Coarse graining is often used for describing the transition from a quantum variable to its observed value or, more grandly, from a quantum universe to a classical one [8–10]. Indeed, our "holographic universe" below will be seen to incorporate into its definition *both* classical (relativistic) and quantum (string) features.

III. SOME PROPERTIES OF THE FISHER-BASED ORDER MEASURE

Inspection of Eq. (2) shows that $\mathbb{R}$ is (a) unitless which, like negentropy, has the benefit of allowing completely different phenomena to be compared for their levels of order; (b) invariant under uniform stretch $y_k = a_k x_k$, $k = 1,\ldots,K$, with the $a_k =$ const., of all its coordinates (showing that $\mathbb{R}$ is an intensive quantity); and (c) measures *the total number* of ordered "details" within the system, rather than the density of their structure. For example, for a one-dimensional system $p(x) = (2/a)\sin^2(n\pi x/a)$, $0 \leqslant x \leqslant a$, the order $\mathbb{R} = 4\pi^2 n^2$. This is explicitly independent of system extension a, and purely a rapidly increasing function of the *total number* n of waves within the system, i.e., its structural complexity.

A. Comparisons with the Kolmogoroff-Chaitin measure

Parameter n is also the number of nodes (zeros) in the above sinusoidal system $p(x)$. The Kolmogoroff-Chaitin (K-C) complexity [11] is another widely used measure of complexity. The K-C is likewise proportional to n^2 in application to the Dijkstra [12] routing algorithm. The K-C measure also equals the *total number n* of statements in a computer program; the *total number n* of switches within a network; or the shortest description of a string in some fixed universal language.

B. Past applications of Fisher order measure $\mathbb{R}$

Past applications of $\mathbb{R}$ have been to biology [4], in evaluating the order of an E coli bacterium undergoing compression, and in cosmology: Despite the common impression that the Hubble expansion of the Universe causes a relentless loss of its order, it was shown [13] that the order $\mathbb{R}$ obeying Eq. (2) actually *remains constant* under Hubble expansion. This assumes the Universe to obey the Robertson-Walker metric

$$ds^2 = (cdt)^2 - a^2(t)(dx^2 + dy^2 + dz^2), \quad (x,y,z) \equiv \mathbf{r}. \tag{4}$$

This constant value of $\mathbb{R}$ was later found [14], assuming flat space and ignoring general relativity (GR). However, any cosmological model includes the presence of mass, and mass implies *space curvature*, by GR [15]. Our model, below, will include the GR effects of space-time curvature in application to (a) a freely falling (geodesic) observer, and (b) a stationary observer.

IV. APPLICATION TO HOLOGRAPHIC UNIVERSE WITH STRING STRUCTURE

A. The holostar

Bekenstein [16] proposed that the entropy of a black hole (bh) is proportional to the area of its event horizon in Planck units ($c = G = \hbar = k = 1$). A recently discovered, exact, non-bh solution to the Einstein field equations shares this property [17]. The solution, called the "holostar," also satisfies Eq. (2) for $\mathbb{R}$, requiring that the system be continuous and of finite size L. It consists of matter with a string equation of state bounded by a *real* spherical membrane located at radius r_H. The metric outside r_H is the Schwarzschild vacuum metric.

To serve as a realistic model for the observable Universe, the radius r_H of the boundary membrane must be larger than the current local horizon (or Hubble) radius of the observable part of the Universe. The junction conditions at the radial position r_H of the boundary membrane require $r_H = 2M + r_0$, where M is the gravitational mass of the holostar, as measured by an observer in the exterior Schwarzschild metric. This boundary membrane has the same properties as the (purely fictitious) "membrane" attributed to a *Schwarzschild bh* by the membrane paradigm.

The interior matter density obeys an equation of state for radially aligned strings. To avoid infinite matter density at the origin $r = 0$, we regard the interior region $r \leqslant r_0$, where $r_0 \sim$ Planck length, as a Minkowski vacuum of *zero* gravitational mass M (by $r_H = 2M + r_0$ with $r_H = r_0$). This also obeys Birckhoff's theorem. This justifies replacement of this region with a Minkowski vacuum region without changing the metric for $r > r_0$. The proviso is a second real (inner) boundary membrane at $r = r_0$, with a fixed level of surface tension. Thus, mass exists only over the radial interval $r_0 \leqslant r \leqslant r_H$.

This holostar model well fits all currently known cosmological data. For example, its nearly unaccelerated expansion is compatible with recent supernova measurements [17]. However, as cautionary notes, (i) the model has not yet been shown to explain milestone effects such as big bang nucleosynthesis constraints or cosmic microwave background features; and the approach here is not fully rigorous because (ii) Eq. (2) for $\mathbb{R}$ is not derived within the context of general relativity and (iii) the inverse-square law form for $\rho(r)$ is (as below) ultimately a heuristic assumption.

The holographic metric within the region of nonzero matter density $r_0 \leqslant r \leqslant r_H$ is given by

$$ds^2 = \left(\frac{r_0}{r}\right)dt^2 - \left(\frac{r}{r_0}\right)dr^2 - r^2 d\Omega. \tag{5}$$

B. A simple model for the Universe

The geometric properties of this new solution allow it to serve *as an alternative model for the Universe*. Far away from the holostar's center, the geodesic motion of massive particles is virtually indistinguishable from that of a uniformly stretched (expanding or contracting) Friedmann-Robertson-Walker (FRW) universe. We want to know the level of order $\mathbb{R}$ that is contained within such a universe over its radial values $r_0 \leqslant r \leqslant r_H$.

In order to accommodate GR, the calculation of order $\mathbb{R}$ must take into account that the holostar space-time is curved. In the metric given by Eq. (5) the curvature produces length distortions only in the radial direction ($ds = \sqrt{r/r_0}\, dr = \sqrt{g_{rr}}\, dr$); the azimuthal ϕ and polar θ coordinates have the same meaning as spherical coordinates in flat space. Therefore all integrations (derivatives) in the second Eq. (2), must be multiplied (divided) by the square root of the radial metric coefficient $\sqrt{g_{rr}} = \sqrt{r/r_0}$.

C. Dependence of order upon relative motion of observer

The metric-induced distortion $ds = \sqrt{r/r_0}\, dr$ along the radial direction must be complemented by *special relativistic* effects due to the motion of the observer within the metric. In Ref. [17] it was shown that a geodesically moving (free-falling) observer moves ultrarelativistically and nearly radially through the coordinate system. Assume that any object falling into the holostar from far away initially has a rather low velocity, i.e., a γ factor of order 1. Then [15] the relative radial velocity component $\beta_r \equiv v_r/c$ in the holostar's interior obeys $\beta_r^2 = 1 - r_0/r$. This approaches 1 (speed of light) whenever $r \gg r_0$ (far from the center r_0), indicating *ultrarelativistic motion* except near the center.

It is convenient to use the special relativistic *gamma* factor obeying $1/\gamma^2 = 1 - \beta^2$, so that $\gamma_r^2 = r/r_0$ in this case. Consequently, its Lorentz contraction will transform any proper radial distance ds measured in the stationary (r,θ,ϕ) coordinate system into a much larger distance $dl = \gamma_r\, ds$ in the system of the *free-falling* observer. Multiplying the radial metric coefficient g_{rr} in the integral for the free-falling observer by a factor $\gamma_r^2 = r/r_0$ converts it into the corresponding integral for the free-falling observer.

As one would expect, the (a) free-fall and (b) stationary observers will "see" different levels of order $\mathbb{R}$. These are found below.

D. Choice of density function $p(r)$

To find $\mathbb{R}$ we need an appropriate density function $p(r)$. The energy density $\rho(r)$ of the string-type matter at any radial position r on interval (r_0,r_H) is the quantity in the 00 slot of the stress-energy tensor, i.e., $\rho(r) = 1/(8\pi r^2)$. Thus, the finiteness of r_0 avoids an infinity in $\rho(r)$.

However, there are problems with representing $p(r)$ by the energy density $\rho(r)$ of the string-type matter. First, ρ is a *component* of the rank-2 stress-energy tensor. In order to compute I we must take the square root of the density function, but the square root of a *single* component of a multi-component tensor is not well defined. Second, in GR a *single* component of a multicomponent tensor does not have any coordinate-independent meaning; the results would then generally depend on the particular coordinate system used. Both objections are overcome by heuristically choosing for $p(r)$ *the trace* of the stress-energy tensor $T = \rho - P_r - 2P_\perp = 1/(4\pi r^2)$. Finally, it is interesting that the density function $p(r) \propto 1/r^2$ agrees with that proposed for a typical galaxy [18].

V. CALCULATION OF ORDER

The calculation of order $\mathbb{R}$ is carried through for the two metric cases g_{rr} of an observer in (a) free fall and (b) stationary in the coordinate system. In both cases we calculate the order over total interval $r_0 \leqslant r \leqslant r_H$, regardless of any limitations imposed on an interior observer due to his local horizon radius. Stepwise:

(1) The density function $p(r) \propto 1/r^2 \equiv A^2/r^2$ is normalized, fixing the constant A. Next, its square root is taken to give the amplitude function $\psi(r)$, assuming that its phase is zero (quantum effects ignored). The latter is justified at regions $r \gg r_0$ of low curvature, where a huge number of quantum strings exist whose quantum states, when coarse grained, behave like classical macroscopic states by Ehrenfest's theorem.

(2) The gradient $\nabla\psi$ is then taken. The result is substituted into Eq. (2) to compute I.

(3) Finally, chord length L is computed, and the results are substituted into the first Eq. (2) to give $\mathbb{R}$.

A. Order of free-falling observer, metric $g_{rr} = (r/r_0)^2$

Here, in case (a), the proper distance dl is obtained by the application of two transformations, one general- and one special-relativistic: First we translate radial coordinate distances dr into proper distances ds, which requires multiplying dr by the square root of the radial metric coefficient, e.g., $ds = \sqrt{r/r_0}\, dr$. However, ds is the proper distance measured by an observer *at rest* in the (r,θ,ϕ) coordinate system. In order to transform to the free-fall frame, we must multiply ds by the special relativistic gamma factor of the motion, $dl = \gamma\, ds$. Coincidentally $\gamma = \sqrt{r/r_0}$ is exactly equal to the radial metric coefficient g_{rr}. Formally we can absorb the second factor into the metric coefficient $g_{rr} \to \gamma^2 g_{rr}$, resulting in a modified radial metric coefficient $g_{rr} = (r/r_0)^2$ for the free-falling observer.

(1) Normalization requires

$$1 \equiv 4\pi \int_{r_0}^{r_H} dr \sqrt{g_{rr}}(r) r^2 p(r) = 4\pi \int_{r_0}^{r_H} dr (r/r_0) r^2 p(r). \quad (6)$$

Using $p(r) \equiv A^2/r^2$ gives

$$p(r) = A^2/r^2, \quad A^2 = \frac{r_0}{2\pi\left(r_H^2 - r_0^2\right)}, \quad (7)$$

and the amplitude

$$\psi(r) \equiv \sqrt{p(r)} = A/r \quad (8)$$

assuming zero phase.

(2) Next, we have to evaluate $\nabla\psi(r) = d\psi/ds = d\psi/(\sqrt{g_{rr}}\, dr) = (r_0/r) d\psi/dr$, so by Eq. (8),

$$\nabla\psi(r) = \frac{r_0}{r}\frac{d}{dr}\left(\frac{A}{r}\right) = -\frac{r_0 A}{r^3}. \quad (9)$$

Then by Eq. (2),

$$I = 4\int d\mathbf{x}\nabla\psi^*\cdot\nabla\psi = 4(4\pi)\int_{r_0}^{r_H} dr(r/r_0)r^2\left(\frac{r_0 A}{r^3}\right)^2 = 4r_H^{-2}. \quad (10)$$

(3) With the central region $r \leqslant r_0$ being a Minkowski vacuum, the longest chord L just passes straight through the center region (with a diameter of $2r_0$), so its length is simply

$$L = 2r_0 + 2\int_{r_0}^{r_H}\sqrt{g_{rr}}dr = \frac{r_H^2 + r_0^2}{r_0}. \quad (11)$$

Finally, using results (10) and (11) in Eq. (2) gives an order

$$\mathbb{R} = 8^{-1}L^2 I = \frac{1}{2}\left(\frac{r_H}{r_0} + \frac{r_0}{r_H}\right)^2 = \frac{1}{2}\left(\frac{r_H}{r_0}\right)^2 \quad (12)$$

since $r_H \gg r_0$. The free-falling observer in the holostar "sees" the order going as the square of the very large ratio r_H/r_0.

As was mentioned, the boundary radius r_H must be larger than the local horizon (Hubble) radius of the currently observable part of the Universe. Not knowing how much larger it is, to get actual numbers we use the Hubble radius for r_H. Since all expressions for order $\mathbb{R}$ increase with r_H, the resulting values for $\mathbb{R}$ will be *lower bounds* to the true order. Using the current values $r_H = 13.76 \times 10^9$ light-years $= 1.302 \times 10^{26}$ m, and $r_0 = 1.00 \times 10^{-35}$ m, the order value (12) is

$$\mathbb{R} = 0.85 \times 10^{120}. \quad (13)$$

B. Order of stationary observer, metric $g_{rr} = (r/r_0)$

In this case (b) the same three steps as above were followed. For brevity, we just give the results:

(1) Here we have $ds = \sqrt{g_{rr}}\,dr$, with the unmodified metric $g_{rr} = (r/r_0)$, giving

$$\psi(r) \equiv \sqrt{p(r)} = A/r. \quad (14)$$

(2) Next, we evaluate $\nabla\psi(r) = d\psi/(\sqrt{g_{rr}}\,dr) = (r_0/r)^{1/2}\,d\psi/dr$. By Eq. (2),

$$I = 4\int d\mathbf{x}\nabla\psi^*\cdot\nabla\psi = 4(4\pi)\int_{r_0}^{r_H} dr(r/r_0)^{1/2}r^2\left(\frac{r_0^{1/2}A}{r^{5/2}}\right)^2 = 4r_0^{-1/2}r_H^{-3/2}. \quad (15)$$

(3) The longest chord length L, as in Eq. (11), is now

$$L = 2r_0 + 2\int_{r_0}^{r_H}\sqrt{g_{rr}}dr = \frac{4}{3}\frac{r_H^{3/2}}{r_0^{1/2}} + \frac{2}{3}r_0. \quad (16)$$

Finally, using results (18) and (19) in Eq. (2) gives

$$\mathbb{R} = \frac{8}{9}\left(\frac{r_H}{r_0}\right)^{3/2}, \quad (17)$$

after using $r_H \gg r_0$. For the stationary observer in the holostar, the order goes as the $3/2$ power of the large ratio (r_H/r_0). This power $3/2$ dependence upon (r_H/r_0) is significantly slower than in Eq. (12) for the freely falling observer.

Using the current values $r_H = 13.76 \times 10^9$ light-years $= 1.302 \times 10^{26}$ m, and $r_0 = 1.00 \times 10^{-35}$ m, the order value (20) is

$$\mathbb{R} = 0.42 \times 10^{90}. \quad (18)$$

C. Comparison of results

The strong difference between theoretical forms (12) and (17) for the order, and their numerical values (13) and (18), show that, as with other relativistic effects, the order depends upon the relative motion of source and observer. Oddly enough, the free-falling observer sees effectively the same level of order whether he is outside or inside the holostar. He also "sees" much more order than the stationary one. Why is this?

By Eq. (2) the order depends upon the values of L^2 and I. Comparing results (11) and (16) indicates that the stationary observer sees the holostar as having an apparent size L that is much less (by factor $\sqrt{r_0/r_H}$) than that of the free falling observer. Then the contribution of L to $\mathbb{R}$ in Eq. (2) gives advantage to the free-falling observer. However, comparing results (10) and (15) for values of I gives advantage to the stationary observer. The key effect is that definition (2) of $\mathbb{R}$ depends much more strongly (quadratically) upon the size of L than upon I, so that values (13) and (18) for $\mathbb{R}$ give advantage to the free-falling observer.

VI. DISCUSSION

Results (12) and (17) show that the structural order in the holostar model of the Universe is a *purely geometrical* property. It depends only upon the holostar's size r_H. Of course all local baryonic and nonbaryonic matter-energy configurations such as clusters of galaxies and stars must inevitably contribute to total order, so that Eqs. (12) and (17)—which ignore these—actually give lower bounds to the total. However, the holostar does include small-scale structure, namely, near its center where $r \to$ Planck length r_0, so that gradients $\nabla\psi \sim r^{-3}$ or $r^{-5/2}$ [cases (a) or (b)] are immense, strongly increasing order (2).

That $\mathbb{R}$ in Eqs. (12) and (17) depends upon *both* boundary values r_0 and r_H is consistent with the "holographic principle," by which the physics within the hologram depends upon the *entire boundary* of the universe.

The relativistic levels $\mathbb{R} = 0.85 \times 10^{120}$ in Eq. (13) and $\mathbb{R} = 0.42 \times 10^{90}$ in Eq. (18) are to be compared with the *much smaller* value $\mathbb{R} = 26 \times 10^{60}$ recently found [14], which ignored all effects of GR. As we find, allowing for the curvature of space-time greatly increases the level of order.

Finally in Ref. [19] the maximum possible negentropy $\mathbb{R}_H$ of *a discrete* universe is computed by Penrose as roughly 10^{120}, assuming it eventually collapses into one massive bh. *Interestingly, this coincides with result (13) for the order* $\mathbb{R}$ *of our continuous holostar universe*. That the two *mathematically different* measures (1) and (2) numerically agree is interesting, suggesting that the Universe contains about equal levels of discrete and continuous structure. There is also a fundamental *physical* agreement since both the holostar and Penrose models obey the Bekenstein entropy-area property.

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