Reanalysis of the hydrodynamic theory of fluid, polar-ordered flocks

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I reanalyze the hydrodynamic theory of fluid, polar-ordered flocks. I find new linear terms in the hydrodynamic equations which slightly modify the anisotropy, but not the scaling, of the damping of sound modes. I also find that the nonlinearities allowed *in equilibrium* do not stabilize long-ranged order in spatial dimensions $d = 2$, in accord with the Mermin-Wagner theorem. Nonequilibrium nonlinearities *do* stabilize long-ranged order in $d = 2$, as argued by earlier work. Some of these were missed by earlier work; it is unclear whether or not they change the scaling exponents in $d = 2$.

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I. INTRODUCTION

One of the most interesting and ubiquitous phenomena exhibited by active matter is collective motion, variously known as flocking [1–6], swarming [7,8], or by a variety of other names. Such motion can be coherent over enormous numbers of self-propelled entities, and a wide range of of length scales: from kilometers (herds of wildebeest) to microns (microorganisms [7,8]; mobile macromolecules in living cells [9,10]). It is also [2] a dynamical version of ferromagnetic ordering. A "hydrodynamic" theory of flocking [3–6] shows among other things that, while useful, the analogy to equilibrium ferromagnets is far from perfect. In particular, flocks do *not* obey the Mermin-Wagner theorem [11]: that is, they *do* spontaneously break a continuous symmetry (rotation invariance) by developing long-ranged order [in the case of flocks, by developing a nonzero average velocity $\langle\vec{v}(\vec{r},t)\rangle \neq \vec{0}$] in spatial dimensions $d = 2$, even with only short-ranged interactions.

The mechanism for this apparent violation of the "Mermin-Wagner" theorem [11] is fundamentally nonlinear. A number of nonlinear terms in the hydrodynamic equations of motion become "relevant," in the renormalization group (RG) sense, as the spatial dimension d is lowered below 4, leading to a breakdown of linearized hydrodynamics [12] which suppresses fluctuations enough to stabilize long-ranged order possible in $d = 2$.

In this paper, I revisit the formulation of the hydrodynamic theory of what I'll call "fluid, polar-ordered flocks", by which I mean flocks that are spatially homogeneous, on average, and have $\langle\vec{v}(\vec{r},t)\rangle \neq \vec{0}$. I find a few differences with the results of [3–6]. Some of these are minor: a few linear terms, that produce only minor modifications of the damping of the propagating sound modes predicted in [3–6], were missed in that earlier work.

My more important conclusions concern the scaling laws of two-dimensional flocks. It was originally argued [3–6] that the exponents characterizing the scaling of fluctuations in flocks that results from the breakdown of hydrodynamics could be determined exactly in $d = 2$. In this paper, I will argue that those arguments were incorrect, because they neglected certain other, equally important, symmetry-allowed nonlinearities in the hydrodynamic equations. These additional nonlinearities invalidate the earlier arguments, and render it impossible to determine the exact scaling laws in $d = 2$, or, indeed, in any spatial dimension $d \leqslant 4$.

If these new nonlinearities should prove to be irrelevant, in the RG sense, in $d = 2$, then the exact exponents predicted by [3–6] *would*, in fact, hold in $d = 2$. At the moment, however, there is no compelling theoretical argument that they are irrelevant, though there is also none that they are not.

All of these nonlinearities involve density fluctuations. Hence, in systems in which density fluctuations are suppressed, it *is* possible to obtain exact exponents in $d = 2$. One class of such systems—flocks with birth and death—has been treated elsewhere [13]; others, such as incompressible systems [14], and systems with long-ranged interactions [15], will be addressed in future work [16].

The reanalysis presented here correctly predicts that one naively relevant nonlinearity in the flocking hydrodynamic equations that is allowed even in equilibrium systems [17] does *not* lead to any corrections to scaling (or, indeed, to any qualitatively new long-wavelength physics whatsoever); this means that the equilibrium systems described by such a model does *not* exhibit long-ranged order in $d = 2$ (in accord with the Mermin-Wagner theorem [11]).

My discussion here is limited to "ordered" flocks moving on a substrate: i.e., one in which the flocking organisms spontaneously pick a direction to move together via purely short-ranged interactions that make neighbors tend to follow each other, but which do *not* pick out any *a priori* preferred direction for this motion. That is, the flocking spontaneously breaks rotation invariance, as equilibrium ferromagnetism does. Flocks moving without a substrate conserve momentum, and so have very different hydrodynamics, which has been considered elsewhere [18]; I will not discuss these here. One specific realization of a flock on a substrate is the Vicsek algorithm [2] in its ordered state.

The remainder of this paper is organized as follows. In Sec. II, I derive the hydrodynamic model for a fluid, polar flock, correcting some mistakes in the analysis of [3–6]. Section III treats the linearized version of the theory presented in Sec. II, while Sec. IV addresses the nonlinear theory and scaling laws.

II. HYDRODYNAMIC MODEL IN THE FLUID, ORIENTATIONALLY ORDERED PHASE

The hydrodynamic theory describes the flock by continuous, coarse grained number density $\rho(\vec{r},t)$ and velocity $\vec{v}(\vec{r},t)$

fields. The hydrodynamic equations of motion governing these fields can in the long-wavelength limit can be written down purely on symmetry grounds [3–6], and are

$$\partial_t \vec{v} + \lambda_1(\vec{v}\cdot\vec{\nabla})\vec{v} + \lambda_2(\vec{\nabla}\cdot\vec{v})\vec{v} + \lambda_3\vec{\nabla}(|\vec{v}|^2) = \alpha\vec{v} - \beta|\vec{v}|^2\vec{v} - \vec{\nabla}P_1 - \vec{v}(\vec{v}\cdot\vec{\nabla}P_2) + D_1\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_T\nabla^2\vec{v} + D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v} + \vec{f}, \tag{2.1}$$

$$\partial_t\rho + \nabla\cdot(\vec{v}\rho) = 0, \tag{2.2}$$

where the parameters $\lambda_i (i = 1 \to 3)$, α, β, and the "isotropic pressure" $P(\rho,|\vec{v}|)$ and the "anisotropic pressure" $P_2(\rho,|\vec{v}|)$ are, in general, functions of the density ρ and the magnitude $|\vec{v}|$ of the local velocity. It is useful to Taylor expand P_1 and P_2 around the equilibrium density ρ_0:

$$P_1 = \sum_{n=1}^{\infty}\sigma_n(|\vec{v}|)(\rho-\rho_0)^n, \tag{2.3}$$

$$P_2 = P_2(\rho,|\vec{v}|) = \sum_{n=1}^{\infty}\mu_n(|\vec{v}|)(\rho-\rho_0)^n. \tag{2.4}$$

Here β, D_1, D_2, and D_T are all positive, and $\alpha < 0$ in the disordered phase and $\alpha > 0$ in the ordered state (in mean field theory).

The α and β terms simply make the local $\vec{v}$ have a nonzero magnitude $v_0 = \sqrt{\frac{\alpha}{\beta}}$ [19] in the ordered phase, where $\alpha > 0$. $D_{1,T,2}$ are the diffusion constants (or viscosities) reflecting the tendency of a localized fluctuation in the velocities to spread out because of the coupling between neighboring "birds." The $\vec{f}$ term is a random driving force representing the noise. It is assumed to be Gaussian with white noise correlations:

$$\langle f_i(\vec{r},t)f_j(\vec{r'},t')\rangle = \Delta\delta_{ij}\delta^d(\vec{r}-\vec{r'})\delta(t-t'), \tag{2.5}$$

where Δ is a constant, and i, j denote Cartesian components. The pressure P tends, as in an equilibrium fluid, to maintain the local number density $\rho(\vec{r})$ at its mean value ρ_0, and $\delta\rho = \rho - \rho_0$. The anisotropic pressure $P_2(\rho,|\vec{v}|)$ in Eq. (2.1) is only allowed due to the nonequilibrium nature of the flock; in an equilibrium fluid such a term is forbidden, since Pascal's law ensures that pressure is isotropic. In the nonequilibrium steady state of a flock, no such constraint applies. In earlier work [3–6], this term was ignored. Here I will show that this term changes none of the predictions of the hydrodynamic theory.

The final equation (2.2) is just conservation of bird number: the birds do not reproduce or die on the wing. The interesting and novel results that arise when this constraint is relaxed by allowing birth and death while the flock is moving have been discussed elsewhere [13].

The hydrodynamic model embodied in Eqs. (2.1)–(2.3) and (2.5) is equally valid in both the "disordered" (i.e., nonmoving) ($\alpha < 0$) and "ferromagnetically ordered" (i.e., moving) ($\alpha > 0$) state. Here I am interested in the ferromagnetically ordered, broken-symmetry phase which occurs for $\alpha > 0$. In this state, the velocity field can be written as

$$\vec{v} = v_0\hat{x}_{\parallel} + \vec{\delta v} = (v_0 + \delta v_{\parallel})\hat{x}_{\parallel} + \vec{v}_{\perp}, \tag{2.6}$$

where $v_0\hat{x}_{\parallel} = \langle\vec{v}\rangle$ is the spontaneous average value of $\vec{v}$ in the ordered phase, and the fluctuations $\delta v_{\parallel}$ and $\vec{v}_{\perp}$ of $\vec{v}$ about this mean velocity along and perpendicular to the direction of the mean velocity are assumed to be small. Indeed, I will shortly be expanding the equation of motion (2.1) in these quantities. (I will also hereafter be using the subscripts $_\parallel$ and $\perp$ to denote components of any vector along and perpendicular to the mean velocity $\langle\vec{v}\rangle$.) Taking $v_0 = \sqrt{\frac{\alpha}{\beta}}$ as discussed above [19], and taking the dot product of both sides of Eq. (2.1) with $\vec{v}$ itself, I obtain

$$\tfrac{1}{2}(\partial_t|\vec{v}|^2 + (\lambda_1+2\lambda_3)(\vec{v}\cdot\vec{\nabla})|\vec{v}|^2) + \lambda_2(\vec{\nabla}\cdot\vec{v})|\vec{v}|^2 = (\alpha-\beta|\vec{v}|^2)|\vec{v}|^2 - \vec{v}\cdot\vec{\nabla}P - |\vec{v}|^2\vec{v}\cdot\vec{\nabla}P_2 + D_1\vec{v}\cdot\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_T\vec{v}\cdot\nabla^2\vec{v} + D_2\vec{v}\cdot((\vec{v}\cdot\vec{\nabla})^2\vec{v}) + \vec{v}\cdot\vec{f}. \tag{2.7}$$

In this hydrodynamic approach, we are interested only in fluctuations $\vec{\delta v}(\vec{r},t)$ and $\delta\rho(\vec{r},t)$ that vary slowly in space and time. [Indeed, the hydrodynamic equations (2.1) and (2.2) are only valid in this limit.] Hence, terms involving space and time derivatives of $\vec{\delta v}(\vec{r},t)$ and $\delta\rho(\vec{r},t)$ are always negligible, in the hydrodynamic limit, compared to terms involving the same number of powers of fields without any time or space derivatives.

Furthermore, the fluctuations $\vec{\delta v}(\vec{r},t)$ and $\delta\rho(\vec{r},t)$ can themselves be shown to be small in the long-wavelength limit. Hence, we need only keep terms in Eq. (2.7) up to linear order in $\vec{\delta v}(\vec{r},t)$ and $\delta\rho(\vec{r},t)$. The $\vec{v}\cdot\vec{f}$ term can likewise be dropped, since it only leads to a term of order $\vec{v}_{\perp}f_{\parallel}$ in the $\vec{v}_{\perp}$ equation of motion, which is negligible (since $\vec{v}_{\perp}$ is small) relative to the $\vec{f}_{\perp}$ term already there.

These observations can be used to eliminate many of the terms in Eq. (2.7), and solve for the quantity

$$U \equiv \alpha(\rho,|\vec{v}|) - \beta(\rho,|\vec{v}|)|\vec{v}|^2; \tag{2.8}$$

the solution is

$$U = \lambda_2\vec{\nabla}\cdot\vec{v} + \vec{v}\cdot\vec{\nabla}P_2 + \frac{\sigma_1}{v_0}\partial_{\parallel}\delta\rho + \frac{1}{2v_0}\left(\partial_t + \gamma_2\partial_{\parallel}\right)\delta v_{\parallel}, \tag{2.9}$$

where I have defined

$$\gamma_2 \equiv (\lambda_1 + 2\lambda_3)v_0. \tag{2.10}$$

Inserting this expression (2.9) for U back into Eq. (2.1) [where U appears by virtue of its definition (2.8)], I find that P_2 and λ_2 cancel out of the $\vec{v}$ equation of motion, leaving

$$\partial_t\vec{v} + \lambda_1(\vec{v}\cdot\vec{\nabla})\vec{v} + \lambda_3\vec{\nabla}(|\vec{v}|^2) = \frac{\sigma_1}{v_0}\vec{v}(\partial_{\parallel}\delta\rho) - \vec{\nabla}P + D_1\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_T\nabla^2\vec{v} + D_2(\vec{v}\cdot\vec{\nabla})^2\vec{v} + \left[\frac{1}{2v_0}(\partial_t + \gamma_2\partial_{\parallel})\delta v_{\parallel}\right]\vec{v} + \vec{f}. \tag{2.11}$$

This can be made into an equation of motion for $\vec{v}_{\perp}$ involving only $\vec{v}_{\perp}(\vec{r},t)$ and $\delta\rho(\vec{r},t)$ by projecting perpendicular to the direction of mean flock motion $\hat{x}_{\parallel}$, and eliminating $\delta v_{\parallel}$ using Eq. (2.9) and the expansion

$$U \approx -\Gamma_1\left(\delta v_{\parallel} + \frac{|\vec{v}_{\perp}|^2}{2v_0}\right) - \Gamma_2\delta\rho, \tag{2.12}$$

where I have defined

$$\Gamma_1 \equiv -\left(\frac{\partial U}{\partial|\vec{v}|}\right)^0_{\rho}, \quad \Gamma_2 \equiv -\left(\frac{\partial U}{\partial\rho}\right)^0_{|\vec{v}|}, \tag{2.13}$$

with, here and hereafter, super- or subscripts 0 denoting functions of ρ and $|\vec{v}|$ evaluated at $\rho = \rho_0$ and $|\vec{v}| = v_0$. I have also used the expansion (2.6) for the velocity in terms of the fluctuations $\delta v_\parallel$ and $\vec{v}_\perp$ to write

$$|\vec{v}| = v_0 + \delta v_\parallel + \frac{|\vec{v}_\perp|^2}{2v_0} + O(\delta v_\parallel^2, |\vec{v}_\perp|^4), \tag{2.14}$$

and kept only terms that an RG analysis shows to be relevant in the long-wavelength limit. Inserting (2.12) into (2.9) gives

$$\begin{aligned} -\Gamma_1\left(\delta v_\parallel + \frac{|\vec{v}_\perp|^2}{2v_0}\right) - \Gamma_2\delta\rho &= \lambda_2\vec{\nabla}_\perp\cdot\vec{v}_\perp + \lambda_2\partial_\parallel\delta v_\parallel + \frac{(\mu_1 v_0^2+\sigma_1)}{v_0}\partial_\parallel\delta\rho \\ &\quad + \frac{1}{2v_0}(\partial_t + \gamma_2\partial_\parallel)\delta v_\parallel, \end{aligned} \tag{2.15}$$

where I have kept only linear terms on the right-hand side of this equation, since the nonlinear terms are at least of order derivatives of $|\vec{v}_\perp|^2$, and hence negligible, in the hydrodynamic limit, relative to the $|\vec{v}_\perp|^2$ term explicitly displayed on the left-hand side.

This equation can be solved iteratively for $\delta v_\parallel$ in terms of $\vec{v}_\perp$, $\delta\rho$, and its derivatives. To lowest (zeroth) order in derivatives, $\delta v_\parallel \approx -\frac{\Gamma_2}{\Gamma_1}\delta\rho$. Inserting this approximate expression for $\delta v_\parallel$ into Eq. (2.15) everywhere $\delta v_\parallel$ appears on the right-hand side of that equation gives $\delta v_\parallel$ to first order in derivatives:

$$\begin{aligned} \delta v_\parallel \approx& -\frac{\Gamma_2}{\Gamma_1}\left(\delta\rho - \frac{1}{v_0\Gamma_1}\partial_t\delta\rho + \frac{\lambda_4\partial_\parallel\delta\rho}{\Gamma_2}\right) \\ &-\frac{\lambda_2}{\Gamma_1}\vec{\nabla}_\perp\cdot\vec{v}_\perp - \frac{|\vec{v}_\perp|^2}{2v_0}, \end{aligned} \tag{2.16}$$

where I have defined

$$\begin{aligned} \lambda_4 &\equiv \frac{(\mu_1 v_0^2+\sigma_1)}{v_0} - \frac{\Gamma_2}{\Gamma_1}\left(\lambda_2 + \frac{\gamma_2}{v_0}\right) \\ &= \frac{(\mu_1 v_0^2+\sigma_1)}{v_0} - \frac{\Gamma_2}{\Gamma_1}(\lambda_1+\lambda_2+2\lambda_3). \end{aligned} \tag{2.17}$$

In deriving the second equality in (2.17), I've used the definition (2.10) of γ_2.

Inserting (2.6), (2.14), and (2.16) into the equation of motion (2.11) for $\vec{v}$, and projecting that equation perpendicular to the mean direction of flock motion $\hat{x}_\parallel$ gives, neglecting "irrelevant" terms,

$$\begin{aligned} \partial_t\vec{v}_\perp + \gamma\partial_\parallel\vec{v}_\perp + \lambda_1^0(\vec{v}_\perp\cdot\vec{\nabla}_\perp)\vec{v}_\perp &= -g_1\delta\rho\partial_\parallel\vec{v}_\perp - g_2\vec{v}_\perp\partial_\parallel\delta\rho - \frac{c_0^2}{\rho_0}\vec{\nabla}_\perp\delta\rho - g_3\vec{\nabla}_\perp(\delta\rho^2) \\ &\quad + D_B\vec{\nabla}_\perp(\vec{\nabla}_\perp\cdot\vec{v}_\perp) + D_T\nabla_\perp^2\vec{v}_\perp + D_\parallel\partial_\parallel^2\vec{v}_\perp \\ &\quad + \nu_t\partial_t\vec{\nabla}_\perp\delta\rho + \nu_\parallel\partial_\parallel\vec{\nabla}_\perp\delta\rho + \vec{f}_\perp \end{aligned} \tag{2.18}$$

where I have defined

$$D_B \equiv D_1 + \frac{2v_0\lambda_3^0\lambda_2^0}{\Gamma_1}, \tag{2.19}$$

$$D_\parallel \equiv D_T + D_2 v_0^2, \tag{2.20}$$

$$\gamma \equiv \lambda_1^0 v_0, \tag{2.21}$$

$$g_1 \equiv v_0\left(\frac{\partial\lambda_1}{\partial\rho}\right)_0 - \frac{\Gamma_2\lambda_1^0}{\Gamma_1}, \tag{2.22}$$

$$g_2 \equiv \frac{\Gamma_2\gamma_2^0}{\Gamma_1 v_0} - \frac{\sigma_1}{v_0}, \tag{2.23}$$

$$g_3 \equiv \sigma_2 + \left(\frac{\Gamma_2}{\Gamma_1}\right)^2\lambda_3^0 - \left(\frac{\partial\lambda_3}{\partial\rho}\right)_0\frac{\Gamma_2 v_0}{\Gamma_1}, \tag{2.24}$$

$$c_0^2 \equiv \rho_0\sigma_1 - \frac{2\rho_0 v_0\lambda_3^0\Gamma_2}{\Gamma_1}, \tag{2.25}$$

$$\nu_t \equiv -\frac{2\Gamma_2\lambda_3^0}{\Gamma_1^2}, \tag{2.26}$$

(2.26)

and

$$\nu_\parallel \equiv \frac{2v_0\lambda_3^0\lambda_4^0}{\Gamma_1} + \frac{\Gamma_2 D_1}{\Gamma_1}. \tag{2.27}$$

Using (2.6) and (2.14) in the equation of motion (2.2) for ρ gives, again neglecting irrelevant terms,

$$\begin{aligned} \partial_t\delta\rho + \rho_o\vec{\nabla}_\perp\cdot\vec{v}_\perp + w_1\vec{\nabla}_\perp\cdot(\vec{v}_\perp\delta\rho) + v_2\partial_\parallel\delta\rho &= D_{\rho_\parallel}\partial_\parallel^2\delta\rho + D_{\rho_\perp}\nabla_\perp^2\delta\rho + D_{\rho v}\partial_\parallel(\vec{\nabla}_\perp\cdot\vec{v}_\perp) \\ &\quad + \phi\partial_t\partial_\parallel\delta\rho + w_2\partial_\parallel(\delta\rho^2) + w_3\partial_\parallel(|\vec{v}_\perp|^2), \end{aligned} \tag{2.28}$$

where I have defined

$$v_2 \equiv v_0 - \frac{\rho_0\Gamma_2}{\Gamma_1}, \tag{2.29}$$

$$\phi \equiv \frac{\Gamma_2\rho_0}{v_0\Gamma_1^2}, \tag{2.30}$$

$$w_2 \equiv \frac{\Gamma_2}{\Gamma_1}, \tag{2.31}$$

$$w_3 \equiv \frac{\rho_0}{2v_0}, \tag{2.32}$$

$$D_{\rho_\parallel} \equiv \frac{\rho_0\lambda_4^0}{\Gamma_1} = \frac{\rho_0}{\Gamma_1}\left(\frac{(\mu_1 v_0^2+\sigma_1)}{v_0} - \frac{\Gamma_2}{\Gamma_1}(\lambda_1^0+\lambda_2^0+2\lambda_3^0)\right), \tag{2.33}$$

and, last but by no means least,

$$D_{\rho v} \equiv \frac{\lambda_2^0\rho_o}{\Gamma_1}. \tag{2.34}$$

The parameter $D_{\rho_\perp}$ is actually zero at this point in the calculation, but I have included it in Eq. (2.28) anyway, because it is generated by the nonlinear terms under the renormalization group, as I shall discuss in Sec. IV. Likewise, the parameter $w_1 = 1$, but I have also included it for convenience in discussing the renormalization group in Sec. IV. I will henceforth focus my attention on the fluid, orientationally ordered state, in which all of the diffusion constants $D_{\rho_\parallel}$, $D_{\rho_\perp}$, $D_{\rho v}$, D_B, $D_\parallel$, and D_T are positive.

III. LINEARIZED THEORY OF THE FLUID, ORIENTATIONALLY ORDERED PHASE

Expanding (2.18) and (2.28) to linear order in the small fluctuations $\vec{v}_\perp$ and $\delta\rho$ gives

$$\partial_t \vec{v}_\perp + \gamma \partial_\parallel \vec{v}_\perp = -\frac{c_0^2}{\rho_0}\vec{\nabla}_\perp \delta\rho + D_B \vec{\nabla}_\perp(\vec{\nabla}_\perp \cdot \vec{v}_\perp) + D_T \nabla_\perp^2 \vec{v}_\perp + D_\parallel \partial_\parallel^2 \vec{v}_\perp + \nu_t \partial_t \vec{\nabla}_\perp \delta\rho + \nu_\parallel \partial_\parallel \vec{\nabla}_\perp \delta\rho + \vec{f}_\perp \quad (3.1)$$

and

$$\partial_t \delta\rho + \rho_o \vec{\nabla}_\perp \cdot \vec{v}_\perp + v_2 \partial_\parallel \delta\rho = D_{\rho_\parallel} \partial_\parallel^2 \delta\rho + D_{\rho_\perp} \nabla_\perp^2 \delta\rho + D_{\rho v} \partial_\parallel (\vec{\nabla}_\perp \cdot \vec{v}_\perp) + \phi \partial_t \partial_\parallel \delta\rho. \quad (3.2)$$

These equations can now readily be solved for the mode structure and correlations by Fourier transforming in space and time; this gives

$$[-i(\omega - \gamma q_\parallel) + \Gamma_L(\vec{q})]v_L + \left[\frac{ic_0^2}{\rho_0} q_\perp - \nu_t q_\perp \omega - \nu_\parallel q_\perp q_\parallel\right]\delta\rho = f_L, \quad (3.3)$$

$$[-i(\omega - \gamma q_\parallel) + \Gamma_T(\vec{q})]\vec{v}_T = \vec{f}_T, \quad (3.4)$$

$$[i\rho_0 q_\perp + D_{\rho v} q_\perp q_\parallel]v_L + [-i(\omega - v_2 q_\parallel) + \Gamma_\rho(\vec{q}) - \phi q_\parallel \omega]\delta\rho = 0, \quad (3.5)$$

where I've defined the wave-vector dependent longitudinal, transverse, and ρ dampings $\Gamma_{L,T,\rho}$:

$$\Gamma_L(\vec{q}) = D_L q_\perp^2 + D_\parallel q_\parallel^2, \quad (3.6)$$

$$\Gamma_T(\vec{q}) = D_T q_\perp^2 + D_\parallel q_\parallel^2, \quad (3.7)$$

$$\Gamma_\rho(\vec{q}) = D_{\rho_\parallel} q_\parallel^2 + D_{\rho_\perp} q_\perp^2, \quad (3.8)$$

with $D_L \equiv D_B + D_T$. I have also separated the velocity $\vec{v}_\perp$ and the noise $\vec{f}_\perp$ into components along and perpendicular to the projection $\vec{q}_\perp$ of $\vec{v}$ perpendicular to $\langle \vec{v} \rangle$ via

$$v_L \equiv \vec{v}_\perp \cdot \vec{q}_\perp / q_\perp, \quad \vec{v}_T \equiv \vec{v}_\perp - v_L \frac{\vec{q}_\perp}{q_\perp}, \quad (3.9)$$

with f_L and $\vec{f}_T$ obtained from $\vec{f}$ in the same way.

These equations differ from the corresponding equations considered in [3–6] only in the $\nu_{t,\parallel}$ terms in (3.3), and the $D_{\rho_\parallel}$ and $D_{\rho v}$ terms in (3.5). These prove to lead only to minor changes in the propagation direction dependence, but not the scaling with wavelength, of the damping of the sound modes found in [3–6], as I will now demonstrate.

I begin by determining the eigenfrequencies of the system, defined in the usual way as the complex, wave-vector dependent frequencies $\omega(\vec{q})$ at which the Fourier transformed hydrodynamic equations (3.3)–(3.5) admit nonzero solutions for $\vec{v}_T$, $\delta\rho$, and v_L when the noise $\vec{f}$ is set to zero. Note that $\vec{v}_T$ is decoupled from v_L and ρ; this implies a pair of "longitudinal" eigenmodes involving just the longitudinal velocity v_L and ρ, and an additional $d-2$ "transverse" mode associated with the transverse velocity $\vec{v}_T$. The longitudinal modes are closely analogous to ordinary sound waves in a simple fluid [20], while the transverse modes are the analog of the diffusive shear modes in such a fluid.

In the hydrodynamic limit (i.e., when wave number $q \to 0$), the longitudinal eigenfrequencies become a pair of underdamped, propagating modes with complex eigenfrequencies

$$\omega_\pm(\vec{q}) = c_\pm(\theta_{\vec{q}})q - i\epsilon_\pm(\vec{q}), \quad (3.10)$$

where the direction-dependent sound speeds $c_\pm(\theta_{\vec{q}})$ are given by exactly the same expression as found in previous work [3–6]:

$$c_\pm(\theta_{\vec{q}}) = \left(\frac{\gamma + v_2}{2}\right)\cos(\theta_{\vec{q}}) \pm c_2(\theta_{\vec{q}}), \quad (3.11)$$

where I have defined

$$c_2(\theta_{\vec{q}}) \equiv \sqrt{\frac{(\gamma - v_2)^2 \cos^2(\theta_{\vec{q}})}{4} + c_0^2 \sin^2(\theta_{\vec{q}})}, \quad (3.12)$$

where $\theta_{\vec{q}}$ is the angle between $\vec{q}$ and the direction of flock motion (i.e., the $x_\parallel$ axis).

As mentioned earlier, the wave-vector dependent dampings $\epsilon_\pm(\vec{q})$ of these propagating sound modes *are* altered slightly from the form found in [3–6]. They remain of $O(q^2)$, as found in previous work, but with a slightly modified dependence on propagation direction $\hat{q}$. More precisely, they are given by

$$\epsilon_\pm \equiv \frac{\text{NUM}}{[2c_\pm(\theta_{\vec{q}}) - (v_2 + \gamma)\cos(\theta_{\vec{q}})]} \quad (3.13)$$

with

$$\text{NUM} \equiv [\Gamma_L(\vec{q}) + \Gamma_\rho(\vec{q}) - \phi c_\pm(\theta_{\vec{q}})\cos(\theta_{\vec{q}})q^2]c_\pm(\theta_{\vec{q}}) - v_2\Gamma_L(\vec{q})\cos(\theta_{\vec{q}}) - \gamma[\Gamma_\rho(\vec{q}) - \phi c_\pm(\theta_{\vec{q}})\cos(\theta_{\vec{q}})q^2] \times \cos(\theta_{\vec{q}}) + \frac{c_0^2}{\rho_0} D_{\rho v} \frac{q_\parallel q_\perp^2}{q} - \rho_0 q_\perp^2[\nu_t c_\pm(\theta_{\vec{q}}) + \nu_\parallel \cos(\theta_{\vec{q}})], \quad (3.14)$$

where I remind the reader that the wave-vector dependent dampings $\Gamma_{L,\rho}$ are $O(q^2)$, and defined earlier in Eqs. (3.6) and (3.8).

The transverse modes have the far simpler character of anisotropic diffusion, with purely imaginary eigenfrequencies

$$\omega_T(\vec{q}) = -i\Gamma_T(\vec{q}) \quad (3.15)$$

with the wave-vector dependent damping Γ_T also $O(q^2)$, and defined earlier in Eq. (3.7).

I now turn to the correlation functions in this linearized approximation. These are easily obtained by first solving the linear algebraic equations (3.3)–(3.5) for the fields $v_L(\vec{q},\omega)$, $\vec{v}_T(\vec{q},\omega)$, and $\rho(\vec{q},\omega)$ in terms of the noises $f_L(\vec{q},\omega)$, and $\vec{f}_T(\vec{q},\omega)$. These solutions are, of course, linear in those noises. Hence, by correlating these solutions pairwise, one can obtain any two field correlation function in terms of the correlations (2.5) of $\vec{f}$. The resulting correlation function for the velocity is

$$C_{ij}(\vec{q},\omega) \equiv \langle v_{\perp i}(\vec{q},\omega) v_{\perp j}(-\vec{q},-\omega)\rangle = \frac{\Delta(\omega - v_2 q_\parallel)^2 L_{ij}^\perp}{\{[\omega - c_+(\theta_{\vec{q}})q]^2 + \epsilon_+^2(\vec{q})\}\{[\omega - c_-(\theta_{\vec{q}})q]^2 + \epsilon_-^2(\vec{q})\}} + \frac{\Delta P_{ij}^\perp}{[(\omega - \gamma q_\parallel)^2 + \Gamma_T^2(\vec{q})]}, \tag{3.16}$$

where I have defined the longitudinal (L) and transverse (T) projection operators in the $\perp$ plane,

$$L_{ij}^\perp(\hat{q}) \equiv \frac{q_{\perp i} q_{\perp j}}{q_\perp^2}, \quad P_{ij}^\perp(\hat{q}) \equiv \delta_{ij}^\perp - L_{ij}^\perp(\hat{q}), \tag{3.17}$$

where $\delta_{ij}^\perp$ is a Kronecker δ in the $\perp$ plane (i.e., it is equal to the usual Kronecker δ if $i \neq \parallel \neq j$, and zero otherwise). These operators project any vector first into the $\perp$ plane, and then either along (L) or orthogonal to (P) $\vec{q}_\perp$ within the $\perp$ plane.

The first term in Eq. (3.16) comes from the "longitudinal" component v_L while the second comes from the $d-2$ "transverse" components of $\vec{v}_\perp$. Clearly, in $d = 2$, only the longitudinal component is present; the second (transverse) term in (3.16) vanishes in $d = 2$.

The density autocorrelations obtained by the procedure described above are given, to leading order in wave vector and frequency, by

$$C_{\rho\rho}(\vec{q},\omega) \equiv \langle \rho(\vec{q},\omega)\rho(-\vec{q},-\omega)\rangle = \frac{\rho_0 q_\perp^2 \Delta}{\{[\omega - c_+(\theta_{\vec{q}})q]^2 + \epsilon_+^2(\vec{q})\}\{[\omega - c_-(\theta_{\vec{q}})q]^2 + \epsilon_-^2(\vec{q})\}}. \tag{3.18}$$

Both the velocity correlations (3.16) and the density correlations (3.18) have the same form, and the same scaling with frequency and wave vector, as those reported in earlier work [3–6]. The only change from those earlier results is the slightly modified form (3.13) and (3.14) of the sound dampings which appear in (3.16) and (3.18).

The same statement is true of the equal-time correlations of $\vec{v}$ and ρ, which can be obtained in the usual way by integrating the spatiotemporally Fourier transformed correlations (3.16) and (3.18) over all frequency ω. These equal time correlations are important, because they determine the size of the velocity and density fluctuations. The size of the velocity fluctuations determines whether or not long-ranged order can exist in these systems, while the size of the density fluctuations determines the presence or absence of giant number fluctuations [6,21].

Integrating (3.16) over all ω and tracing over the Cartesian components i, j gives the equal-time correlation of $\vec{v}$:

$$\langle|\vec{v}_\perp(\vec{q},t)|^2\rangle = \frac{1}{2}\left(\frac{\Delta[c_+(\theta_{\vec{q}}) - v_2\cos(\theta_{\vec{q}})]^2}{\epsilon_+(\vec{q})[c_+(\theta_{\vec{q}}) - c_-(\theta_{\vec{q}})]^2} + \frac{\Delta[c_-(\theta_{\vec{q}}) - v_2\cos(\theta_{\vec{q}})]^2}{\epsilon_-(\vec{q})[c_+(\theta_{\vec{q}}) - c_-(\theta_{\vec{q}})]^2} + \frac{(d-2)\Delta}{\Gamma_T(\vec{q})}\right). \tag{3.19}$$

Note that these scale like $1/q^2$ for all directions of wave vector $\vec{q}$. This scaling is precisely the same as that found in the linearized theory of [3–6]; only the precise form of the dependence on the direction of $\vec{q}$ is slightly changed by the presence of the new linear terms ν_t, $\nu_\parallel$, and ϕ that I have found here that were missed in the treatment of [3–6].

This $1/q^2$ scaling of $\vec{v}_\perp$ fluctuations with q in Fourier space implies that the real space fluctuations,

$$\langle|\vec{v}_\perp(\vec{r},t)|^2\rangle = \int \frac{d^d q}{(2\pi)^d}\langle|\vec{v}_\perp(\vec{q},t)|^2\rangle, \tag{3.20}$$

diverge in the infrared ($q \to 0$ or system size $L \to \infty$) limit in all spatial dimensions $d \leqslant 2$. This in turn implies that long-ranged order [i.e., the existence of a nonzero $\langle\vec{v}_\perp(\vec{r},t)\rangle$] is not possible in $d = 2$, *according to the linearized theory*.

This result, which is simply the Mermin-Wagner [11] theorem, is actually overturned by nonlinear effects, which stabilize the long-ranged order in $d = 2$ [i.e., make the existence of a nonzero $\langle\vec{v}(\vec{r},t)\rangle$ possible], as first noted by [3–6]. I shall show in Sec. IV that nonlinear effects still stabilize long-ranged order in this way even when the additional nonlinearities I have found here, which were missed in [3–6], are included.

The equal time density autocorrelations can likewise be obtained by integrating Eq. (3.18) over frequency ω; this gives

$$\langle|\delta\rho(\vec{q},t)|^2\rangle = \frac{1}{2}\left(\frac{\Delta\rho_0 q_\perp^2}{[c_+(\theta_{\vec{q}}) - c_-(\theta_{\vec{q}})]^2 q^2}\right) \times \left(\frac{1}{\epsilon_+(\vec{q})} + \frac{1}{\epsilon_-(\vec{q})}\right). \tag{3.21}$$

This also scales like $1/q^2$ for all directions of $\vec{q}$. This divergence implies "giant number fluctuations" [22]: the RMS fluctuations $\sqrt{\langle\delta N^2\rangle}$ of the number of particles within a large region of the system scale like the mean number of particles $\langle N\rangle$ faster than $\sqrt{\langle N\rangle}$; specifically, $\sqrt{\langle\delta N^2\rangle} \propto \langle N\rangle^{\phi(d)}$, with $\phi(d) = 1/2 + 1/d$ in spatial dimension d. Note that this means in particular that $\sqrt{\langle\delta N^2\rangle} \propto \langle N\rangle$ in $d = 2$.

Again, I emphasize that this is the prediction of the linearized theory. It once again coincides with the results of the linearized treatment of [3–6]. Both the prediction that long-ranged orientational order is destroyed in $d = 2$, and the value $\phi(d) = 1/2 + 1/d$ of the exponent $\phi(d)$ for $d < 4$ prove, when nonlinear effects are taken into account, to be incorrect, as first noted by [3]. I now turn to the treatment of those nonlinear effects.

IV. NONLINEAR EFFECTS AND THE ABSENCE OF AN ARGUMENT FOR EXACT SCALING LAWS

One of the most interesting features of the hydrodynamic theory of flocking is that nonlinearities and fluctuations completely change their scaling behavior at long distances, as first noted by [3–6]. In this section, I show that, while this fundamental conclusion of [3–6] is correct, certain more detailed predictions they make are invalidated by the additional nonlinearities found here, which were missed by them.

Equally noteworthy are the nonlinear terms that are missing from (2.18) and (2.28): all nonlinearities arising from the anisotropic pressure P_2 and the λ_2 nonlinearity drop out of (2.18) and (2.28). This in particular has the very important consequence of saving the Mermin-Wagner theorem. This is because the λ_2 term is allowed even in equilibrium systems [17]. The incorrect treatment in [3–6] suggested that this term *by itself* could stabilize long-range order in $d = 2$. Given that this term is allowed in equilibrium, this would imply that the Mermin-Wagner theorem would fail for such an equilibrium system. The correct treatment I have done here shows that this is not the case: the λ_2 term by itself cannot stabilize long-ranged order in $d = 2$, since the nonlinearities associated with it drop out of the long-wavelength description of the ordered phase.

Returning now to the nonlinearities in (2.18) and (2.28) that were missed by [3–6], I will now show that *all* of them become relevant, in the renormalization group (RG) sense [23], for spatial dimensions $d \leqslant 4$. To assess the effect of the new nonlinear terms I have found here, I shall analyze Eqs. (2.18) and (2.28) using the dynamical renormalization group [12].

The dynamical RG starts by averaging the equations of motion over the short-wavelength fluctuations: i.e., those with support in the "shell" of Fourier space $b^{-1}\Lambda \leqslant |\vec{q}| \leqslant \Lambda$, where Λ is an "ultraviolet cutoff," and b is an arbitrary rescaling factor. Then, one rescales lengths, time, $\delta\rho$ and $\vec{v}_\perp$ in Eqs. (2.18) and (2.28) according to $\vec{v}_\perp = b^\chi \vec{v}_\perp'$, $\delta\rho = b^\chi \delta\rho'$, $\vec{r}_\perp = b\vec{r}_\perp'$, $r_\parallel' = b^\zeta (r_\parallel')'$, and $t = b^z t'$ to restore the ultraviolet cutoff to Λ [24]. This leads to a new pair of equations of motion of the same form as (2.18) and (2.28), but with "renormalized" values (denoted by primes below) of the parameters given by

$$D_{B,T}' = b^{z-2}(D_{B,T} + \text{graphs}), \tag{4.1}$$

$$D_{\parallel,\rho\parallel}' = b^{z-2\zeta}(D_{\parallel,\rho\parallel} + \text{graphs}), \tag{4.2}$$

$$\Delta' = b^{z-\zeta-2\chi+1-d}(\Delta + \text{graphs}), \tag{4.3}$$

$$\left(\lambda_1^0\right)' = b^{z+\chi-1}\left(\lambda_1^0 + \text{graphs}\right), \tag{4.4}$$

$$g_{1,2}' = b^{z+\chi-\zeta}(g_{1,2} + \text{graphs}), \tag{4.5}$$

$$g_3' = b^{z+\chi-1}(g_3 + \text{graphs}), \tag{4.6}$$

$$\phi' = b^{z+\chi-1}(\phi + \text{graphs}), \tag{4.7}$$

$$w_{1,2,3}' = b^{z+\chi-\zeta}(w_{1,2,3} + \text{graphs}), \tag{4.8}$$

where "graphs" denotes contributions from integrating out the short-wavelength degrees of freedom.

I have focused on the particular linear parameters $D_{B,T,\parallel,\rho\parallel}$ and Δ since, as is clear from Eqs. (3.19) and (3.21), they determine the size of the fluctuations [25] in the linearized theory.

One proceeds by seeking fixed points of these recursion relations. One simple fixed point is the linear fixed point, at which all of the nonlinear coefficients λ_1^0, $g_{1,2,3}$, and $w_{1,2,3}$ are zero. At such a fixed point, the graphical corrections [denoted "graphs" in Eqs. (4.1)–(4.8)] vanish, since, without nonlinearities, Fourier modes at different wave vectors and frequencies do not interact. It is then straightforward to determine from Eqs. (4.1)–(4.8) the values of the rescaling exponents z, ζ, and χ that will keep $D_{B,T,\parallel,\rho\parallel}$ and Δ [and, hence, the size of the fluctuations) fixed: simply those that make the exponents in (4.1)–(4.8)] vanish. That is, we must choose [26]

$$z - 2 = 0 \text{ (linear fixed point)} \tag{4.9}$$

to keep D_B and D_T fixed,

$$z - 2\zeta = 0 \text{ (linear fixed point)}, \tag{4.10}$$

to keep $D_\parallel$ and $D_{\rho\parallel}$ fixed, and

$$z - \zeta - 2\chi + 1 - d = 0 \text{ (linear fixed point)}, \tag{4.11}$$

to keep Δ fixed under the RG. The solutions to these three conditions (4.9)–(4.11) are trivially found to be

$$z = 2 \text{ (linear fixed point)}, \tag{4.12}$$

$$\zeta = 1 \text{ (linear fixed point)}, \tag{4.13}$$

and

$$\chi = (2-d)/2 \text{ (linear fixed point)}. \tag{4.14}$$

Let us now consider the stability this linear fixed point against the effect of the nonlinear terms λ_1^0, $g_{1,2,3}$, and $w_{1,2,3}$. Because, as mentioned earlier, I have chosen the rescaling exponents so as to keep the magnitude of the fluctuations the same on all length scales, a given nonlinearity has important effects at long distances if it grows upon renormalization with this choice (4.12)–(4.14) of the rescaling exponents z, ζ, and χ provided that it grows upon renormalization; contrariwise, if it gets smaller upon renormalization with this choice of the rescaling exponents, it is unimportant at long distances [27]. Using the exponents (4.12)–(4.14) in the recursion relations (4.15), (4.16), (4.6), (4.7), and (4.8), and ignoring the graphical corrections, which are higher than linear order in λ_1^0, $g_{1,2,3}$, and $w_{1,2,3}$, I find that all seven of these nonlinearities have identical renormalization group eigenvalues of $(4-d)/2$ at the linearized fixed point; that is,

$$\left(\lambda_1^0\right)' = b^{(4-d)/2}\lambda_1^0, \tag{4.15}$$

$$g_{1,2,3}' = b^{(4-d)/2}g_{1,2,3}, \tag{4.16}$$

$$w_{1,2,3}' = b^{(4-d)/2}w_{1,2,3}. \tag{4.17}$$

Thus, for $d > 4$, all of the nonlinearities flow to zero, and so become unimportant, at long length and time scales. Hence, the linearized theory is correct at long length and time scales, for $d > 4$. For $d < 4$, however, all of these nonlinearities grow, and the linear theory breaks down at sufficiently long length and time scales.

Both this analysis and its conclusion that nonlinear effects invalidate the linear theory for $d < 4$ are *almost* identical to those of [3–6]. However, whereas they found only four nonlinearities ($\lambda_{1,2}$, w_1, and g_3 in the notation I am using here) that became relevant as d is decreased below $d = 4$, I find seven such nonlinearities. More importantly, the vector structure of some of the new nonlinearities differs from that of those studied in [3–6] in crucial ways. In particular, all of the nonlinearities considered in [3–6] could, in $d = 2$, be written as total $\perp$ derivatives. This implies that these nonlinearities can only renormalize terms which themselves involved $\perp$ derivatives (i.e., $D_{B,T}$); hence, all of the terms that did *not* involve $\perp$ derivatives (i.e., $D_{\parallel,\rho\parallel},\Delta$) were incorrectly argued in [3–6] to get no graphical corrections. This leads to the incorrect conclusion that, in order to obtain a fixed point, one had to choose the rescaling exponents z, ζ, and χ to make the exponents in (4.2) and (4.3) vanish; i.e., that in $d = 2$,

$$z - 2\zeta = 0, \quad z - \zeta - 2\chi + 1 - d = z - \zeta - 2\chi - 1 = 0. \tag{4.18}$$

The earlier work of [3–6] went on to incorrectly argue that there were no graphical corrections λ_1^0 either, because the equations of motion (2.18) and (2.28) have, in $d = 2$ and in the absence of the extra relevant nonlinearities $g_{1,2}$ and $w_{1,2,3}$ found here, an exact "pseudo-Galilean invariance" symmetry [28]: they remain unchanged by a pseudo-Galilean the transformation,

$$\vec{r}_\perp \to \vec{r}_\perp - \lambda_1 \vec{v}_1 t, \quad \vec{v}_\perp \to \vec{v}_\perp + \vec{v}_1, \tag{4.19}$$

for arbitrary constant vector $\vec{v}_1 \perp \hat{x}_\parallel$. Note that if $\lambda = 1$, this reduces to the familiar Galilean invariance in the x direction. Since such an exact symmetry must continue to hold upon renormalization, with the *same* value of λ_1, λ_1 cannot be graphically renormalized in the absence of the extra relevant nonlinearities $g_{1,2}$ and $w_{1,2,3}$ found here. Requiring that $\lambda_1' = \lambda_1$ in (4.15), and setting graphs = 0, implies that

$$\chi = 1 - z \tag{4.20}$$

in $d = 2$. This and (4.18) form three independent equations for the three unknown exponents χ, z, and ζ, whose solution in $d = 2$ is

$$z = 6/5, \tag{4.21}$$

$$\zeta = 3/5, \tag{4.22}$$

and

$$\chi = -1/5, \tag{4.23}$$

which are the exponents purported in [3–6] to be exact in $d = 2$.

The presence of the extra nonlinearities $g_{1,2,3}$ and $w_{1,2,3}$ invalidates every essential ingredient of the above argument: these nonlinearities are not total $\perp$ derivatives, so one can *not* argue that $D_{\parallel,\rho\parallel}$ and Δ get no graphical corrections. This invalidates the exact scaling relations (4.18), and makes it impossible to obtain exact exponents in $d = 2$.

I have been unable to come up with alternative arguments that give exact exponents in the presence of these additional terms. Now, *if* these additional nonlinearities were irrelevant in $d = 2$ under a full dynamical RG, then the exact exponents of [3–6] would be correct in $d = 2$.

There is a precedent for this (that is, for terms that appear relevant by simple power counting below some critical dimension d_c actually proving to be irrelevant once "graphical corrections"—i.e., nonlinear fluctuation effects—are taken into account). One example of this is the cubic symmetry breaking interaction [29] in the $O(n)$ model, which is relevant by power counting at the Gaussian fixed point for $d < 4$, but proves to be irrelevant, for sufficiently small n, at the Wilson-Fisher fixed point that actually controls the transition for $d < 4$, at least for $\epsilon \equiv 4 - d$ sufficiently small.

Unfortunately, doing a similar $4 - \epsilon$ analysis of the relevance of these new nonlinearities in the flocking problem would tell us nothing about whether or not these terms are relevant in $d = 2$, since 2 is far below the critical dimension $d_c = 4$ of the flocking problem. Hence, whether or not the exact exponents predicted by [3–6] are correct remains an open question. They could be; numerical experiments [4–6,21,30,31] and some real experiments [32] suggest they are, but we really do not know at this point.

Not all of the predictions of [3–6] become questionable in the light of the existence of these new nonlinearities, however. In particular, the claim that long-ranged orientational order can exist even in $d = 2$ is unaffected. I know this because the nonlinear terms clearly make positive contributions to the damping coefficient corrections to the velocity diffusion "constants" D_B and D_T are positive, and that they are relevant in the RG sense, which means they must change the scaling of the velocity fluctuations from that predicted by the linearized theory. I know that they are relevant by the following proof by contradiction: if all of the nonlinear effects were irrelevant, then simple power counting would suffice to determine their relevance. But simple power counting says that *all* of the nonlinearities are *relevant* for $d < d_c = 4$, which contradicts the original assumption that they are all irrelevant. Thus, the nonlinearites *must* change the scaling of the velocity fluctuations. Since the effect of the nonlinearities is to renormalize the velocity diffusion constants D_B and D_T upwards, and since this tends to reduce velocity fluctuations, the growth of velocity fluctuations with length scale must be suppressed (more precisely, its *scaling* must be suppressed; i.e., it must grow like a smaller power of length scale L) than is predicted by the linearized version of the equations of motion (2.18) and (2.28). But those linearized equations predict [3–6] only *logarithmic* divergences of velocity fluctuations with length scale. Hence, the real fluctuations, including nonlinear effects, must be smaller than logarithmic by some power of length scale [33], which means they must be *finite* as $L \to \infty$. This boundedness of velocity fluctuations means that long-ranged order *is* possible in a two-dimensional flock, in contrast to equilibrium systems with continuous symmetries.

Note that all of the troublesome nonlinearities that make it impossible to determine exact exponents in $d = 2$ involve the fluctuation $\delta\rho$ of the density ρ. Therefore, if these fluctuations could somehow be "frozen out," it would be possible to determine exact exponents in $d = 2$.

There are a number of types of flocks in which precisely such a freezing out of density fluctuations occurs. One class of such systems—flocks with birth and death—has been treated elsewhere [13]; others, such as incompressible systems [14] and systems with long-ranged interactions [15], will be addressed in future work [16]. In all of these systems, exact scaling exponents can be found in $d = 2$.

V. CONCLUSION

In conclusion, I have reanalyzed the hydrodynamic theory of fluid, polar-ordered flocks. In addition to identifying certain previously missed linear terms in the hydrodynamic equations for such systems, which slightly modify the anisotropy, but not the scaling, of the damping of sound modes in flocks, I have also found that certain nonlinearities that are allowed in equilibrium, and that were predicted by earlier work [3–6] to stabilize long-ranged order in $d = 2$, in fact do not. Other nonlinearities missed by earlier work could potentially change the scaling exponents from those predicted earlier [3–6], but it is also possible that they do not.

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[25] Although c_0, γ, and v_2 also enter the equal time correlations (through the direction-dependent sound speed $c_\pm(\theta_{\vec{q}})$, it is easy to see that it is only *ratios* of these velocities that enter the equal time correlation functions. It is straightforward to check that these ratios do not change under the rescaling used here.

[26] This choice is, of course, arbitrary, but provides the most convenient way to assess the relevance of the nonlinear terms. See [27] for a further discussion.

[27] Other choices of rescaling force one to find suitable "dimensionless couplings;" that is, ratios of the nonlinear parameters (e.g., λ^0) and suitable powers of the linear parameters (e.g., Δ, D_B) that actually give the ratios of the renormalizations of the linear parameters by fluctuations to their linear values. These necessarily have the same RG eigenvalues as those we found here for λ^0, etc., alone, independent of the arbitrary choice of z, χ, and ζ.

[28] In order for *both* equations of motion (ρ's and $\vec{v}_\perp$'s) to have this pseudo-Galilean invariance, it is actually also necessary that $w_1 = \lambda_1^0$. However, by a more complicated argument, one can show that the existence of a pseudo-Galilean invariance for one value of w_1 ensures the same relation between exponents for all values of w_1 as that obtained when $w_1 = \lambda^0$.

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