Balance between absorbing and positive fixed points in resource consumption models

Hilla Behar,[1] Nadav Shnerb,[2] and Yoram Louzoun[1,*]
[1]*Department of Mathematics, Bar Ilan University, Ramat Gan, Israel*
[2]*Department of Physics, Bar Ilan University, Ramat Gan, Israel*
(Received 29 April 2012; revised manuscript received 13 August 2012; published 28 September 2012)

The effect of resource usage on economic growth has been studied in multiple models. However, the generic effect of improving resource usage efficacy through improved technical skills has not been studied in detail. We here analyze a model incorporating resource usage by capital and the parallel production of technical skill in order to study the effect of improving the efficacy of resources usage with advanced technologies. We show that a practically inevitable result of such a model is that improving the resource usage efficacy leads to a lower steady-state level of resources. A surprising conclusion from ordinary differential equations realization of the model is an extreme sensitivity to parameters, where a small parameter change can lead to an irreversible state through a hysteresis mechanism between a scenario of a collapse of the economy and a scenario of sustainable economy. This sensitivity is lost when spatial stochastic simulations are performed. In the stochastic regime the two scenarios coexist, with different fractions of the lattice residing in each state. Changing parameters smoothly changes the fraction of lattice sites in each state. The transition between the collapsed economy and the sustainable one is not symmetrical. Escape from the collapsed situation can only occur through diffusion from neighboring sustained lattice sites. On the other hand, the collapse can occur even in the absence of diffusion. This difference leads to diffusion dependent capital growth, where an optimal capital is obtained for middiffusion values. Such a transition may actually be generic phenomena in ecological and economic systems.

DOI: 10.1103/PhysRevE.86.031146 PACS number(s): 64.60.ah, 05.10.Gg, 88.05.Lg

I. INTRODUCTION

A central debate in the effect of economic growth on resource usage revolves around the possible role of technological change [1]. One often encounters the idea, especially among economists, that technological improvements allow a better use of resources and substitution among different resources for the same purpose [2–5]. The alternative and contrasting view is that we are facing an unprecedented ecological crisis and that technological progress only accelerates resource depletion [6]. Thus, a central open question at the interface between ecology and economics is the role of scientific and technical progress on the ecological system. A related issue is the equity among world regions: Would a stable world exhibit a narrower wealth distribution than our present growing economies? As discussed in Bairoch [7] and followers, the industrial revolution strongly increased economic disparities between different regions (in contrast with the disparity within a given region, which is not studied here). Before the industrial revolution, the average wealth of the western world circa 1750 was comparable to the average wealth of Asian countries, while this ratio was increased by a 10-fold factor at the end of the 20th century. Thus, one can ask if technical skills and technological knowledge are the source of disparity.

The parallel development of technical skills, inequality, and resource depletion can be understood through a pair of coupled feedback loops. The positive feedback loop between technical skill and capital and the negative feedback loop between capital growth and resource depletion. The first loop has been extensively studied, through the two parallel processes of the effect of the technical skill on growth [8] and the parallel increase in technical skill and knowledge as a function of capital investment [9–11]. The second loop describes a negative feedback between capital growth and resource depletion. The effect of capital on resource depletion has also been extensively studied and modeled [12–14]. We here show that the coupling of these two feedback loops leads to surprising conclusions.

We use a discrete time stochastic simulation on a lattice to study the stochastic dynamics of the above-mentioned feedback loops. We extend a simulation methodology that we have used in a wide variety of models [15–20] to the current resource depletion analysis and show that the proposed feedback loops do indeed lead to unequal wealth distribution through a novel mechanism of jumps between two stable fixed points. The model presented here is highly simplified. We do not claim that this model is a precise description of the economic reality. Instead, we propose a general approach based on the generic properties of the different fixed points. The results presented here are not sensitive to the details of the model and, as such, represent a more general claim about a sustainable economy.

A simple version of these opposing feedback loops via three ordinary differential equations is given in Sec. III A. The mean-field approximation of this model is discussed in Sec. III B. In Sec. III C the bifurcation properties of the model are described. We then study the stochastic counterpart of the ordinary differential equations (ODEs). The stochastic model is discussed in Sec. III D, and the simulations results are examined in Sec. III E. The explanations of the results are described in Sec. III F. A simpler model with a same behavior is presented in Sec. III G. In Secs. III H and III I, we analyze the effect of the diffusion and the effect of the dimensions in the first model.

*Corresponding author: louzouy@math.biu.ac.il

II. METHODS

A. ODE and PDE

The ODEs were solved numerically using the Matlab fourth-order Runge Kutta [21], as applied in the MATLAB [22] ode45 function assuming nonstiff equations. The partial differential equations (PDEs) were solved using a fourth-order Runge Kutta. The diffusion scheme that was used was a second-order leapfrog scheme [23].

B. Simulation

Monte Carlo simulations of the systems studied have been performed on one- and two-dimensional lattices. We initiated the reactants at random positions and enacted each reaction separately. We computed at each lattice point the probability of each reaction and performed reactions according to the prescribed probabilities. At high reaction rates, we used a Poisson approximation [24]. The simulation updating was synchronous. The dynamics were simulated for different parameter values. The lattice size used was 100×100 in the two-dimensional lattice and 1000×1 in the one-dimensional lattice, unless otherwise noted. The simulation was described in detail in previous publications [15].

III. RESULTS

A. Model description

In order to study the feedback between resource consumption and technical skill, we present a simple model containing three elements: technical skill (denoted by A), capital (denoted by B), and resources (denoted by C). Capital, skill, and resources interact through the following schematic interactions:

(a) Investment of capital in research and development increases the technical skill (technical coefficient).

(b) Knowledge has a natural creation and a natural decay. For example, some knowledge can lose its relevance after some time.

(c) Capital has a natural decay (e.g., inflation).

(d) Production consumes resources and requires existing capital and technical skills. In parallel, production uses resources and thus decreases the total amount of available resources.

(e) Resources have a natural creation, and a natural decay. This is obviously not true for all types of resources, but we here focus on renewable resources.

These interactions can be schematized by the following mass action based reactions:

(a) $\varnothing \xrightarrow{\beta_A} A$ (A has a natural creation rate of β_A)

(b) $A \xrightarrow{\delta_A} \varnothing$ (A decays with a rate of δ_A)

(c) $B \xrightarrow{\alpha_A} B + A$ (A is created by B with a rate of α_A)

(d) $A + B + C \xrightarrow{\beta_B} A + 2B + C$ (B is created by A, B, and C with a rate of β_B)

(e) $B \xrightarrow{\delta_B} \varnothing$($B$ decays with a rate of δ_B)

(f) $\varnothing \xrightarrow{\beta_C} C$ (C has a natural creation rate β_C).

(g) $A + B + C \xrightarrow{k\beta_B} A + B$ (C is used by A, B, and C with a rate of $k\beta_B$)

(h) $C \xrightarrow{\delta_C} \varnothing$ (C decays with a rate of δ_C).

B. Mean-field approximation

This seemingly simple system leads to nontrivial conclusions about the relations between technical skill, capital, and resources. We first study an ODE model with three variables that does not take into account possible spatial inhomogeneities or the possible discrete aspect of the interactions (e.g., capital investments are performed in large sums, and technical skills are acquired through the presence of specialists). The mass action approximation of the expected population dynamics is

$$\begin{aligned} \frac{dA}{dt} &= \beta_A - \delta_A A + \alpha_A B \\ \frac{dB}{dt} &= \beta_B ABC - \delta_B B \\ \frac{dC}{dt} &= \beta_C - k\beta_B ABC - \delta_C C. \end{aligned} \quad (1)$$

This system can have up to three non-negative fixed points that we denote by Eqs. (A3), (A4), and (A5) (see Appendix A for a detailed analysis of their properties). Equation (A3) is the $B = 0$ fixed point that exists for all the parameters. One or two of the other fixed points exist in different parts of the phase space. When positive fixed points exist, the steady state values of A and C are always inversely correlated. In this formalism, every mechanism used to increase A will automatically decrease C. In other words, A acts as a "superpredator" on C, while B only serves from a dynamical point of view as an intermediate between A and C. For example, increasing the basal rate of knowledge production will actually decrease resources. If A represents technical skills and C represents resources, the implication from this model is that while the capital growth depends on a large number of parameters, the technical skills are always the inverse of the resources. Increasing technical skills automatically increases the possibility of using resources, which leads to lower steady-state values of the resources. This is obviously a claim about the steady state, and long transients can emerge. Such a correlation is clearly observed where most resources are concentrated in countries with limited technology [25,26]. Note, however, that the observed correlation can be the result of many other mechanisms.

A similar phenomenon is observed for the parameter k. k represents the number of C's that are used to create a single B. The leading idea behind a green economy is that fewer resources will be used per unit of production [27–29]. This is translated in the current schematic model to a low value of k. The basic notion behind this idea is that lowering the usage rate of resources will keep more resources available. However, this idea fails when one reaches a steady state. In steady state smaller values of k lead to lower values C. The result (presented in detail in Appendix B) is the opposite result of what sustainable economy scientists are trying to reach. The difference between the green economy theory and this model is that a green economy looks at transients, while here we study the steady state. As will be further shown, these conclusions are not specific to this model but apply to a wide range of similar ones.

C. Bifurcation properties

As mentioned above, the ODE system in Eq. (1) can have three possible combinations of fixed points (Fig. 1): a single

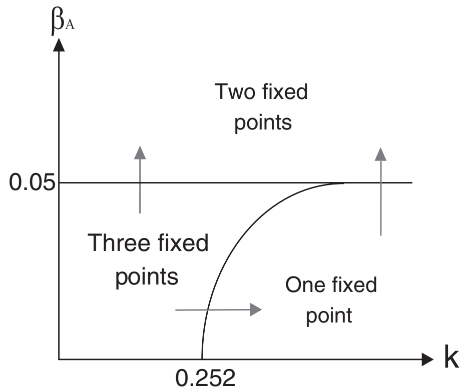

FIG. 1. The number of the fixed points in system (1) according to β_A and k when $\delta_A = 0.05$, $\alpha_A = 0.01$, $\beta_B = 0.01$, $\delta_B = 0.1$, $\beta_C = 0.5$, and $\delta_C = 0.05$. The system has three transitions: from three fixed points to one fixed point (as a function of k), three fixed points to two fixed points, and one fixed point to two fixed points (as a function of β_A).

stable fixed point [Eq. (A3) in Appendix A], two fixed points [Eqs. (A3) and (A5)], or three fixed points. One can study the bifurcation properties of this system through three possible parameter variations, as described by the arrows in Fig. 1: The transition between three and one fixed point happens when the resource utilization efficacy is decreased (higher k values). Beyond some level, the efficacy is too low and the entire system collapses (to the zero capital steady state). The transition from three to two fixed points happens when the technical skill production rate is increased and the central fixed point reaches zero. In such a case, the zero fixed point becomes unstable, and the only remaining stable fixed point is the high capital fixed point. The third bifurcation happen when the technical skill production rate is increased and the resource usage efficacy is low (high k), a high fixed point appears, and the zero capital fix point becomes unstable. The bifurcations occurring following the three different transitions in Fig. 1 are described in Fig. 2. This system has a hysteresis [30] for all the parameter variations but not for all values of the parameters (e.g., no hysteresis occurs when k is high and β_A is changed). Economically, this means that if a state is stabilized around the high fixed point, and this country slightly reduces its resource utilization efficiency, the whole country can lose its capital in a short time and stabilize around the zero capital fixed point. If this country wants to increase the capital back, it would have more efficient resource utilization, yet the state cannot go back to the high capital fixed point.

The inverse relation between A and C, the reduction in the resources steady state following an increased efficacy, and the hysteresis are not artifacts of our specific model (model 1). These results are obtained in a quite general model as follows:

$$\begin{aligned}\frac{dA}{dt} &= \beta_A - \delta_A A + f(B)\\ \frac{dB}{dt} &= g(A,C)B - \delta_B B \qquad (2)\\ \frac{dC}{dt} &= \beta_C - kg(A,C)B - \delta_C C,\end{aligned}$$

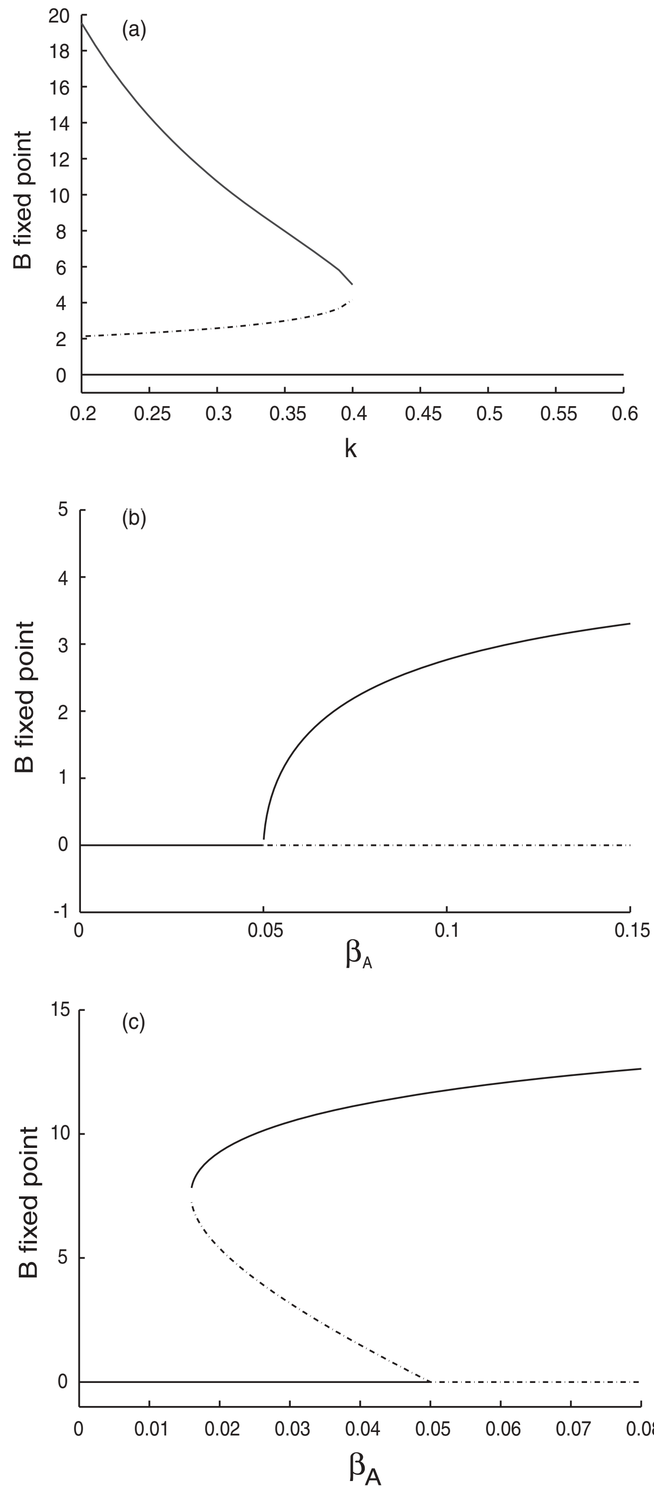

FIG. 2. (a) The fixed points as a function of k with $\delta_A = 0.05$, $\alpha_A = 0.01$, $\beta_B = 0.01$, $\delta_B = 0.1$, $\beta_C = 0.5$, $\delta_C = 0.05$, and $\beta_A = 0.035$. The solid line is stable fixed points, and the dashed line is unstable fixed points. (b) The fixed points according to β_A when $\delta_A = 0.05$, $\alpha_A = 0.01$, $\beta_B = 0.01$, $\delta_B = 0.1$, $\beta_C = 0.5$, $\delta_C = 0.05$, and $k = 1.1$. (c) The fixed points according to β_A when $\delta_A = 0.05$, $\alpha_A = 0.01$, $\beta_B = 0.01$, $\delta_B = 0.1$, $\beta_C = 0.5$, $\delta_C = 0.05$, and $k = 0.3$. The solid line is a stable fixed points, and the dashed line is an unstable fixed points. The system exhibits hysteresis for all the parameter variations.

where $f(B)$ and $g(A,C)$ are increasing monotonic concave or linear functions of B and A,C, respectively, that have a value of zero when the appropriate variables are zero (Appendix C). We show for that the same phenomena occur in a set of systems than described by Eq. (2) in Appendices D and E.

While the first-order transition occurring in the model may be realistic, and, indeed, dramatic changes in the economical state and in the resource usage of countries are observed [31], the question remains whether this phenomenon is indeed the direct conclusion of the model or whether it is an artifact of the specific formalism used. To test this, we simulated the same reactions with a stochastic system, which takes into account the effect of space and the quantization in the value of the different agents (e.g., technical skill is represented by skilled workers and capital is invested in relatively big chunk). We have used a simulation formalism that we have extensively used and tested in the past [15,17,18,32].

D. Stochastic model

We run stochastic simulations on one- and two-dimension lattices. The simulations include diffusion between the cells (for all agent types A, B, and C) and stochastic fluctuations. These fluctuations originate from the fact that the reactions in the simulation are treated as independent random events and that each lattice site has a discrete number of agents (and not a continuous number as in the ODE).

An interesting aspect of the simulation is the interaction between regions that are in the different steady states of the ODE system through diffusion. In the economic interpretation of this model, there are two stable states: either a country is rich in money and poor in resources or vice versa. The simulation provides an insight into the interaction between the two steady states (through diffusion).

E. Simulation results

We first run one- and two-dimensional stochastic simulation. This stochastic simulation stabilizes around a different average from the ODE. Figure 3 shows the mean of the B agent density in the stochastic simulation, when k is varied. The transition between one fixed point and three fixed points occurs for different values of k in the stochastic simulation and the ODE. The same occurs for the variation in all other parameters.

However, the more important difference is that the first-order transition between the numbers of the fixed points in the ODE is replaced by a continuous transition between the numbers of the fixed points in the stochastic simulation. Another difference between the ODE and the stochastic simulation is that in the simulations, in contrast with the ODE results [33], the initial conditions have a very limited effect on the steady state reached (except obviously when all sites are initialized at the absorbing state).

The lattice size has no effect on our results (see Appendix F). Beyond a minimal size, all lattice based simulations behave similarly.

The difference between the ODE and the stochastic simulation could be the result of multiple elements: The inclusion of diffusion and a spatial distribution of A, B, and C or the effect of replacing the deterministic interactions by stochastic

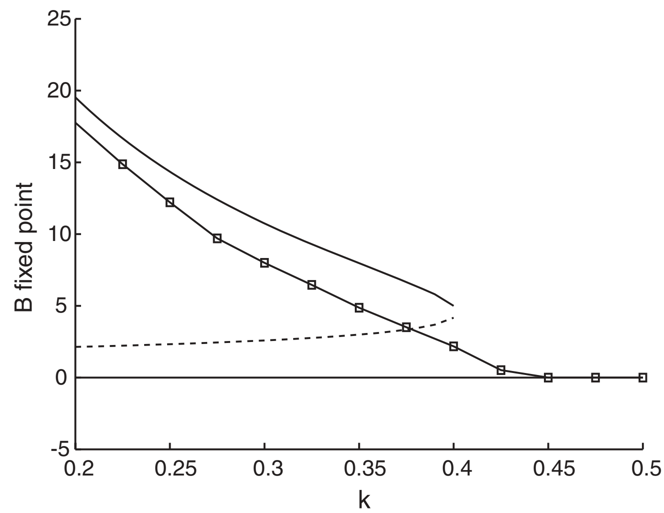

FIG. 3. The averages of B as a function of k with parameters $\delta_A = 0.005$, $\alpha_A = 0.001$, $\beta_B = 0.001$, $\delta_B = 0.01$, $\beta_C = 0.05$, $\delta_C = 0.005$, and $\beta_A = 0.00333$ with diffusion rates of 0.001 for A, B, and C, and initial conditions of $A = 2$, $B = 5$, $C = 6$ on a 100×100 lattice. The solid line is stable fixed points in the ODE. The dashed line is unstable fixed points in the ODE. The square line is the equilibrium in the stochastic system. The results of the stochastic simulation are continuous, and the transition point in the stochastic simulation occurs at higher k values than in the ODE.

ones. In order to show that the stochastic effects are the origin of the difference, we analyzed a spatial deterministic system, using a system of PDEs. The interactions and the parameters in this system are completely equivalent to the parameters in the ODE. The only difference is the addition of diffusion. When starting from a nearly uniform distribution, the ODE and PDE results are similar. However, when a split grid approach is used, where half the grid is initiated in the higher fixed point and the other half in the lower fixed point, a slow yet consistent transition to one of the fixed points occurs, through an invasion wave. Such a behavior has been shown to be generic for system with two possible stable fixed points [34]. Note, however, that the PDE maintains the hysteresis of the ODE. In steady state, either the entire space will be in the higher or in the lower fixed point. Thus, the smooth transition between the states must be the result of the stochastic interactions.

F. Nonsymmetrical transfer between two states

The effect of stochasticity in this system can be understood via a simple schematic drawing. Figure 4 describes the transition between two fixed points, in a simplified two-variable system that will be described in the next section. This system has two stable fixed points: an absorbing state and a positive fixed point and one unstable fixed point in between (precisely like the current system). After a short time, all the values of the cells in the lattice go to the line that connects these three fixed points. Stochastic fluctuations can then transfer lattice sites from one stable fixed point to the other. This transfer is, however, nonsymmetrical since the low fixed point is an absorbing state. If in a given lattice site the B population is extinct [i.e., it goes to the low fixed point (A3)], it cannot leave this point purely through stochastic fluctuations. The only way

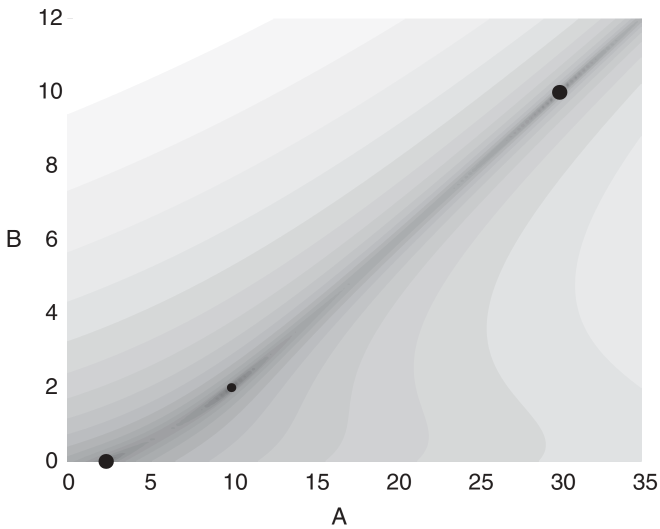

FIG. 4. Absolute value of gradient as a function of A and B values. Darker colors are closer to zero. One can clearly see that the gradient rises sharply beyond the thin line connecting the three fixed points. The two extreme fixed points are stable (large circles), while the intermediate fixed point (small circle) is not stable.

that this lattice point can be reoccupied by B agents is through diffusion from neighboring lattice sites. In contrast, a lattice site that goes to the high fixed point can jump to the other side of the schematic line through loss to its neighbors induced by diffusion or following stochastic fluctuation.

G. Simpler toy model AB with competition (C only mediates competition between A and itself and B and itself)

In order to study this transition, we developed a simpler model that contains the same inherent feedback loops. A first simplification of the original model [Eq. (1)] is to assume a quasi steady state (QSS) on C (i.e., to assume that the dynamics of C are so rapid that we can assume that C has reached a local equilibrium between every time step of A and B).

The QSS model of Eq. (1) is model (3),

$$\frac{dA}{dt} = \beta_A - \delta_A A + \alpha_A B,$$
$$\frac{dB}{dt} = \frac{\beta_B \beta_C AB}{k\beta_B AB + \delta_C} - \delta_B B; \tag{3}$$
$$C = \frac{\beta_C}{k\beta_B AB + \delta_C}.$$

A second possible simplification is to take into account that C serves as a competition mechanism of the B and the A agents over themselves, leading to the following simplified model:

$$\frac{dA}{dt} = \beta_A - \delta_A A + \alpha_A B - \mu A^2,$$
$$\frac{dB}{dt} = (\beta_B A - \delta_B)\, B - \varepsilon B^2. \tag{4}$$

The results of both models are similar. We report the QSS model result in Appendix G and the simpler model with competition on A and B in the following sections. Note that simplified models, with competition over only A or only B

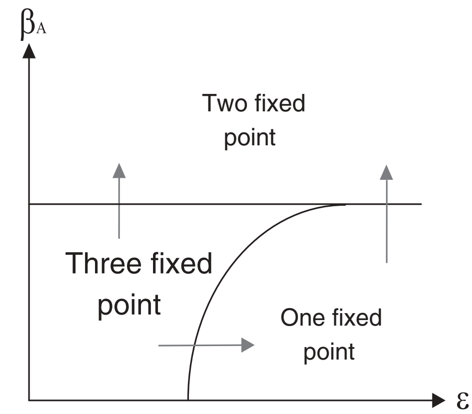

FIG. 5. The number of the fixed points as a function of β_A and ε for model (4). The transitions in the system are three fixed points to one fixed point (by changing ε), three fixed points to two fixed points, and one fixed point to two fixed points (by changing β_A).

[e.g., Eq. (5)] have different bifurcation properties and, thus, are not studied here.

$$\frac{dA}{dt} = \beta_A - \delta_A A + \alpha_A B,$$
$$\frac{dB}{dt} = (\beta_B A - \delta_B)B - \varepsilon B^2, \tag{5a}$$
$$\frac{dA}{dt} = \beta_A - \delta_A A + \alpha_A B - \mu A^2,$$
$$\frac{dB}{dt} = (\beta_B A - \delta_B)B. \tag{5b}$$

As is the case in the original ABC model [Eq. (1)], system (4) has a positive feedback between A and B. High technical

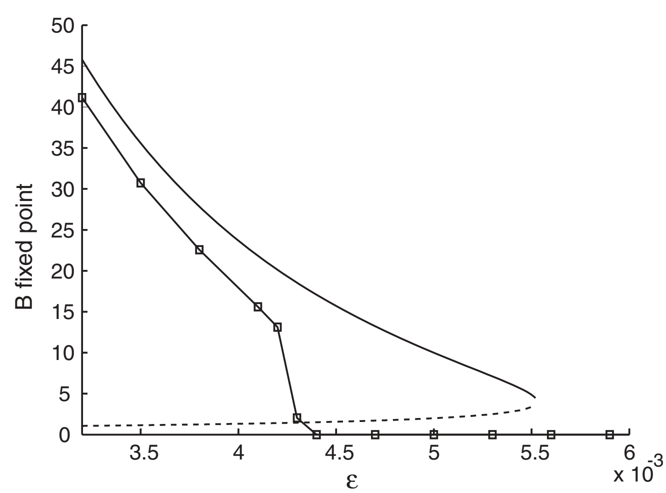

FIG. 6. The average of B as a function ε with the parameters $\delta_A = 0.02$, $\alpha_A = 0.01$, $\beta_B = 0.002$, $\delta_B = 0.01$, $\mu = 0.00005$, and $\beta_A = 0.005$, with diffusions of A and B at 0.001, initial conditions of $A = 30$, $B = 10$, on a 100×100 lattice. The solid line is the stable fixed points in the ODE. The dashed line is the unstable fixed points in the ODE. The square line is the fixed point in the stochastic system. The results of the stochastic simulation are continuous, and the transitions in the stochastic simulation are shifted compared with the ODE.

skills increase the capital growth rates, and capital can be used to produce new technical skills. The system in Eq. (4) has three possible combinations of fixed points (Fig. 5), and the bifurcation types of system (4) and system (1) are equivalent when all the parameters of C (δ_C,k and β_C) in system (1) are translated to μ and ε [the competitions of A and B in system (4)]. Finally, the stochastic realizations of system (4) produced transition similar to the ones of the full ABC model in one and two dimensions (see Fig. 6 for a two-dimensional simulation).

We split the one dimensional lattice into two regions, each in a different stable fixed point. Half the lattice was initialized at the higher stability fixed point, and the other half at the lower stability fixed point (the absorbing state). In this split network analysis, a new division of the parameter space emerges. When the transition probability from the lower to the higher fixed point is higher than the opposite transition probability, a fisher wave is observer [35], where the high-density region converts the low-density region at a constant speed [see linear increase in the total A population in Fig. 7 lower-right and the step by step growth of the region occupied by the higher fixed point in Fig. 7 upper-right]. If, on the other hand, the transition probability from the upper to the lower fixed point is higher, a natural decay occurs simultaneously over the entire region that is in the upper fixed point, as can be seen in the approximately exponential decrease in the total A population [Fig. 7 lower-left] and in the local collapse of the A population [Fig. 7 upper-left].

To summarize, the simple two variable model shows that the difference between the hysteresis expected from the ODE model and the stochastic simulation is the result of the nonsymmetric transition between the upper fixed point and the absorbing state. In intermediate regions, the balance between these two mechanisms produces an inhomogeneous distribution that has a stable average, which is based on a constantly changing spatial distribution of the A and B concentrations (Fig. 8).

H. Survival depends on intermediate values of the diffusion

The results of the two variable model show that diffusion plays a crucial role in these models. If, indeed, the transition to the upper fixed point is purely diffusion dependent while the opposite transition does not require diffusion, then the total B (and A) population is expected to increase as we increase the diffusion rate. At a zero diffusion rate, the only possible transition is to the absorbing state, and the system will always collapse to this state. On the other hand, if the diffusion is very high, the system converges to its mean-field approximation. Thus, we expect two regimes. When the mean-field approximation leads to a single positive fixed point, we expect the system to have a high B population for all high-enough nonzero diffusion rates, while when the system has two stable fixed points, we expect the B population to

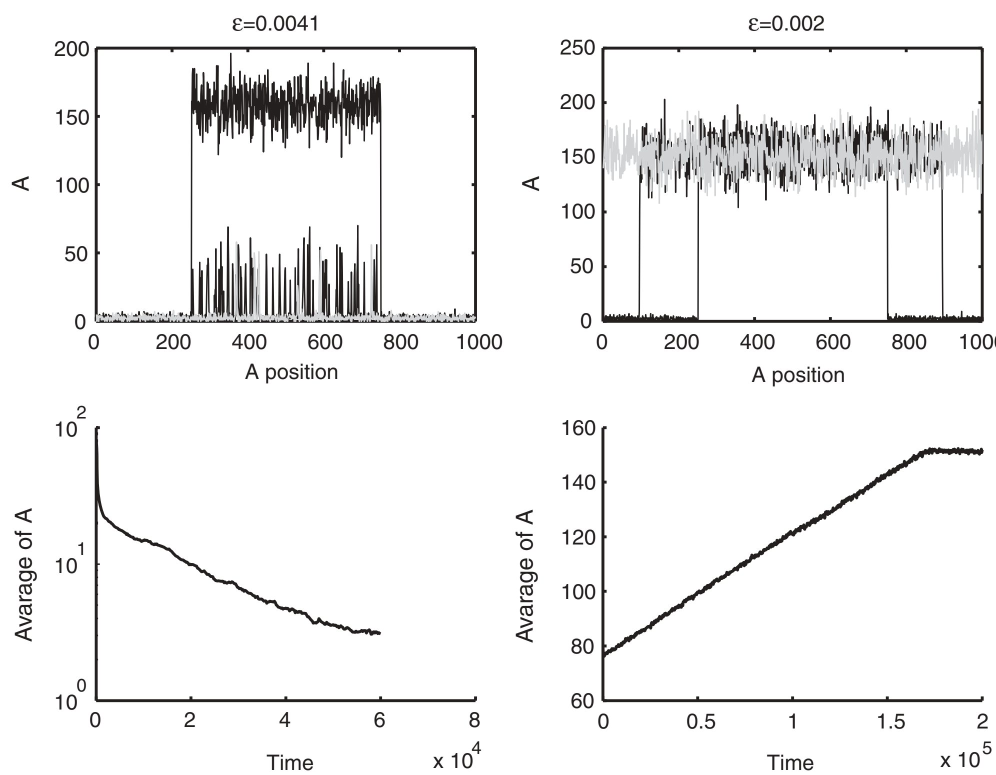

FIG. 7. Snapshot after 200 000 iterations of the simulation with the parameters $\delta_A = 0.002$, $\beta_A = 0.005$, $\beta_B = 0.002$, $\delta_B = 0.01$, $\alpha_A = 0.01$, $\mu = 0.00005$, $\varepsilon = 0.002$, and $\varepsilon = 0.0041$, diffusion of A and B of 0.001, on a 1000×1 lattice. The black line is the initial distribution, the dark gray line is the distribution after 100 000 iterations, and the light gray line is after 200 000 iterations. The lower figures are the averages of A as a function of time. When the system goes to the higher steady state, all the lower points (absorbing state) go to the high steady state in order, and the average of the system goes linearly to the higher steady state, since the absorbing state goes to higher value through diffusion. When the system goes to the lower steady state, all the lattice sites residing in the high steady state can transfer in parallel to the low steady state, and the average of the system is decreasing approximately exponentially.

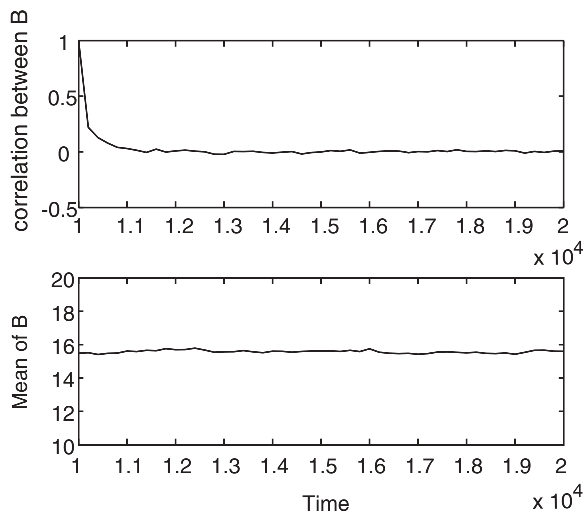

FIG. 8. (Upper) The correlation between the B agents spatial distribution as a function of time, with the following parameters: $\delta_A = 0.002$, $\beta_A = 0.005$, $\beta_B = 0.002$, $\delta_B = 0.01$, $\alpha_A = 0.01$, $\mu = 0.00005$, and $\varepsilon = 0.0041$. (Lower) Average B concentration as a function of time. The average B concentration is constant over time, while the correlation goes to zero.

increase with diffusion and then collapse when we approach the mean-field approximation. We have tested that this is indeed the case (in the full *ABC* model), by varying β_A and the diffusion rates (Fig. 9). One can clearly see a rapid increase in B as the diffusion is increased until it collapses back to zero when the diffusion is too high.

I. Dimension effect

Another direct conclusion from the proposed mechanism is the clear effect of the dimension. If the transition between the absorbing state and the upper fixed point is induced by diffusion, we expect its probability to grow, as we increase the dimension. Figure 10 shows the difference between one- and two-dimension simulations and the simulation without diffusion. In one dimension, each lattice site has two neighbors, and, in two dimensions, four neighbors. As expected, the region in parameter space where the system is mostly in the upper fixed point grows as we increase the dimension (Fig. 10).

IV. DISCUSSION

We have presented here a model describing the relationship among capital, knowledge, and resources. In this model, we took into account the effect of improved technical skill on the resources usage efficacy. A surprising result of this model (that seems to be in accordance with reality) is that improving the resources usage efficacy only reduces the steady-state value of the resources. This is a direct result from the feedback loops within the model and not a specific result of the precise formalism used.

Simply stated, since the capital reaches a steady state that is affected by the product of the technical skill and the resources, the resources and the technical skill must be inversely proportional in steady state.

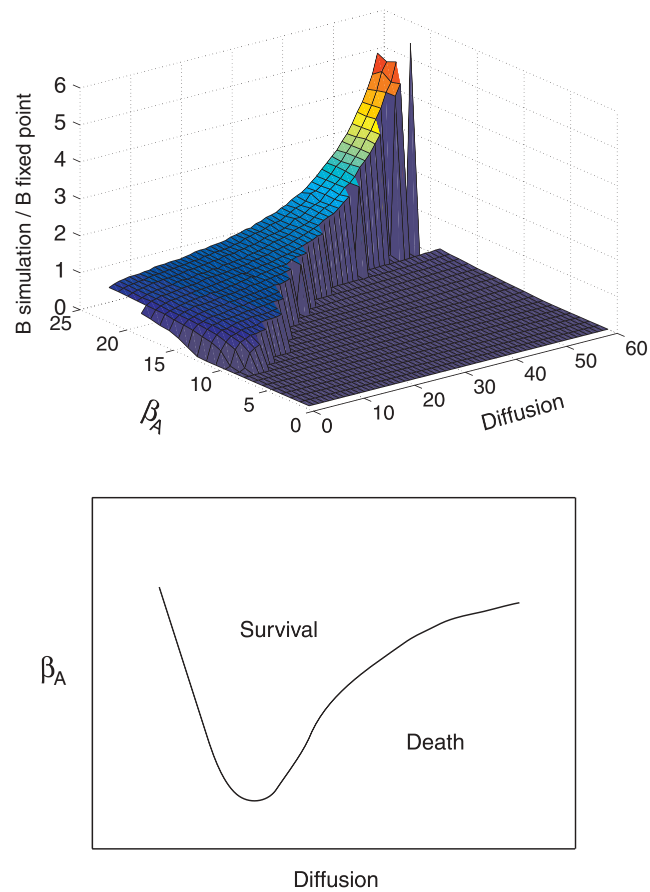

FIG. 9. (Color online) (Upper) The ratio between the average of B in the simulation and the expected high B fixed point, in the ODE as a function of β_A and the diffusion, with the parameters $\delta_A = 0.05$, $\alpha_A = 0.01$, $\beta_B = 0.01$, $\delta_B = 0.1$, $\beta_C = 0.5$, $\delta_C = 0.05$, and $k = 0.3$ on a 100×100 lattice. The diffusion and growth rate are given in arbitrary units. The diffusion is equal in A, B, and C. When the diffusion $= 0$ the system collapses for all β_A. For low positive diffusions, the system survives for a wide range of β_A values. For high diffusions, the survival range in β_A is smaller. When the diffusion is very high, the simulation is equal to the ODE and the β_A values that the system survives are limited to the ones expected in the ODE (Lower).

Another interesting aspect of the deterministic description of this model is the emergence of hysteresis for all the parameters. In other words, the total capital (and following it the resources and the technical skill) exhibit a discontinuous steady-state value as a function of a small change in a given parameter. The discontinuous transitions are irreversible, where a small parameter change leads to a drastic change in the capital, but the opposite change does not return the system to the initial capital values.

Since a sharp transition is surprising, we checked whether the stochastic counterpart of the same system has a similar behavior. In the stochastic model, the transitions are continuous. This raises the question of the mechanism leading to the smooth transition between the two sides of the expected hysteresis. We introduced a simpler system that has only two variables but similar feedback loops. This system describes the relationship between knowledge and capital only, with competition between A and itself and B and itself. The phase diagram

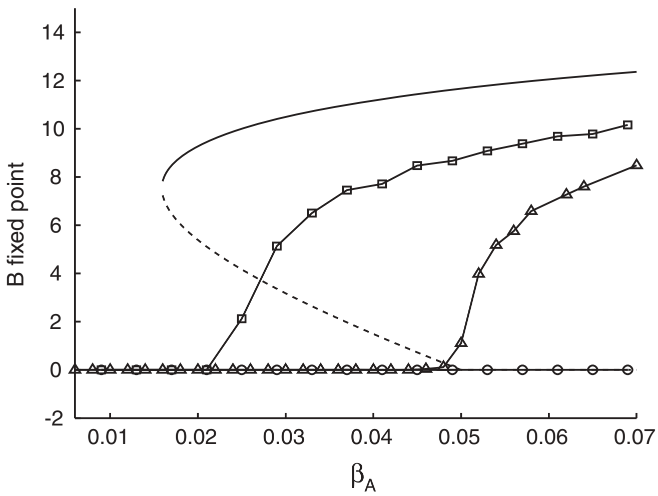

FIG. 10. The average of B as a function of β_A with the parameters $\delta_A = 0.05$, $\alpha_A = 0.01$, $\beta_B = 0.01$, $\delta_B = 0.1$, $\beta_C = 0.5$, $\delta_C = 0.05$, and $k = 0.3$, diffusion rates of A, B, and C at 0.01, initial conditions of $A = 1$, $B = 1$, $C = 9.4$ on a 100×100 lattice. The solid line is the stable fixed points in the ODE. The dashed line is the unstable fixed points in the ODE. The squares are the equilibrium values in a two dimensional stochastic system with a diffusion rate of 0.01. The triangles are the equilibrium values in a one-dimensional stochastic system with a diffusion rate of 0.01. The circles are the equilibrium value with no diffusion.

of the AB model is similar to the ABC model, and it has similar bifurcations and a similar behavior in the stochastic realization. In the simplified system, the dynamics is limited to the narrow path between the two stable fixed points and is characterized by the transition probabilities between the two states. This transition is not symmetrical. The low stable fixed point (A3) is an absorbing state, while the positive high stable fixed point is not. Thus, in order to move from the lower fixed point to the upper one, diffusion is required. This asymmetry makes the model results sensitive to the diffusion rate and to the dimension, where escape from the absorbing state can only occur for midterm diffusion values and for high-enough dimensions. Similar simpler models with a mean-field hysteresis replaced by a continuous transition in the stochastic simulation have been studied [16,36]. However, this paper presents, to the best of our knowledge, the first systematic analysis of this transition in the current context.

The proposed model has clear implications for the effect of efficacy on resource depletion; mainly that resource conservation should be performed by directly preserving resources and not by increasing the efficacy of their usage. Moreover, we show that the effect of resource depletion is reversible in contrast with the ODE expectations. This model is obviously an extreme simplification of the economic reality, and the diffusion process does not resemble the economic transition between regions. Similarly, the simple mass action formalism does not take into account the intricacies of economics. However, our conclusions are not the result of any specific detail in the model but, rather, from the feedback loop structure and the difference between the two types of fixed points. One can, thus, assume that the qualitative conclusions from the model hold in reality.

ACKNOWLEDGMENTS

We thank Gerard Weisbuch for the economic insight and for the help with the introduction and Miriam Beller for comments and assistance.

APPENDIX A

In this Appendix, we find the fixed points of system (1). System (1) can be scaled as follows:

$$\frac{dA}{d\tilde{t}} = \widetilde{\beta_A} - \widetilde{\delta_A} A + \widetilde{B},$$
$$\frac{d\widetilde{B}}{d\tilde{t}} = A\widetilde{B}\widetilde{C} - \widetilde{B}, \quad \text{(A1)}$$
$$\frac{d\widetilde{C}}{d\tilde{t}} = \widetilde{\beta_C} - \tilde{k}A\widetilde{B}\widetilde{C} - \widetilde{\delta_C}\widetilde{C}.$$

Through the following parameter and variable rescaling:

$$\tilde{t} = \delta_B t, \quad \widetilde{C} = \frac{\beta_B}{\delta_B} C, \quad \widetilde{\delta_A} = \frac{\delta_A}{\delta_B}, \quad \tilde{k} = \frac{k\beta_B}{\alpha_A},$$
$$\widetilde{B} = \frac{\alpha_A}{\delta_B} B, \quad \widetilde{\beta_A} = \frac{\beta_A}{\delta_B}, \quad \widetilde{\beta_C} = \frac{\beta_C \beta_B}{(\delta_B)^2}, \quad \widetilde{\delta_C} = \frac{\delta_C}{\delta_B}. \quad \text{(A2)}$$

Equation (A1) has three fixed points,

$$A = \frac{\widetilde{\beta_A}}{\widetilde{\delta_A}}, \quad \widetilde{B} = 0, \quad \widetilde{C} = \frac{\widetilde{\beta_C}}{\widetilde{\delta_C}}, \quad \text{(A3)}$$

$$\widetilde{B} = \frac{(\widetilde{\beta_C} - \tilde{k}\widetilde{\beta_A}) - \sqrt{(\tilde{k}\widetilde{\beta_A} + \widetilde{\beta_C})^2 - 4\tilde{k}\widetilde{\delta_A}\widetilde{\delta_C}}}{2\tilde{k}},$$
$$A = \frac{\widetilde{\beta_A} + \widetilde{B}}{\widetilde{\delta_A}}, \quad \widetilde{C} = \frac{1}{A}, \quad \text{(A4)}$$

$$\widetilde{B} = \frac{(\widetilde{\beta_C} - \tilde{k}\widetilde{\beta_A}) + \sqrt{(\tilde{k}\widetilde{\beta_A} + \widetilde{\beta_C})^2 - 4\tilde{k}\widetilde{\delta_A}\widetilde{\delta_C}}}{2\tilde{k}},$$
$$A = \frac{\widetilde{\beta_A} + \widetilde{B}}{\widetilde{\delta_A}}, \quad \widetilde{C} = \frac{1}{A}. \quad \text{(A5)}$$

The (A5) fixed point is always stable when it exists, and it is positive. The (A4) fixed point is unstable (again as long as it exists and is positive). The only fixed point whose stability is sensitive to the parameters is (A3). In all regions of the parameter space, where all three fixed points exist, the values of A and B are larger in (A5) than in (A4), which, in turn, has higher values than (A3). The values of C are inversely proportional to A and, thus, follow an opposite order.

APPENDIX B

In this Appendix, we explain the effect of k on the resources steady state. k represents the number of C's that are used to create a single B. The leading idea behind green economy is that fewer resources will be used per unit of production [27–29]. This is translated in the current schematic model to a low value of k. The basic notion behind this idea is that lowering the usage rate of resources will keep more resources available. We show that this idea fails when one reaches a steady state.

According to system (1), in steady state the following equation is obtained,

$$C^2(\alpha_A\beta_B\delta_C) + C(-k\beta_A\beta_B\delta_B - \alpha_A\beta_B\beta_C) + \left(k\delta_A\delta_B^2\right) = 0. \quad \text{(B1)}$$

The high solution of Eq. (B1) is

$$C_1 = \frac{k\beta_A\beta_B\delta_B + \alpha_A\beta_B\beta_C + \sqrt{(k\beta_A\beta_B\delta_B + \alpha_A\beta_B\beta_C)^2 - 4\alpha_A\beta_B\delta_C k\delta_A\delta_B^2}}{2\alpha_A\beta_B\delta_C}. \quad \text{(B2)}$$

This solution obeys $\frac{dC_1}{dk} > 0$ for any k that has three fixed points, since

$$\frac{dC_1}{dk} = \frac{\beta_A\beta_B\delta_B}{2\alpha_A\beta_B\delta_C} + \frac{2\,(k\beta_A\beta_B\delta_B + \alpha_A\beta_B\beta_C)\,\beta_A\beta_B\delta_B - 4\alpha_A\beta_B\delta_C\delta_A\delta_B^2}{4\alpha_A\beta_B\delta_C\sqrt{(k\beta_A\beta_B\delta_B + \alpha_A\beta_B\beta_C)^2 - 4\alpha_A\beta_B\delta_C k\delta_A\delta_B^2}}$$

$$= \frac{\beta_A\delta_B}{2\delta_C\alpha_A}\left[1 + \frac{(k\beta_A\beta_B\delta_B + \alpha_A\beta_B\beta_C)\,\beta_B - 2\alpha_A\delta_C\delta_A\delta_B/\beta_A}{\sqrt{(k\beta_A\beta_B\delta_B + \alpha_A\beta_B\beta_C)^2 - 4\alpha_A\beta_B\delta_C k\delta_A\delta_B^2}}\right]. \quad \text{(B3)}$$

The nominator of the last term is positive when three solutions exist, and all other parts of the expression are always positive.

In other words, smaller values of k induce lower C steady-state values. The more efficiently the resources are used (either through better technical skills or through more green technologies), the lower the steady-state value of the resources will be. This is exactly the opposite result of what sustainable economy scientists are trying to reach.

APPENDIX C

In this Appendix, we prove that the general model (model 2) exists for the following properties:

(a) There is inverse relation between A and C;

(b) There is reduction of the C value in the high steady state as we reduce k;

(c) The bifurcation tables of model (2) are similar to the ones in system 1,

where $f(B)$ and $g(A,C)$ are monotonic increasing concave or linear functions of B and A,C respectively that have a value of zero when the appropriate variables are zero.

In steady state, model (2) leads to

$$\beta_A - \delta_A A + f(B) = 0,$$
$$g(A,C)B - \delta_B B = 0, \quad \text{(C1)}$$
$$\beta_C - kg(A,C)B - \delta_C C = 0.$$

Equations (C1) have two sets of solutions: If $B = 0$, then A and C are stable around there equilibrium. The second solution is where $B \neq 0$. In this case, the following relation holds:

$$f^{-1}(\delta_A A - \beta_A) = B,$$
$$g(A,C) = \delta_B, \quad \text{(C2)}$$
$$\beta_C - k\delta_B f^{-1}(\delta_A A - \beta_A) - \delta_C C = 0.$$

These equations should be written as an equilibrium between A and C, when B does not play a role,

$$(a)g(A,C) = \delta_B,$$
$$(b)C = \frac{\beta_C}{\delta_C} - k\frac{\delta_B}{\delta_C}f^{-1}(\delta_A A - \beta_A). \quad \text{(C3)}$$

(a) $f(B)$ is a monotonic increasing function of B, so $f^{-1}(\delta_A A - \beta_A)$ is a monotonic increasing function of A. Thus, C is a monotonic decreasing function of A in equation (C3b). So there is an inverse relation between A and C.

(b) Decreasing k reduces the slope of $C(A)$ in equation (C3b). The steady state is at the intersection of (C3a) and (C3b). In addition, we assume that $g(A,C)$ is an increasing monotonic concave or linear function of A and C, thus, equation (C3b) must be concave. Increasing k leads to a lower value of A and, thus, a higher value of C at the intersection for the high fixed point (Fig. 11). So there is a reduction of the C value in the high steady state as we reduce k.

(c) The bifurcation table of this function is influenced by two parameters: k and β_A. For low $\beta_A$and high k, there is no intersection between equations (C3a) and (C3b), thus there is only one fixed point ($B=0$). Decreasing k leads to a transition from no positive solutions to two positive solutions (Fig. 11). Namely, for low β_A, decreasing k leads to a transition from one fixed point to three fixed points. In addition, increasing β_A leads to two positive solutions for equations (C3). One of the solutions is obtained at a very low value of A and a high value of C. The appropriate value of B in this fixed point is $B = f^{-1}(\delta_A A - \beta_A)$. This is an increasing function, so for a high value of β_A and a low value of A the value of

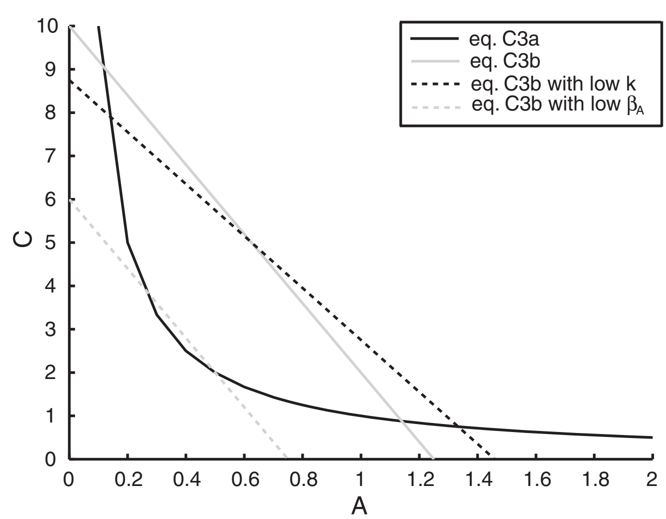

FIG. 11. The behavior of equations (C3) for high and low k and high and low β_A. The black solid line is an example for equation (C3a) $AC = 1$. An example for equation (C3b) is $C = 5 - k(A - \beta_A)$. The gray solid line is the equation (C3b) while $k = 8$, $\beta_A = \frac{5}{8}$, the black dashed line is equation (C3b) with low k: $k = 6$, $\beta_A = \frac{5}{8}$, and the gray dashed line is equation (C3b) with low β_A: $k = 8$, $\beta_A = \frac{1}{8}$.

B will be negative. Thus, for any k, increasing β_A leads to a transition from one or three fixed points to two fixed points. These transitions are equal to the bifurcation table of system 1 (Fig. 1).

APPENDIX D

We here present one possible extension of model (2):

$$\frac{dA}{dt} = \beta_A - \delta_A A + \alpha_A B,$$
$$\frac{dB}{dt} = \beta_B \frac{A}{A+v} BC - \delta_B B, \quad \text{(D1)}$$
$$\frac{dC}{dt} = \beta_C - k\beta_B \frac{A}{A+v} BC - \delta_C C,$$

and show that

(a) The inverse relation between A and C holds.

(b) The reduction in the C value steady state as we reduce k also holds.

(c) The bifurcation tables are similar to the one in system 1.

This system is a specific application of system (2), where $f(B) = \alpha_A B$ and $g(A,C) = \beta_B \frac{AC}{A+v}$.

The functions $f(B), g(A,C)$ are monotonic concave or linear functions of B and A,C respectively, and they have a value of zero when the appropriate variables are zero, thus, according to Appendix C, all three properties exist.

APPENDIX E

We here present one possible extension of model (2),

$$\frac{dA}{dt} = \beta_A - \delta_A A + \frac{\alpha_A B}{B+v},$$
$$\frac{dB}{dt} = \beta_B ABC - \delta_B B, \quad \text{(E1)}$$
$$\frac{dC}{dt} = \beta_C - k\beta_B ABC - \delta_C C,$$

and show that

(a) The inverse relation between A and C holds.

(b) The reduction in the C value steady state as we reduce k also holds.

(c) The bifurcation tables are similar to the one in system 1.

This system is a specific application of system (2), where $f(B) = \frac{\alpha_A B}{B+v}$ and $g(A,C) = \beta_B AC$.

The functions $f(B), g(A,C)$ are monotonic concave or linear functions of B and A,C respectively, and they have a value of zero when the appropriate variables are zero, Thus, according to Appendix C, all three properties exist.

APPENDIX F

In this Appendix, we show that the lattice size has no effect of the stochastic results. We run the same stochastic simulation on larger lattice size and smaller lattice size.

The size of the small lattice that we analyzed is 25 × 25. The results of this simulation are similar to the results of the simulation on 100 × 100 lattice size, as shown in Fig. 12(a). Similarly, the results obtained in a large lattice (400 × 400) are equal to the results in 100 × 100 lattice size [see Fig. 12(b)].

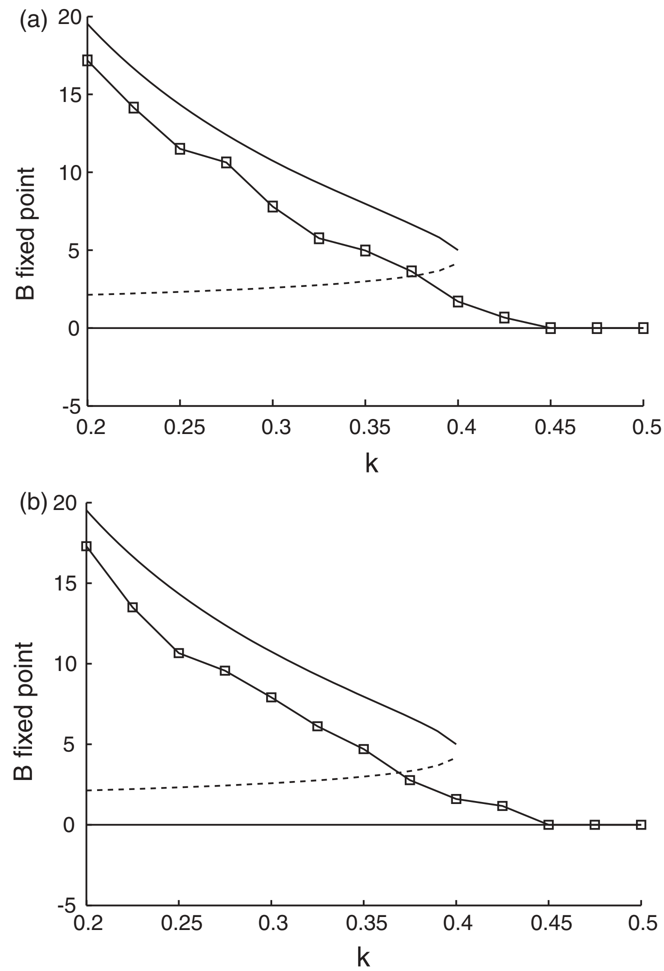

FIG. 12. (a) The average of B as a function k with the parameters $\alpha_A = 0.001$, $\delta_A = 0.005$, $\beta_A = 0.00333$, $\beta_B = 0.001$, $\delta_B = 0.01$, $\delta_C = 0.005$, $\beta_C = 0.05$, with diffusions of A and B at 0.001, with initial conditions of $A = 2$, $B = 5$, $C = 6$, on a 25 × 25 lattice. The solid line is the stable fixed points in the ODE. The dashed line is the unstable fixed points in the ODE. The square line is the fixed point in the stochastic system. (b) The average of B as a function k with the parameters $\alpha_A = 0.001$, $\delta_A = 0.005$, $\beta_A = 0.00333$, $\beta_B = 0.001$, $\delta_B = 0.01$, $\delta_C = 0.005$, $\beta_C = 0.05$, with diffusions of A and B at 0.001, initial conditions of $A = 2$, $B = 5$, $C = 6$, on a 400 × 400 lattice. The solid line is the stable fixed points in the ODE. The dashed line is the unstable fixed points in the ODE. The square line is the fixed point in the stochastic system.

In both cases the results of the stochastic simulation are continuous, and the transitions in the stochastic simulation are shifted compared with the ODE. Note that beyond the larger lattice sizes used, the diffusion time from one part of the lattice to the other parts approaches the time scale of the simulation, and we can treat a very large lattice as an ensemble of independent lattices. Thus, above a minimal size, the lattice size has no effect of the results in model (1).

APPENDIX G

In this Appendix, we show that the behavior if the QSS model (model 3) is similar to the behavior of model (1). The QSS model is developed through the following steps:

First, we find the steady-state value of C according to the third equation of model (1):

$$\beta_C - k\beta_B ABC - \delta_C C = 0, \quad C = \frac{\beta_C}{k\beta_B AB + \delta_C}. \tag{G1}$$

We set this value of C in the other equations of model (1),

$$\frac{dA}{dt} = \beta_A - \delta_A A + \alpha_A B, \quad \frac{dB}{dt} = \frac{\beta_B \beta_C AB}{k\beta_B AB + \delta_C} - \delta_B B. \tag{G2}$$

Model (G2) is the QSS model of model (1) assumption of C. The QSS model has three fixed points,

$$A = \frac{\beta_A}{\delta_A}, \quad B = 0, \tag{G3}$$

$$B = \frac{\beta_C - k\beta_A - \sqrt{(\beta_C - k\beta_A)^2 - 4k(\delta_A\delta_C - \beta_A\beta_C)}}{2k}, \quad A = \frac{\beta_A + B}{\delta_A}, \tag{G4}$$

$$B = \frac{\beta_C - k\beta_A + \sqrt{(\beta_C - k\beta_A)^2 - 4k(\delta_A\delta_C - \beta_A\beta_C)}}{2k}, \quad A = \frac{\beta_A + B}{\delta_A}. \tag{G5}$$

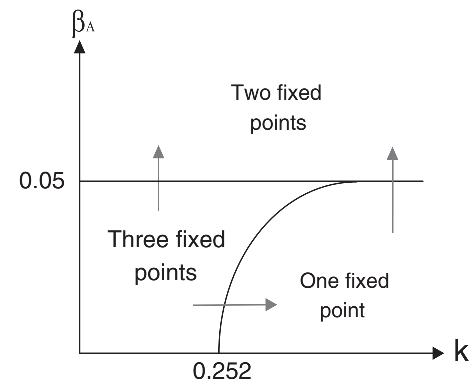

FIG. 13. The number of the fixed points of the QSS model according to β_A and k when $\delta_A = 0.05$, $\alpha_A = 0.01$, $\beta_B = 0.01$, $\delta_B = 0.1$, $\beta_C = 0.5$, and $\delta_C = 0.05$ .The system has three transitions: from three fixed points to one fixed point (as a function of k), three fixed points to two fixed points, and one fixed point to two fixed points (as a function of β_A). The bifurcation table of the QSS model is equal to the bifurcation table of model (1).

The bifurcation table of the QSS model is equal to the bifurcation table of model (1). This table is shown in Fig. 13. Thus, the behavior of the QSS model is equal to the behavior of model (1).

[1] S. Gomulka, *The Theory of Technological Change and Economic Growth* (Routledge, London, 1990).
[2] T. Berger, Agr. Econ. **25**, 245 (2001).
[3] N. H. Stern, *The Economics of Climate Change: The Stern Review*, Vol. 7 (Cambridge University Press, 2007).
[4] M. L. Weitzman, J. Environ. Econ. Manage. **60**, 1 (2010).
[5] W. D. Nordhaus, J. Econ. Lit. **45**, 686 (2007).
[6] P. Leigh, Ethics Sci. Environ. Politics **2005**, 3 (2005).
[7] P. Bairoch, *Victoires et déboires: histoire économique et sociale du monde du XVIe siècle à nos jours* (Gallimard, Paris, 1997).
[8] E. J. Malecki, *Technology and Economic Development: The Dynamics of Local, Regional, and National Change* (Longman Scientific and Technical, Essex, 1991).
[9] A. D. Foster and M. R. Rosenzweig, Am. Econ. Rev. **86**, 931 (1996).
[10] T. Koizumi and K. J. Kopecky, J. Int. Econ. **7**, 45 (1977).
[11] N. Rosenberg, Oxford Econ. Pap. **15**, 217 (1963).
[12] C. Groth and P. Schou, J. Environ. Econ. Manage. **53**, 80 (2007).
[13] A. Dixit, P. Hammond, and M. Hoel, Rev. Econ. Stud. **47**, 551 (1980).
[14] J. R. Vincent, T. Panayotou, and J. M. Hartwick, J. Environ. Econ. Manage. **33**, 274 (1997).
[15] A. Agranovich, Y. Louzoun, N. Shnerb, and S. Moalem, J. Theor. Biol. **241**, 307 (2006).
[16] A. Windus and H. J. Jensen, Theor. Popul. Biol. **72**, 459 (2007).
[17] N. M. Shnerb, Y. Louzoun, E. Bettelheim, and S. Solomon, Proc. Natl. Acad. Sci. USA **97**, 10322 (2000).
[18] Y. Louzoun, N. M. Shnerb, and S. Solomon, Eur. Phys. J. B **56**, 141 (2007).
[19] Y. Louzoun, S. Solomon, H. Atlan, and I. R. Cohen, Physica A **297**, 242 (2001).
[20] Y. Louzoun, S. Solomon, H. Atlan, and I. R. Cohen, Bull. Math. Biol. **65**, 375 (2003).
[21] A. Jameson, W. Schmidt, and E. Turkel, AIAA J. **81**, 1259 (1981).
[22] D. Hanselman and B. C. Littlefield, *Mastering MATLAB 5: A Comprehensive Tutorial and Reference* (Prentice Hall PTR, 1997).
[23] R. Alexander, SIAM J. Numer. Anal. **14**, 1006 (1977).
[24] J. P. Aparicio and H. G. Solari, Phys. Rev. Lett. **86**, 4183 (2001).
[25] J. D. Sachs and A. M. Warner, Europ. Econ. Rev. **45**, 827 (2001).
[26] T. Gylfason and G. Zoega, World Economy **29**, 1091 (2006).
[27] G. Seyfang, Environ. Planning A **33**, 975 (2001).
[28] H. Henderson and S. Sethi, *Ethical Markets: Growing the Green Economy* (Chelsea Green, White River Junction, VT, 2006).
[29] P. Krugman, N. Y. Times Mag. **5**, 34 (2010).
[30] M. A. Krasnosel'ski and A. V. Pokrovski, *Systems with Hysteresis* (Springer Verlag, Berlin, 1989).
[31] J. M. Polimeni, *The Jevons Paradox and the Myth of Resource Efficiency Improvements* (Earthscan, London, 2008).
[32] N. M. Shnerb, E. Bettelheim, Y. Louzoun, O. Agam, and S. Solomon, Phys. Rev. E **63**, 021103 (2001).
[33] R. Durrett and S. A. Levin, Philos. Trans. R. Soc. London B **343**, 329 (1994).
[34] J. Dockery, V. Hutson, K. Mischaikow, and M. Pernarowski, J. Math. Biol. **37**, 61 (1998).
[35] Y. Kan-On, Nonlinear Anal. **28**, 145 (1997).
[36] M. Roy, K. Harding, and R. D. Holt, J. Theor. Biol. **255**, 152 (2008).