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Large rare fluctuations in systems with delayed dissipation

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We study the probability distribution and the escape rate in systems with delayed dissipation that comes from the coupling to a thermal bath. To logarithmic accuracy in the fluctuation intensity, the problem is reduced to a variational problem. It describes the most probable fluctuational paths, which are given by acausal equations due to the delay. In thermal equilibrium, the most probable path passing through a remote state has time-reversal symmetry, even though one cannot uniquely define a path that starts from a state with given system coordinate and momentum. The corrections to the distribution and the escape activation energy for small delay and small noise correlation time are obtained in explicit form.

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I. INTRODUCTION

Large fluctuations play an important role in many physical phenomena, an example being spontaneous switching between coexisting stable states of a system, such as switching between the magnetization states in magnets, or voltage and current states in Josephson junctions, or macromolecule configurations or populations. Typically, large fluctuations are rare events on the dynamical time scale of the system. A theoretical analysis of such events goes back to Kramers [1], who considered the rate of switching of a Brownian particle from a potential well. The problem of the switching rate and the probability distribution becomes more complicated for systems away from thermal equilibrium, as in this case the probability distribution is no longer of the Boltzmann form. A rigorous mathematical approach to the problem was developed and many results have been obtained for dynamical systems without delay driven by white Gaussian noise and for Markovian reaction and population systems (see Refs. [2–14]). More recently the problem of large rare fluctuations in white-noise-driven systems with delay was addressed in the mathematical literature [15,16].

Delay naturally arises in dissipative dynamical systems. In such systems dissipation results from the coupling to a reservoir: Motion of the system causes changes in the reservoir, which in turn affect the motion. The underlying reaction of the reservoir is generically delayed. Perhaps the best-known example is the delay of the viscous force on an accelerated particle in hydrodynamics [17,18]. Along with the dissipative force, the reservoir exerts a random force on the system. If dissipation is delayed, the random force has a finite correlation time. These effects have been attracting much attention in the context of optomechanics and dynamical backaction [19] as well as systems with delayed active feedback, including those used for gravitational wave detection [20]. Delayed dissipation can play an important role also in ion channels [21]. A description based on delayed dissipation is used to reduce the number of dynamical variables, in particular in the analysis of cold-atom-based systems [22].

In this paper we develop a formalism for studying large rare fluctuations in classical systems with delayed dissipation. Much of the analysis refers to the case of Gaussian noise, but the results can be immediately extended to non-Gaussian noise as well. An important part of the paper, which allows us to test the general formulation, is the analysis of coupling to a reservoir in thermal equilibrium, where the noise and the dissipative force are connected by the fluctuation-dissipation relation [23].

Central for the analysis is the idea of the optimal fluctuation. In a large rare fluctuation the system is brought from its stable state to a remote state in phase space. This requires a large deviation of the noise from its root-mean-square value. Different noise realizations can result in the same outcome, but they have different probability densities (in the space of noise trajectories). For Gaussian noise, the difference is exponentially large. The overall probability of a large fluctuation of the system is determined by the most probable or optimal appropriate realization of the noise.

As a consequence, in a fluctuation to a remote point in phase space or in switching the system is most likely to move along a well-defined (optimal) trajectory, which corresponds to the optimal noise realization. Using the approach [24] and its extensions, the narrow peak in the distribution of the trajectories has indeed been seen in simulations and in the experiments (see Refs. [25–32]).

An important feature of large rare fluctuations in Markovian (no-delay) systems in thermal equilibrium with a bath is that the optimal fluctuational path is the time-reversed path in the absence of noise [33,34]. This can be understood from the argument that, in relaxation in the absence of noise, the energy of the system goes into the entropy of the thermal reservoir, whereas in a large fluctuation the entropy of the reservoir goes into the system energy. The minimal entropy change corresponds to a time-reversed process [23]. In other words, the optimal trajectory for a large fluctuation corresponds to the noise-free trajectory with the inverse sign of the friction coefficient. One can view this property also as a consequence of the symmetry of transition rates in systems with detailed balance discussed for diffusive systems described by the Fokker-Planck equation by Kolmogorov [35] (optimal fluctuational paths and the path distribution were not discussed in Ref. [35]).

We show below that the situation is more complicated if the dissipative force is delayed. We consider linear coupling to a thermal reservoir, which leads to a delayed viscous friction. The model is described in Sec. II below. A variational problem for finding optimal fluctuational paths in systems with delayed dissipation in the presence of Gaussian noise is formulated in Sec. III. Both the problems of the tail of the probability distribution and escape from a metastable state are considered. In Sec. IV we show that if the noise has thermal origin, the tail of the probability distribution remains to be of the Boltzmann form in the presence of dissipation delay. We also consider the time-reversal symmetry of the most probable trajectories. In Sec. V the results are illustrated using an exponentially correlated thermal noise. In Sec. VI we give explicit expressions for the logarithm of the probability distribution and the escape activation energy for the case where the correlation time of the noise and the dissipation delay are short. Section VII contains concluding remarks.

II. A SYSTEM LINEARLY COUPLED TO A THERMAL BATH

Delayed dissipation of a classical system coupled to a thermal bath has been actively discussed starting from the mid 1960s [36,37] and several delay-related features of classical fluctuations have been found (see Refs. [38–41] and references therein). Here we sketch the derivation of the equation of motion in the presence of delay. The analysis refers to a one-dimensional dynamical system (a particle) coupled to a bath with a quasicontinuous excitation spectrum; no special model of the bath is used, except that the coupling is assumed to be sufficiently weak and linear in the particle coordinate q. We set the particle mass equal to one. The total Hamiltonian of the particle and the bath is

$$H = H_0 + H_b + H_i, \quad H_0 = \tfrac{1}{2}p^2 + U(q), \quad H_i = qh_b. \tag{1}$$

Here p is the momentum of the particle and $U(q)$ is its potential; H_b is the Hamiltonian of the bath in the absence of the interaction; h_b is a function of the dynamical variables of the bath only.

Perhaps the best-known example of a microscopic model of dissipation is the dissipation coming from a bath that consists of harmonic oscillators, with h_b linear in the oscillator coordinates q_k [42–45] (see also Refs. [46–50] and references cited in Refs. [14,40,41]),

$$H_b = \frac{1}{2}\sum_k \left(p_k^2 + \omega_k^2 q_k^2\right), \quad h_b = \sum_k \varepsilon_k q_k; \tag{2}$$

the extension of this model to nonlinear in q coupling was discusses in Ref. [51]. However, the description of the backaction of the bath in terms of delayed dissipation outlined below is not limited to this model; it immediately extends also to the case where H_i is linear in the particle momentum p as well as to the case of nonlinear in q and p coupling.

A simple way to obtain delayed dissipation is to note that in the equation of motion of the system

$$\ddot{q}(t) + U'(q(t)) + h_b(t) = 0, \tag{3}$$

the function $h_b(t)$ itself depends on $q(t')$ with $t' \leqslant t$ because the bath is perturbed by the system. If the interaction is weak, the response of the bath to the motion of the system can be described using the generalized susceptibility α_h. It determines the mean value of h_b if instead of the coupling to the considered dynamical system the bath were driven by a time-dependent force $F(t)$, with energy $-F(t)h_b$. In the considered case the role of $F(t)$ is played by $-q(t)$,

$$h_b(t) \approx h_b^{(0)}(t) - \int_{-\infty}^{t} dt'\alpha_h(t-t')q(t'), \tag{4}$$

where we assumed that the interaction was adiabatically turned on at $t \to -\infty$. In the model (2) one immediately finds $\alpha_h(t) = \sum_k (\varepsilon_k^2/\omega_k)\sin\omega_k t$. Equation (4) applies in this case for an arbitrarily strong coupling.

In Eq. (4), $h_b^{(0)}(t)$ is the value of $h_b(t)$ in the absence of the interaction with the system. It is a random function of time; for example, in the model (2) the randomness comes from the randomness of the amplitudes and phases of the noninteracting oscillators [42]. We set $\langle h_b^{(0)}(t)\rangle = 0$. The power spectrum of $h_b^{(0)}$ can be written as $2\,\mathrm{Re}\,\Phi_h(\omega)$, where

$$\Phi_h(\omega) = \int_0^\infty dt\, \exp(i\omega t)\phi_h(t), \qquad \phi_h(t) = \langle h_b^{(0)}(t)h_b^{(0)}(0)\rangle. \tag{5}$$

The power spectrum is related to the susceptibility α_h by the fluctuation-dissipation theorem [23]. The approximations involved in deriving Eq. (4) are outlined in the Appendix.

Using that from the fluctuation-dissipation theorem $\alpha_h(t) = -\beta\dot{\phi}_h(t)$ for $t > 0$ $(\beta = 1/k_BT)$, one obtains from Eqs. (1) and (4) the equation of motion of the system coupled to the bath in the form

$$\ddot{q}(t) + U_h'(q(t)) + \beta\int_0^\infty dt'\phi_h(t')\dot{q}(t-t') = f(t), \qquad U_h(q) = U(q) - (\beta/2)q^2\phi_h(0). \tag{6}$$

Here $f(t)$ is a random force; if the only source of this force is the coupling to the bath, then $f(t) = h_b^{(0)}(t)$. However, often in the experiment noise comes from external sources that are not in thermal equilibrium with the system, and $f(t)$ accounts for such noise.

The integral term in Eq. (6) describes dissipation of the system due to the coupling to a thermal bath. The friction force is delayed. Within the linear response approximation it is linear in the velocity of the system, but depends on the velocity history. The coupling to the bath leads also to the renormalization of the potential of the system $U(q) \to U_h(q)$. We note that it is natural to count q off from a minimum of the bare potential $U(q)$ (a constant shift of q can be incorporated into H_b). Then the renormalization (6) corresponds to softening of the potential near this minimum since $\phi_h(0) > 0$.

A. Stationary states

It follows from Eq. (6) that the stationary states of the system in the absence of noise (where $\dot{q} = \ddot{q} = 0$) are located at the extrema of $U_h(q)$. Near its extremum q_0 the potential U_h can be linearized in $\delta q = q - q_0$ and the solution of Eq. (6) can be sought in the form $\delta q(t) \propto \exp(\lambda t)$ with λ given by equation

$$\lambda^2 + U_h''(q_0) + \beta\lambda\Phi_h(i\lambda) = 0. \tag{7}$$

1. Underdamped systems

Equation (7) is simplified if the coupling to the bath is so weak that the last term is a perturbation [at least for the solutions with $\Phi_h = 0$ (see below)]. In this case, if U_h has a minimum at $q_0 \equiv q_a$, with $U_h''(q_a) \equiv \omega_a^2 > 0$,

$$\lambda_{a\pm} \approx \pm i\tilde{\omega}_a - \Gamma, \quad \Gamma = (\beta/2)\mathrm{Re}\Phi_h(\omega_a), \tag{8}$$

where $\tilde{\omega}_a = \omega_a + (\beta/2)\mathrm{Im}\Phi_h(\omega_a)$. Equation (8) applies provided $\beta|\Phi_h(\omega_a)| \ll \omega_a$.

Since the thermal noise power spectrum $2\,\mathrm{Re}\Phi_h(\omega)$ is non-negative, $\Gamma \geqslant 0$. Therefore, in the presence of delay a minimum of U_h still corresponds to an asymptotically stable state of the system, an attractor. The parameter Γ is the characteristic relaxation rate of the system. Since $\Gamma \ll \omega_a$, the stable state is a focus on the phase plane (q,p) and the motion near $(q_a, p_a = 0)$ is underdamped.

Generally, systems with delay have an infinite-dimensional phase space. Therefore Eq. (7) has more than two solutions. However, the function $\Phi_h(\omega)$ is analytical for Im $\omega > 0$. This is a consequence of causality, which underlies the Kramers-Kronig relations for the susceptibility [23]. Then $\Phi_h(i\lambda)$ does not diverge for Re$\lambda > 0$, and if the coupling is small, the last term in Eq. (7) remains small for Re$\lambda > 0$, which means that there are no solutions of Eq. (7) with Re$\lambda > 0$ in the weak-coupling limit. In turn, this means that a minimum of $U_h(q)$ is an attractor.

In many cases of physical interest the correlator $\phi_h(t)$ exponentially decays for large times: $\phi_h(t) \propto \exp(-t/t_c)$ for $t \to \infty$, where t_c is the correlation time of bath fluctuations. Then for a very weak coupling Eq. (7) has a root Re$\lambda \approx -1/t_c$, which describes a comparatively fast relaxation of the bath when the system is close to the attractor, $\Gamma t_c \ll 1$. We will not discuss in this paper the case of a power-law decay of correlations in the bath, which has attracted much attention in the context of quantum tunneling [52], although some of the results apply to this case too.

A local maximum of $U_h(q)$ is a saddle point; we will use the notation $q_{\mathcal{S}}$ for this point. Within a naive perturbation theory, near the saddle point in the limit of small $|\Phi_h|$ the eigenvalues are

$$\lambda_{\mathcal{S}\pm} \approx \pm\Omega_{\mathcal{S}} - (\beta/2)\Phi_h(\pm i\Omega_{\mathcal{S}})), \tag{9}$$

where $\Omega_{\mathcal{S}} = |U''(q_{\mathcal{S}})|^{1/2}$. The coupling-induced renormalization of the root $\lambda_{\mathcal{S}+}$, which describes moving away from $q_{\mathcal{S}}$, is small for small $|\Phi_h|$. However, the change of the root $\lambda_{\mathcal{S}-}$ can be significant even in the small-$|\Phi_h|$ limit. Indeed, if $\phi_h(t) \propto \exp(-t/t_c)$ for $t \to \infty$ and $t_c^{-1} < |U_h''(q_{\mathcal{S}})|^{1/2}$, the correction to $\lambda_{\mathcal{S}-}$ proportional to Φ_h in Eq. (9) diverges: Formally, $\Phi_h(\omega)$ diverges for $-\mathrm{Im}\omega > 1/t_c$. Physically, the bare system moves in the inversely parabolic potential near the saddle point too quickly for the bath to follow it. In this case Re$\lambda_{\mathcal{S}-}$ approaches $-1/t_c$, where the coupling constant is small (i.e., coupling of an arbitrarily weak strength leads to a finite change of the stable eigenvalue near the saddle point). The small-t_c^{-1} analysis applies to the case of the coupling to a bath of harmonic oscillators (2).

A dramatic change of the dynamics can occur also for the coupling of the form (2) if the bath frequencies form a finite-width band [41]. We will not consider this case here.

2. Overdamped systems

Even where the coupling to the bath is weak in the sense that the decay of the system is slow compared to the decay of correlations in the bath, the relaxation rate of the system can exceed the frequency ω_a. In this case, for $\Phi_h(\omega)$ smooth near $\omega = 0$, the motion near the potential minimum is overdamped. From Eq. (7)

$$\lambda_{a+} \approx -\omega_a^2/\beta\Phi_h(0), \quad \lambda_{a-} \approx -\beta\Phi_h(0), \tag{10}$$

with $|\lambda_{a-}| \gg \omega_a \gg |\lambda_{a+}|$. Obviously, the eigenvalues $\lambda_{a\pm}$ are real and negative, indicating that the potential minimum remains an attractor. This can be thought of as the small-inertia (small-mass) regime. Indeed, if we incorporate the mass of the system m into the Hamiltonian H_0 and define $U_h''(q_a) = m\omega_a^2$, we see that $\lambda_{a-} \propto 1/m$, whereas $\lambda_{a+} \propto m$. The root λ_{a+} characterizes the slow motion of the system coupled to the bath; other degrees of freedom relax much faster.

Similarly, near $q_{\mathcal{S}}$ for small inertia we have

$$\lambda_{\mathcal{S}-} \approx -\beta\Phi_h(0), \quad \lambda_{\mathcal{S}+} \approx |U_h''(q_{\mathcal{S}})|/\beta\Phi_h(0) \tag{11}$$

with $|\lambda_{\mathcal{S}-}| \gg |U_h''(q_{\mathcal{S}})|^{1/2} \gg |\lambda_{\mathcal{S}+}|$. The potential maximum remains a saddle point, with motion away from it being slow compared to the motion toward it. Approaching the saddle point is characterized essentially by the relaxation rate of the bath when the system is at the saddle point. Of relevance to the motion of the small-mass system is primarily the root $\lambda_{\mathcal{S}+}$.

If the coupling is described by the model (2) and is not weak, we have not found a simple explicit expression for the eigenvalues λ. However, one can expect that the minima of $U_h(q)$ remain stable states. This is a consequence of the condition Re$\Phi_h(\omega) \geqslant 0$ for Im $\omega = 0$ (i.e., the condition that the power spectrum of the bath is non-negative); we assume that Re$\Phi_h(\omega_a) > 0$ in the frequency range of physical interest. Because of this condition Eq. (7) has no roots with Re$\lambda = 0$ for $q_0 = q_a$. Since for weak coupling Re$\lambda_a < 0$ and given that the dependence of λ_a on the coupling strength is smooth, as Eq. (7) suggests, as we increase the coupling strength the roots λ_a will never cross the axis Re$\lambda_a = 0$. Hence the state (q_a, p_a) will remain stable (see an example in Sec. V).

III. VARIATIONAL PROBLEM FOR OPTIMAL FLUCTUATIONS

Large rare fluctuations in systems with delayed dissipation and with nonwhite noise can be analyzed by extending the approach developed for systems with no delay. The underlying idea is that, before a large fluctuation occurs, the system performs small-amplitude fluctuations about its initially occupied stable state. These small fluctuations persist for a long time that largely exceeds the relaxation time. When ultimately there occurs a large fluctuation to a given point (q,p) in phase space, the system is most likely to move along the optimal trajectory that corresponds to the most probable appropriate realization of the noise, as outlined in the Introduction.

We will consider large rare fluctuations assuming that the noise $f(t)$ in Eq. (6) is Gaussian and stationary. We will use a path-integral technique. For dynamical systems with no delay several path-integral-based approaches to colored-noise-induced fluctuations were proposed earlier (see, for example, Refs. [53–62]; more references can be found in the reviews [63,64]). We will extend to systems with delay the approach developed in Ref. [62], which allows one to study both the probability distribution and the rate of switching between metastable states and, as we show, makes it possible to reveal the nonanalytic features of fluctuations related to the delay.

For a stationary zero-mean Gaussian noise $f(t)$ with correlation function $\phi_f(t-t')=\langle f(t)f(t')\rangle$ the probability density functional of noise realizations is $\mathcal{P}_f[f(t)]=\exp\{-\mathcal{R}_f[f(t)]/D\}$ [45], where

$$\mathcal{R}_f[f(t)]=\frac{1}{4}\int_{-\infty}^{\infty}dt\,dt'f(t)\mathcal{F}(t-t')f(t'),$$
$$\int_{-\infty}^{\infty}dt_1\mathcal{F}(t-t_1)\phi_f(t_1-t')=2D\delta(t-t'). \tag{12}$$

Here D is the characteristic noise intensity; in the case of thermal noise, $f(t)=h_b^{(0)}(t)$, $\phi_f(t)=\phi_h(t)$, and $D=k_BT$. We assume that D is small, so that on average the amplitude of fluctuations of the system about its attractor is small compared to the distance where the nonlinearity of the motion near the attractor becomes substantial. The function $\mathcal{F}(t)/2D$ is the inverse of the noise correlator $\phi_f(t)$.

A. Probability distribution

In the spirit of the theory of large rare fluctuations [33], to logarithmic accuracy the probability distribution $\rho(q,p)$ for a given (q,p) is determined by the probability density for the fluctuating system to reach the state (q,p) for the first time,

$$\rho(q,p)=\text{const}\times\exp[-R(q,p)/D]. \tag{13}$$

Here $R(q,p)$ is the minimum of the functional $\mathcal{R}_f$ with respect to the noise trajectories that bring the system from the attractor to the state (q,p). By appropriately extending the arguments of Ref. [62], one can show that $R(q,p)$ is given by a solution of the variational problem

$$R(q,p)=\min\Big\{\mathcal{R}_f[f(t)]+\int_{-\infty}^{\infty}dt\,\chi(t)\Big[\ddot q(t)+U_h'(q(t))$$
$$+\beta\int_{-\infty}^{t}dt'\phi_h(t-t')\dot q(t')-f(t)\Big]\Big\}, \tag{14}$$

where the minimum is taken with respect to functions $q(t)$, $f(t)$, and $\chi(t)$. The functions $q(t)$ and $f(t)$ that provide the minimum to $R(q,p)$ describe the coupled optimal fluctuational trajectories of the system and the noise.

Equation (14) can be readily understood. The function $\chi(t)$ is a Lagrange multiplier: The extremum with respect to $\chi(t)$ gives the equation of motion of the system (6). The extremum with respect to $f(t)$ corresponds to finding the most probable realization of $f(t)$ that drives the system to the targeted state. The extremum with respect to $q(t)$ couples $f(t)$ and $\chi(t)$ so as to minimize the overall value of $R(q,p)$ and thus to find the leading-order term in the exponent of the distribution $\rho(q,p)$.

The boundary conditions for the trajectory followed by the system as it moves from the attractor $(q_a,p_a=0)$ occupied for $t\to-\infty$ to a given (q,p), arriving at a given time t (we set $t=0$), read

$$f(t),\chi(t)\to 0,\quad q(t)\to q_a,\quad p(t)\to p_a\quad \text{for}\quad t\to-\infty;$$
$$f(t)\to 0\quad \text{for } t\to\infty;\quad \chi(t)=0\quad \text{for } t>0;$$
$$q(t=0)=q,\quad p(t=0)=p. \tag{15}$$

The boundary condition (15) for $t\to-\infty$ allows for the fact that $f(t)$ has to go to zero for $t\to-\infty$, otherwise $\mathcal{R}_f[f]$ would diverge; at the same time, for zero noise the system goes to the attractor. As the noise increases, the system moves away from the attractor. This motion is characterized by an exponential time dependence, as seen from the equations of motion on the optimal trajectory discussed in Sec. III B.

The boundary condition for $t=0$ corresponds to the picture of optimal fluctuation to a point (q,p) in which, once the system has reached this point for the first time, its further dynamics is no longer relevant. Respectively, the force should evolve for $t>0$ so as to minimize $\mathcal{R}_f$ independently of how its evolution affects the system. This is formally described by setting $\chi=0$ for $t>0$. Alternatively, one can set the upper limit of the integral over t in Eq. (14) equal to zero (see Sec. III B). Since $\mathcal{R}_f$ is positive definite, its minimum is reached for $f(t)=0$ and therefore $f\to 0$ for $t\to\infty$.

B. Optimal trajectory

It follows from Eqs. (14) that on the optimal trajectory

$$\frac{1}{2}\int_{-\infty}^{\infty}dt'\mathcal{F}(t-t')f(t')=\chi(t),$$
$$\ddot\chi(t)+U_h''(q(t))\chi(t)-\beta\int_{t}^{\infty}dt'\phi_h(t'-t)\dot\chi(t')=0, \tag{16}$$

while the interrelation between $f(t)$ and $q(t)$ is of the form (6). An interesting and somewhat counterintuitive feature of Eq. (16) is that the time evolution of $\chi(t)$ is acausal: The value of $\chi(t_1)$ depends on $\chi(t_1')$ with $t_1'>t_1$. This does not make the equations ill defined since we are solving a boundary-value problem, where we know where the system arrives at $t=0$ and what happens to the noise and $\chi(t)$ after that.

Equations (6) and (16) are simplified near the attractor, where $U_h'(q)\approx\omega_a^2(q-q_a)$ and the equations become linear. The solution in this range allows one to write the variational equations for the optimal trajectories in the form convenient for numerical integration.

For weak coupling to the bath, the function $\chi(t)$ for $t\to-\infty$ has the form

$$\chi(t)=\chi_-\exp(-\lambda_{a-}t)+\chi_+\exp(-\lambda_{a+}t), \tag{17}$$

where $\lambda_{a\pm}$ are given by Eq. (8) or (10) and $\chi_\pm$ are arbitrary constants. Since $\mathrm{Re}\lambda_{a\pm}<0$, the solution (17) satisfies the boundary condition (15) for $t\to-\infty$. Generally, there are also other terms in $\chi(t)$, but as $t\to-\infty$ they decay faster than Eq. (17) if the correlation time of the bath fluctuations is small compared to $|\mathrm{Re}\lambda_{a\pm}|^{-1}$, as assumed in Eq. (17).

From Eqs. (12) and (16) the optimal realization of the noise is simply expressed in terms of $\chi(t)$,

$$f(t)=\frac{1}{D}\int_{-\infty}^{\infty}dt'\phi_f(t-t')\chi(t'). \tag{18}$$

If the noise correlation time is shorter than the relaxation time of the system, as assumed in Eq. (17), then for $t \to -\infty$

$$f(t) = \sum\nolimits_{\nu=\pm} f_\nu \exp(-\lambda_{a\nu} t),$$
$$f_\pm = D^{-1}\chi_\pm[\Phi_f(i\lambda_{a\pm}) + \Phi_f(-i\lambda_{a\pm})]. \tag{19}$$

Here $\Phi_f(\omega) = \int_0^\infty dt \phi_f(t) \exp(i\omega t)$; the function $2\,\mathrm{Re}\Phi_f(\omega)$ is the power spectral density of the noise $f(t)$. Equation (19) shows that on the optimal trajectory $f(t) \to 0$ for $t \to -\infty$, in agreement with the boundary condition (15).

The deviation of the coordinate and momentum of the system from (q_a, p_a) is also exponential in time for $t \to -\infty$; from Eqs. (6) and (19) for a short noise correlation time

$$q(t) - q_a = -\sum\nolimits_{\nu=\pm} f_\nu \exp(-\lambda_{a\nu} t) \times \{\beta\lambda_{a\nu}[\Phi_h(i\lambda_{a\nu}) + \Phi_h(-i\lambda_{a\nu})]\}^{-1}. \tag{20}$$

From Eqs. (17)–(20) the optimal trajectory near the stable state is determined by the parameters χ_- and χ_+. They must be found from the boundary condition for $t = 0$. In the numerical analysis this allows one to start at a finite time t_i at a finite distance from the stable state, so that the potential $U(q(t_i))$ is still parabolic; then the duration of motion along the optimal trajectory is finite.

Important general features of fluctuations in systems away from thermal equilibrium are that (i) not all points within the basin of attraction to a given stable state (q_a, p_a) can be reached by optimal trajectories that go from this stable state [12] and (ii) several trajectories given by Eqs. (6) and (16) can come to the same state (q, p) [4,9,10,12,62]. As for systems without delay, in the case of multiple extreme trajectories arriving at the same state, one should choose the trajectory that gives the absolute minimum of $R(q, p)$. We expect on physical grounds that, in analogy to systems without delay [12], if a state cannot be reached from one attractor, it can be reached from another attractor or from a saddle point: It is important that the initial state on the optimal trajectory be stationary, but it does not necessarily have to be stable.

1. Singular behavior of the auxiliary function $\chi(t)$

The function $\chi(t)$ is discontinuous at the instant $t = 0$ when the system reaches the targeted state. Therefore, generally, along with the part that is smooth for both $t < 0$ and $t > 0$, the function $\dot\chi(t)$ has a singular term $-\chi(-\varepsilon)\delta(t + \varepsilon)$ $(\varepsilon \to 0+)$.

Alternatively, one can formulate the variational problem of reaching a given state in such a way that the term proportional to $\chi(t)$ in Eq. (14) is integrated from $-\infty$ to the instant of observation $t = 0$. Then the variational equation for $\ddot\chi$ reads

$$\ddot\chi(t) + U_h''(q(t))\chi(t) - \beta \int_t^0 dt' \phi_h(t' - t)\dot\chi(t') + \beta\phi_h(t)\chi(0) = 0, \quad t < 0. \tag{21}$$

This equation coincides with Eq. (16) for $\ddot\chi$ if one allows for the aforementioned δ function in $\dot\chi$ in Eq. (16).

C. Escape problem

Noise can also lead to escape of the system from the initially occupied attractor and switching to another attractor. To find the probability of the corresponding large fluctuation per unit time one should minimize the functional $\mathcal{R}_f$ [Eq. (12)] with respect to noise realizations that lead to escape. The key here is to note that after the noise $f(t)$ and the memory kernel $\phi_h(t)$ will have decayed, the system should be outside the basin of attraction of the initially occupied attractor or at least on the boundary of this basin [62].

A correlated noise decays in time smoothly; it takes an infinite amount of time to decay to zero, as is also the case, generally, for the memory kernel. In contrast, for $t \to \infty$ the system approaches a stationary state. From Eq. (17) such a state may not be an attractor since $\chi(t)$ would diverge there for $t \to \infty$. Therefore, it must be a saddle point $q_{\mathcal{S}}$, a local maximum of the potential $U_h(q)$.

From the above arguments, to logarithmic accuracy, the escape rate has the form

$$W_e \propto \exp(-R_A/D), \tag{22}$$

where the effective activation energy R_A is given by the solution of the variational problem (14) with the boundary conditions for $t \to -\infty$ of the form of Eq. (15), whereas the other boundary conditions are

$$f(t), \chi(t) \to 0, \quad q(t) \to q_{\mathcal{S}} \quad \text{for } t \to \infty. \tag{23}$$

It follows from Eq. (16) linearized near $q_{\mathcal{S}}$ that for weak coupling, $\chi(t)$ decays as $\exp(-\lambda_{\mathcal{S}+}t)$ for $t \to \infty$. If the correlation time of $f(t)$ is smaller than $1/\lambda_{\mathcal{S}+}$, on the optimal escape trajectory $f(t)$ and $q(t) - q_{\mathcal{S}}$ decay in the same way. Otherwise the decay of $f(t)$ and $q(t)$ is controlled by the decay of $f(t)$ as given by Eq. (18) or the decay of $\phi_h(t)$. This completes the formulation of the problem of the activation energy of escape in the presence of delay.

We note that the explicit expression for $f(t)$ in terms of $\chi(t)$ [Eq. (18)] allows eliminating $f(t)$ and reducing the variational problem (14) to that for two coupled functions $\chi(t)$ and $q(t)$ (see Sec. VI). In fact, it corresponds to integrating over realizations of $f(t)$ in the functional integral that determines the rate of the rare event; the resulting functional (besides the q-dependent part) is the characteristic functional of the noise $\tilde{\mathcal{P}}_f[i\chi(t)]$. For systems without delay this approach to the escape problem was used earlier [55–57]. Although mathematically equivalent, it becomes less transparent than the above approach when it is necessary to formulate the boundary conditions for the optimal trajectories and to see the difference between the problems of reaching a given state and escaping from the basin of attraction; the above approach is also more convenient for the analysis of the motion of the system after the targeted state has been reached.

We note also that we do not discuss here the prefactor in the escape rate. For white-noise-driven systems in thermal equilibrium this problem was studied by Kramers [1]. A generalization to systems with delay was considered in Refs. [38,39] by assuming that the exponential factor is of the Boltzmann form.

IV. LARGE FLUCTUATIONS INDUCED BY THERMAL NOISE

Optimal trajectories of the system and the noise can be found and the logarithm of the probability distribution can be obtained in an explicit form in the case where the noise $f(t)$ is of purely thermal origin, $f(t) = h_b^{(0)}(t)$. The probability distribution should be of the Boltzmann form. Here we show that this is indeed the case and address the question of how the distribution is formed, dynamically, and whether there is a difference between the patterns of optimal trajectories in systems with and without delay.

For thermal noise $\phi_f(t) = \phi_h(t)$ and the noise intensity $D = k_BT \equiv 1/\beta$. One can show that the variational equations (6), (16), and (21) for the trajectory that arrives at the target point (q,p) at $t = 0$ have a solution

$$\chi(t) = \dot q(t), \quad t \leqslant 0,$$
$$\ddot q(t) + U_h'(q(t)) = \beta \int_t^0 dt' \phi_h(t'-t)\dot q(t'); \tag{24}$$

the function $f(t)$ is given by Eq. (18) with $\chi(t) = \dot q(t)$ for $t \leqslant 0$ and $\chi(t) = 0$ for $t > 0$. Equation (24) applies also to the problem of escape, except that the integral on the right-hand side of the second equation goes from t to ∞; Eq. (24) holds in the whole range $-\infty < t < \infty$ in this case.

Noting that in the variational problem (14) $\mathcal{R}_f[f(t)] = \frac{1}{2}\int_{-\infty}^{\infty} dt\, \chi(t) f(t)$ and expressing $f(t)$ in this expression in terms of q from the equation of motion of the system (6), one obtains

$$R(q,p) = \frac{1}{2}p^2 + U_h(q) - U_h(q_a),$$
$$R_A = U_h(q_S) - U_h(q_a). \tag{25}$$

Thus, as expected, in thermal equilibrium the probability distribution of the system (13) is of the Boltzmann form, with a renormalized potential due to the interaction. The activation energy of escape is given by the renormalized height of the potential barrier. This is a nice confirmation of the consistence of the developed approach.

We now discuss the form of the optimal trajectories (24) for δ-correlated fluctuations of the thermal bath, where $\phi_h(t) = 4\Gamma k_B T\delta(t)$, with Γ being the coefficient of viscous friction. In this case the trajectory $q(t)$ [Eq. (24)] differs by time inversion $t \to -t, \dot q \to -\dot q$ from the trajectory $q(t)$ in the absence of noise, which is given by equation $\ddot q + 2\Gamma \dot q + U''(q) = 0$ (see Ref. [33]). As mentioned in the Introduction, this symmetry can be understood by noting that in a fluctuation the system gets energy at the expense of the decrease of entropy of the thermal bath. Taking a time-reversed path minimizes the required entropy change.

The time-reversal symmetry of the optimal and noise-free trajectories in Markov systems in thermal equilibrium was observed in simulations by Luchinsky and McClintock [25]. They studied an overdamped system in the stationary regime and the distribution of the trajectories followed on the way to a given point q and then back to the stable state. The distribution peaked at a trajectory that was symmetric with respect to time reversal if time was counted off from the instant of reaching the chosen state. In other words, the segments of the most probable trajectory, which correspond to approaching the state and to moving away from it, are symmetrical. For equilibrium Markov systems with inertia such symmetry should hold for the most probable fluctuational trajectory to a state (q,p) and the noise-free trajectory from the state $(q,-p)$ (see Fig. 1 below).

A. Symmetry of the most probable trajectories

The time-reversal argument must be modified in the general case of delayed dissipation. Indeed, just to define a noise-free trajectory it is insufficient to specify the starting point in phase space (q,p). Instead one has to specify the whole history of motion before the state (q,p) is reached. However, one can still consistently study the distribution of trajectories that go through a given state starting from the vicinity of the attractor.

We consider the most probable trajectory that passes through the state (q,p) at $t = 0$ (obviously, this instant is arbitrary). The segment of the trajectory for $t < 0$ is the optimal fluctuational trajectory. In thermal equilibrium it is described by Eq. (24). The most probable trajectory after the state (q,p) has been reached is described by Eq. (6). In contrast to Markov systems, the force $f(t)$ in this equation is generally nonzero because the noise is correlated. Since it was nonzero for $t < 0$, where it drove the system against the potential gradient, it does not instantly go to zero for $t > 0$. The most probable value of $f(t)$ for $t > 0$ is given by Eq. (18) with $D = k_BT$.

From Eqs. (6) and (18) with account taken of the relations $\chi(t) = \dot q$ for $t < 0$ and $\chi(t) = 0$ for $t > 0$, we obtain that the most probable trajectory of the system after the state (q,p) has been reached is described by equation

$$\ddot q(t) + U_h'(q(t)) = -\beta \int_0^t dt' \phi_h(t-t')\dot q(t') \quad (t > 0). \tag{26}$$

A comparison of Eqs. (24) and (26) shows that for the most probable trajectory to a state with $p = 0$, the segments from the attractor to the state [Eq. (24)] and from the state to the attractor [Eq. (26)] are symmetric with respect to time reversal $t \to -t, \dot q \to -\dot q$. This is no longer true for states with $p \neq 0$. The symmetry and the lack of it are illustrated in Fig. 1 below.

A related important distinction of systems with delay from Markov system is that the most probable trajectory that reaches a point (q,p) is smooth at this point even where $p \neq 0$. This is also illustrated in Fig. 1.

V. THERMAL NOISE WITH EXPONENTIALLY CORRELATED POWER SPECTRUM

To illustrate the general results we will discuss the case where the noise spectrum has a simple form

$$\phi_h(t) = C_h \varkappa k_B T \exp(-\varkappa|t|), \tag{27}$$

where $\varkappa = 1/t_c$ is the reciprocal correlation time and C_h characterizes the noise intensity; for the coupling model (2) C_h is independent of temperature. The limit $\varkappa \to \infty$ corresponds to a δ-correlated noise.

For the model (27), Eq. (7) for the eigenvalues that characterize noise-free motion near the attractor takes the form

$$(\lambda + \varkappa)\left(\lambda^2 + \omega_a^2\right) + C_h\varkappa\lambda = 0. \tag{28}$$

We first consider the case where the coupling is weak. Here two of the roots of Eq. (28) $\lambda_{1,2}$ are given by Eq. (8) for $C_h\varkappa\omega_a^{-1} \ll (\varkappa^2 + \omega_a^2)^{1/2}$ [in this case $\Gamma = -\mathrm{Re}\lambda_{1,2} \approx$

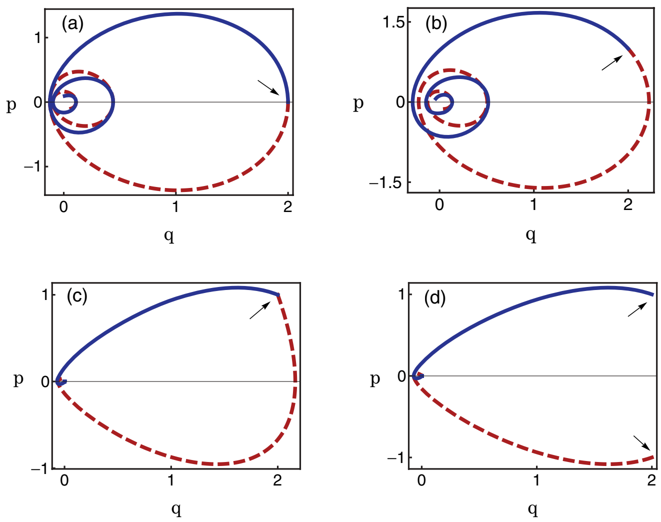

FIG. 1. (Color online) Most probable trajectories of thermal fluctuations in which the system reaches targeted states (q,p) (solid lines). The dashed lines in (a)–(c) show the sections of the trajectories after the targeted states have been reached. Plots (a) and (b) refer to a system in a parabolic potential $U_r = \omega_a^2 q^2/2$ and the exponentially correlated noise (27) with $\varkappa/\omega_a = 1$ and $C_h/\omega_a = 1.5$. The targeted states are (a) $(q = 2, p = 0)$ and (b) $(q = 2, p/\omega_a = 1)$, as indicated by the arrows. Panel (c) shows the same trajectories as in (b) in the Markov limit $\varkappa \to \infty$. Panel (d) shows the symmetry of the most probable trajectories to a state (q,p) and from the state $(q,-p)$ in Markov systems in thermal equilibrium.

$C_h\varkappa^2/(\varkappa^2 + \omega_a^2)]$ or Eq. (10) for $\varkappa \gg C_h \gg \omega_a$ [in this case $\lambda_1 \approx -\omega_a^2/C_h, \lambda_2 \approx -C_h$], respectively. In addition, Eq. (28) has a root with a much larger (in the absolute value) real part

$$\lambda_3 \approx -\varkappa + C_h\varkappa^2/(\varkappa^2 + \omega_a^2), \quad C_h \ll \varkappa$$

$(-\lambda_3 \gg -\text{Re}\lambda_{1,2})$.

In the opposite limit $C_h \gg \varkappa \gg \omega_a$, which can be of relevance for the model (2), the roots become $\lambda_1 \approx -\omega_a^2/C_h$ and $\lambda_{2,3} \approx -[\varkappa \mp i\sqrt{4\varkappa C_h - \varkappa^2}]/2$. For $C_h \sim \varkappa \sim \omega_a$ the real parts of all three roots $\lambda_{1,2,3}$ are of the same order of magnitude. It is easy to see that $|\text{Re}\lambda_{1,2,3}| < \varkappa$ i.e., correlations of the bath decay faster than $q(t)$.

An explicit solution of Eq. (24) for the optimal trajectory to a state (q,p) can be obtained for a harmonic potential $U_h(q) = \omega_a^2 q^2/2$. It reads

$$q(t) = \sum_i q_i e^{-\lambda_i t}, \quad \sum_i \lambda_i(\varkappa + \lambda_i)^{-1} q_i = 0,$$
$$\sum_i q_i = q, \quad \sum_i \lambda_i q_i = -p \quad (i = 1,2,3). \tag{29}$$

For the coupling to the bath of the form (2), this solution applies for an arbitrary coupling strength. It shows that the optimal fluctuational trajectory is a superposition of three exponentials. The relation between $q_{1,2,3}$ given by the second of Eqs. (29) follows from the condition that the sum of the exponentials satisfies the integro-differential equation (24).

The most probable trajectories for reaching a given state (q,p) and then moving back to the attractor for the model (27) are shown in Fig. 1. The sections of the trajectories to and from the targeted state are symmetric for $p = 0$. The symmetry is lost if $p \neq 0$, but the overall trajectory to and from the state is smooth. In contrast, for Markov systems, on the most probable trajectory that reaches a state (q,p) the derivative dp/dq is discontinuous at this state for $p \neq 0$. However, in such systems the most probable trajectory to state (q,p) has a symmetric noise-free counterpart that starts from $(q,-p)$ and goes to the attractor. Such a counterpart is not generally defined for systems with delay.

For a nonparabolic potential $U(q)$ one can find the trajectory to a given state (q,p) numerically by integrating Eq. (24) backward in time from $t = 0$, taking into account the time-reversal symmetry and using the values of $q(0) = q$ and $\dot{q}(0) = p$. To find the optimal escape trajectory, one can differentiate Eq. (24) over time, which leads to equation

$$\dddot{q} + \varkappa\ddot{q} + [U_h''(q) + C_h\varkappa]\dot{q} - \varkappa U_h'(q) = 0. \tag{30}$$

One can then seek the solution of this equation by the shooting method, starting from the vicinity of the stable state, where $q(t) - q_a$ is a sum of three exponentials [see Eq. (29)]. The coefficients $q_{1,2,3}$ have to be found from the condition that the trajectory arrives at $(q_S, p_S = 0)$ for $t \to \infty$ and that the solution satisfies the initial integro-differential equation (24).

VI. SHORT NOISE CORRELATION TIME

An explicit solution for the probability distribution and the escape rate can be obtained in the case of short correlation time of the noise and short delay time of the bath compared to the relaxation time of the system, when functions $\phi_f(t)$ and $\phi_h(t)$ are close to δ functions. To do this it is convenient to eliminate $f(t)$ from the functional (14). Then the variational problem for reaching a given state (q,p) takes the form

$$R(q,p) = \min\Bigg\{ -\frac{1}{D}\int_{-\infty}^{0} dt \int_{-\infty}^{t} dt'\chi(t)\phi_f(t-t')\chi(t') + \int_{-\infty}^{0} dt\,\chi(t)\Big[\ddot{q}(t) + U_h'(q(t)) + \beta\int_{-\infty}^{t} dt'\phi_h(t-t')\dot{q}(t')\Big]\Bigg\}. \tag{31}$$

The variational problem for the activation energy of escape is given by Eq. (31) in which the integrals over t (but not over t') go from $-\infty$ to ∞ rather than from $-\infty$ to 0.

If the dissipation has no delay and the noise is white, $\phi_f(t) = 4\Gamma D\delta(t)$ and $\phi_h(t) = 4\Gamma k_B T\delta(t)$, the optimal trajectories for the variational problem (31) are

$$\ddot{q} + U_h' - 2\Gamma\dot{q} = 0, \quad \chi = \dot{q}. \tag{32}$$

In this approximation, which is of zeroth order in delay and correlation,

$$R^{(0)}(q,p) = \frac{1}{2}p^2 + U_h(q) - U_h(q_a),$$
$$R_A^{(0)} = U_h(q_S) - U_h(q_a). \tag{33}$$

If the correlation times of ϕ_f and ϕ_h are short compared to the relaxation time of the system, one can write

$$\phi_{f,h}(t) = 2\Phi_{f,h}(0)\delta(t) + [\phi_{f,h}(t) - 2\Phi_{f,h}(0)\delta(t)]$$

and then consider the term in square brackets as a perturbation. We choose Γ and D in the same way as for δ-correlated $\phi_{f,h}(t)$,

$$2\Phi_f(0) \equiv \int_{-\infty}^{\infty} dt\, \phi_f(t) = 4\Gamma D,$$
$$2\Phi_h(0) \equiv \int_{-\infty}^{\infty} dt\, \phi_h(t) = 4\Gamma k_B T.$$

To first order in the delay and correlation perturbation, the correction to $R(q,p)$ can be calculated using the zeroth-order trajectories (32). In the integrals over t' in Eq. (31) it is convenient to expand $\chi(t') \approx \chi(t) + (t'-t)\dot\chi(t)$ and $\dot q(t') \approx \dot q(t) + (t'-t)\ddot q(t)$. Substituting this into Eq. (31) one obtains

$$R(q,p) \approx R^{(0)}(q,p) + \Gamma(\overline{t_f} - \overline{t_h})p^2,$$
$$\overline{t_{f,h}} = \int_0^{\infty} t\phi_{f,h}(t)dt \Big/ \int_0^{\infty} \phi_{f,h}(t)dt. \quad (34)$$

The parameters $\overline{t_f}$ and $\overline{t_h}$ characterize the widths of the correlators $\phi_f(t)$ and $\phi_h(t)$, respectively. From Eq. (34) the correction to $R(q,p)$ is of first order in these widths. For overdamped systems without delay a correction of first order in the noise correlation was found in Ref. [65] for the exponentially correlated noise [with a correlator of the form of Eq. (27)] and in Ref. [62] for the noise with an arbitrary spectrum.

The broadening of the noise correlator $\phi_f(t)$ and the dissipation delay [the broadening of $\phi_h(t)$] act in opposite directions, as seen from Eq. (34). The larger $\overline{t_f}$ is, the smaller the probability (13) is to reach a remote state, whereas an increase of $\overline{t_h}$ increases this probability. Note that $\overline{t_f}$ and $\overline{t_h}$ can be positive or negative. In thermal equilibrium the two effects compensate for each other.

From Eq. (34) there is no correction to the activation energy of escape R_A of first order in the width of the correlators $\phi_{f,h}(t)$ because $p=0$ at the saddle point. The lowest-order correction appears in second order. It can be calculated similarly to the above procedure, taking into account that the integrals over t in Eq. (31) now run to infinity. In the integrals over t' one should expand $\chi(t')$ and $\dot q(t')$ about $\chi(t)$ and $\dot q(t)$, respectively, to second order in $t-t'$. Then Eq. (32) gives

$$R_A \approx R_A^{(0)} + \Gamma\left(\overline{t_f^2} - \overline{t_h^2}\right) \int_{-\infty}^{\infty} dt\, \ddot q^2(t),$$
$$\overline{t_{f,h}^2} = \int_0^{\infty} t^2\phi_{f,h}(t)dt \Big/ \int_0^{\infty} \phi_{f,h}(t)dt, \quad (35)$$

where $\ddot q(t)$ is given by Eq. (32) with boundary conditions $q(t) \to q_a$ for $t \to -\infty$ and $q(t) \to q_{\mathcal{S}}$ for $t \to \infty$. For overdamped systems without delay a quadratic in the noise correlation correction to the activation energy of escape was found in Refs. [55,65] for the exponentially correlated noise and in Ref. [62] for the noise with an arbitrary spectrum.

The time moments $\overline{t_{f,h}^2}$ in Eq. (35) are determined by the curvature of the corresponding power spectra for $\omega = 0$: $\overline{t_{f,h}^2} = -[d^2\Phi_{f,h}/d\omega^2]_{\omega=0}/\Phi_{f,h}(0)$. Therefore, they can be positive or negative. The contributions to the escape activation energy R_A due to noise correlations and dissipation delay enter with opposite signs and compensate for each other for thermal noise. An extension of the approach leading to Eq. (35) to overdamped systems was compared with numerical simulations for a model system and good agreement was found in the whole range where the delay time was smaller than the relaxation time [66].

VII. CONCLUSION

In this paper we have considered large rare fluctuations induced by Gaussian noise in systems with delayed dissipation. The dissipation comes from coupling to a thermal bath; the corresponding friction force depends on the history of the system motion and is described by an integral of the velocity with the coupling-dependent kernel. The noise, along with the part that comes from the thermal bath, can have another source. The analysis refers to systems in thermal equilibrium as well as nonequilibrium systems.

The proposed formulation reduces the problems of finding the logarithm of the probability distribution over the phase space of the system and the effective activation energy of escape from a metastable state to variational problems [Eqs. (14), (15), and (23)]. The extreme trajectories of the respective variational functionals provide the most probable paths followed by the system in a fluctuation to a given state or in escape and also the most probable corresponding realizations of the noise. Interestingly, the variational equations that describe these trajectories turned out to be acausal in systems with delay and therefore their numerical solution poses an extra challenge compared to systems without delay.

As a consistency test, we show that if the noise is coming from the thermal bath responsible for the dissipation, the logarithm of the probability distribution has a familiar Boltzmann form, with a renormalized potential due to the coupling to the bath. Although not unexpected, this property is shown here for arbitrary nonlinear systems with delayed dissipation.

For systems in thermal equilibrium we have found simple equations for the most probable trajectories followed in large rare fluctuations. We describe the portions of the trajectory both leading to a state in phase space and followed after the state had been reached. It is shown that if the state corresponds to zero momentum, these portions are time-reversal symmetric. In contrast to Markov systems, however, the phase trajectories are smooth at the observation point. This is a clear signature of the dissipation delay. Also, in contrast to Markov systems, one may not compare the most probable trajectory to a given point (q,p) in phase space of the system with the trajectory from this or time-inversion symmetric point $(q,-p)$ since the latter trajectory is not defined for a system with delayed dissipation. Explicit solutions are obtained for the exponentially correlated in time noise and are used to illustrate the general properties of the most probable trajectories in systems in thermal equilibrium.

The case of short correlation times of the dissipation and generally nonthermal noise is analyzed. It is shown that the logarithm of the probability distribution has corrections of first order in the widths of the noise time correlation function and the dissipation kernel [Eq. (34)]. In contrast, the effective activation energy of switching has only second-order corrections in these widths [Eq. (35)]. These explicit results can be immediately extended also to systems additionally driven by moderately strong periodic fields, which lead to linear in the field amplitude terms in the corresponding exponents, as in the case of systems with no delay [64].

Delayed dissipation is commonly encountered in physical systems, as indicated in the Introduction. In a broad class of such systems of current interest dissipation comes from the coupling to an environment with pronounced dispersion of the density of states. Such an environment is provided by decaying cavity modes or Josephson junctions, for example. Advantageous for the experiments will be strongly nonlinear high-Q nanomechanical or cavity modes coupled to such an environment, as their vibration frequency significantly varies depending on the energy, so that it goes through the regions of different density of states of the environment, which makes the decay non-Markovian. At the same time, such systems in the case where their dynamics in slow time is Markovian have already been used to study not only the probability distribution far from thermal equilibrium, but also the distribution of the trajectories followed in rare large fluctuations, as in Ref. [31]. Developing the means for calculating such distributions in the presence of delay is beyond the scope of the present paper, but the results show where the maximum of this distribution should be located.

Note added. Recently, different problems of large rare fluctuations in systems with delay were addressed and a different approach was used in Ref. [67].

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APPENDIX: THE WEAK-COUPLING APPROXIMATION

In the general case the coupling to the bath H_i is nonlinear in the dynamical variables of bath excitations. Equation (4) then applies for weak coupling. It is obtained by calculating the correction to the bath variables of the first order in H_i and expanding h_b to the first order in this correction. To keep track of the orders of the perturbation theory, we formally introduce a small dimensionless parameter $\varepsilon \ll 1$, which characterizes the coupling strength, $H_i \propto \varepsilon$. The function $\alpha_h(t-t') \propto \varepsilon^2$ is determined by the mean value of the factor multiplying $q(t')$ in the expression for $h_b(t)$. In the same order in ε there remains in h_b a random part, also proportional to $q(t')$, which is averaged out when α_h is calculated. The order of this random part with respect to ε is thus higher than that of the random term $h_b^{(0)} \propto \varepsilon$ and therefore this part is disregarded.

The justification of Eq. (4) is based on the assumption that for the weak coupling, the relaxation time of the system t_r largely exceeds the correlation time of fluctuations of the bath $t_{\rm corr}$. Of importance is the correlation time of the fluctuations at frequencies close to the dynamical frequencies of the system; therefore, we use for $t_{\rm corr}$ a notation that differs from the decay time of the correlator $\phi_h(t)$ for $t\to\infty$. From Eq. (6), $t_r^{-1} \propto \varepsilon^2 t_{\rm corr}$. We are interested in the evolution of the system over time $\Delta t \gtrsim t_r \gg t_{\rm corr}$. Over this time the effect of the integral term in Eq. (6) accumulates and becomes of order unity. Also, if the observation time $t \gg t_{\rm corr}$, the lower limit of the integral over dt' in Eq. (4) (i.e., the instant where the system-to-bath coupling is turned on) becomes immaterial; sometimes it is set equal to zero [51].

Consider now the higher-order response term that is quadratic in $q(t')$. This term will be a double integral, with the structure

$$\iint dt_1 dt_2 \alpha_{hh}(t-t_1, t-t_2) q(t_1) q(t_2), \tag{A1}$$

where α_{hh} is the corresponding nonlinear susceptibility. By construction, it is proportional to ε^3. Compared to the linear in q term kept in Eq. (4), the extra time integral in Eq. (A1) gives an extra factor proportional to $t_{\rm corr}$. Therefore, overall, a contribution of the term (A1) over the time interval $\Delta t \gg t_{\rm corr}$ has an extra factor proportional to $\varepsilon t_{\rm corr} \propto (t_{\rm corr}/t_r)^{1/2} \ll 1$ compared to that of the q-dependent term in Eq. (4). The above argument immediately extends to the terms of higher order in q.

We now show that the noise $h_b^{(0)}$ in Eq. (6) is asymptotically Gaussian. It was argued previously [40] that if the delayed friction is linear in $\dot q$ and the noise is thermal, it should be Gaussian. The argument can be shortened to the observation that the noise is known to be Gaussian in the case of linear coupling to a bath of harmonic oscillators [43]. One can choose the density of states of the oscillator bath in such a way that function $\alpha_h(t)$ for this bath coincides with $\alpha_h(t)$ for the bath we consider. Then, if both baths are at the same temperature, the dynamics of the system should be the same whether the system is coupled to one bath or the other. Therefore, the noise $h_b^{(0)}$ should be Gaussian as well.

To be fully consistent, one should start from the Hamiltonian (1) and show that the approximations of the linear response of the bath [Eq. (4)] and of the noise $h_b^{(0)}(t)$ being Gaussian follow from the same assumptions. To see that this is indeed the case we note first that the correlator $\langle h_b^{(0)}(t_1)h_b^{(0)}(t_2)\rangle$ becomes small for $|t_1-t_2| \gg t_{\rm corr}$. We now consider the correlator $\langle h_b^{(0)}(t_1)h_b^{(0)}(t_2)h_b^{(0)}(t_3)h_b^{(0)}(t_4)\rangle$. We are interested in the long-time behavior of this correlator, where at least some of the time differences $|t_i-t_j|$ are $\sim t_r$ and largely exceed $t_{\rm corr}$ (here i,j run through $1,\ldots,4$). Since correlations decay over time $t_{\rm corr}$, this behavior is described by equation

$$\begin{aligned}&\langle h_b^{(0)}(t_1)h_b^{(0)}(t_2)h_b^{(0)}(t_3)h_b^{(0)}(t_4)\rangle\\ &\quad\approx \frac{1}{2}\sum_{i<j,k<l}\langle h_b^{(0)}(t_i)h_b^{(0)}(t_j)\rangle\langle h_b^{(0)}(t_k)h_b^{(0)}(t_l)\rangle,\end{aligned} \tag{A2}$$

where i,j,k,l run through $1,\ldots,4$, with $i\neq j\neq k\neq l$. In Eq. (A2) the intrapair intervals $|t_i-t_j|$ and $|t_k-t_l|$ must be $\lesssim t_{\rm corr}$, otherwise the correlator would decay. However, the interpair intervals, such as $|t_i-t_k|$, can be arbitrary, including $|t_i-t_k|\gtrsim t_r$, the case we are interested in and presumed in Eq. (A2).

The decoupling (A2) is characteristic for Gaussian noise. Similar decoupling applies for the long-time behavior of higher-order correlators, obviously. Therefore, the noise $h_b^{(0)}(t)$

is indeed asymptotically Gaussian. We note the similarity of the above analysis to the standard in the theory of quantum transport approximation of keeping only Feynman diagrams with nonintersecting lines. However, in the present case, the analysis could be done prior to switching to the interaction representation and for non-Markovian dynamics.

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