

Structure of trajectories of complex-matrix eigenvalues in the Hermitian–non-Hermitian transition

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The statistical properties of trajectories of eigenvalues of Gaussian complex matrices whose Hermitian condition is progressively broken are investigated. It is shown how the ordering on the real axis of the real eigenvalues is reflected in the structure of the trajectories and also in the final distribution of the eigenvalues in the complex plane.

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I. INTRODUCTION

A few years after Wigner’s [1] introduction of the Gaussian ensemble of Hermitian random matrices as a basis of a statistical theory of spectra of complex many-body systems, Ginibre investigated properties of the Gaussian matrices when the Hermitian condition is removed [2]. The Hermitian ones have found applications [3], especially since the link with the characterization of the manifestation of chaos in quantum mechanics has been established [4]. More recently, the non-Hermitian class has attracted great interest [5].

The interpolation between universality classes of matrices is a standard subject of investigation in random matrix theories [6]. The Wigner-Ginibre transition, which is our present object of study, has already been a matter of investigation. It has been established that asymptotically, its density of eigenvalues, starting from Wigner’s semicircle law distribution on the real axis, evolves as the transition proceeds through an elliptic shape, ending up in a uniform circular distribution at the Ginibre situation [7]. In addition, an asymptotic weak non-Hermiticity regime [8] with important applications to quantum open systems [9] has been found in the intermediate regime of transition.

From the physical point of view, this transition can be seen as the passage of a gas from a one-dimensional (1D) configuration to an isotropic two-dimensional (2D) configuration. In fact, it is well known that the positions of eigenvalues of Gaussian ensembles can be considered as those of charges of a Coulomb gas. It also has been recognized that the real eigenvalues of the Hermitian matrices have properties of a 1D gas of bosons, the so-called Girardeau gas [10]. On the other hand, the joint distribution of the complex eigenvalues of the complete non-Hermitian matrices, that is, the Ginibre matrices, have the same structure of the Laughlin wave function of a 2D gas [11].

The behavior of individual eigenvalues of Hermitian random matrices has been a matter of investigation since the pioneering works that described the distribution of largest eigenvalues, an achievement followed by several applications [12]. Recently, the order statistics problem of obtaining the distribution of all eigenvalues considered an ordered sequence of random variables has been addressed [13]. Here we are interested in studying how that ordering on the real axis reflects on the eigenvalue trajectories in the complex plane as the Hermitian condition is progressively removed and also in the final eigenvalue distribution. In other words, do the

complex eigenvalues retain a memory of their initial positions on the real axis? To do this investigation, we resort to a system of differential equations that describes the motion of the eigenvalues as a function of the parameter breaking the Hermitian condition.

II. TRANSITION EQUATIONS AND RESULTS

The Ginibre ensemble consists of complex random matrices S whose joint distribution of elements is given by

$$P(S) = \exp[-\text{tr}(S^\dagger S)] \quad (1)$$

in which no Hermitian condition is imposed. Especially for the case in which we are interested, namely, complex matrices of size N , their eigenvalues, for large values of N , are uniformly distributed in a disk of radius \sqrt{N} . Beyond this circle, at a radius r , the density decays as [14]

$$\frac{\sqrt{\pi} \exp[-u^2]}{2u}, \quad (2)$$

where $u = r - \sqrt{N}$, such that eigenvalues can be found up to $\sim\sqrt{N} + 2$.

Taking a matrix out of this ensemble, we define a new matrix $H(t)$ by the relation

$$H(t) = \left(\frac{S + S^\dagger}{2}\right) + t\left(\frac{S - S^\dagger}{2}\right), \quad (3)$$

where the parameter t varies from 0 to 1. It is relevant to mention that others’ parametrizations have been used to study this transition [9]; however, for our purpose of investigating trajectories, the parameter t defined by Eq. (3) is more convenient. Of course, our t is related to those other parameters by a simple transformation. With $t > 0$, Eq. (3), together with its adjoint, can be inverted to express S in terms of H and H^\dagger as

$$S = \left(\frac{1+t}{2t}\right)H - \left(\frac{1-t}{2t}\right)H^\dagger. \quad (4)$$

Substituting Eq. (4) in Eq. (1), we obtain the density distribution of the t -dependent matrix elements of H :

$$P(H) = K_N(t) \exp\left(-\text{tr}\left[\frac{1+t^2}{2t^2}(H^\dagger H) - \frac{1-t^2}{4t^2}(HH + H^\dagger H^\dagger)\right]\right). \quad (5)$$

For $t = 0$ the matrices $H(0)$ are Hermitian such that, in the limit $t \rightarrow 0$, Eq. (5) becomes

$$P[H(0)] = \exp(-\text{tr}[H^2(0)]). \quad (6)$$

Therefore, $H(0)$ belongs to the Gaussian unitary ensemble whose eigenvalues are distributed on the real axis according to the Wigner semicircle law [14]:

$$\rho(x) = \frac{1}{\pi} \sqrt{2N - x^2}. \quad (7)$$

The joint distribution of the eigenvalues apart from a constant is given by

$$\begin{aligned} P(z_1, z_2, \dots, z_N) &= \exp \left[- \sum_{k=1}^N \left(x_k^2 + \frac{y^2}{t^2} \right) \right] \prod_{j>i} |z_j - z_i|^2 \\ &= \exp[-W(z_1, \dots, z_N)], \end{aligned} \quad (8)$$

where

$$W(z_1, \dots, z_N) = \sum_{k=1}^N \left(x_k^2 + \frac{y^2}{t^2} \right) - 2 \sum_{j>i} \log |z_j - z_i|. \quad (9)$$

Therefore, the eigenvalues can be considered as positions of N point charges under the action of a confining potential harmonic in the two Cartesian axis and a repulsive 1D Coulomb force.

For intermediate values of t , that is, $0 < t < 1$, asymptotically, that is, when N becomes large, the eigenvalues fill an ellipse [7] whose axes for our parametrization are

$$a = \sqrt{\frac{2N}{1+t^2}} \quad (10)$$

and

$$b = t^2 \sqrt{\frac{2N}{1+t^2}}. \quad (11)$$

As $S = H(1)$ belongs to the Ginibre ensemble of non-Hermitian matrices, $H(t)$ undergoes a transition from the Wigner to the Ginibre ensemble with eigenvalues moving along trajectories in the complex plane. Their motion is governed by a system of differential equations that can be deduced in the following way [15]: The matrices $H(t)$ are diagonalized by the similarity transformation

$$D = Q^{-1} H Q, \quad (12)$$

where D is a diagonal matrix whose diagonal contains the complex eigenvalues while Q is a matrix of size N that contains the eigenvectors. However, the adjoint of Eq. (14),

$$D^\dagger = Q^\dagger H^\dagger (Q^{-1})^\dagger, \quad (13)$$

shows that the inverse of the adjoint of Q diagonalizes the Hermitian of H . All of these matrices are functions of the parameter t . Taking the derivative (denoted by a dot) with respect to t of Eq. (12), we obtain

$$\dot{D} = [D, U] + P, \quad (14)$$

where

$$P = Q^{-1} \dot{H} Q \quad (15)$$

and

$$U = Q^{-1} \dot{Q} = -\dot{Q}^{-1} Q, \quad (16)$$

while for the derivative of P we have

$$\dot{P} = [P, U] + Q^{-1} \ddot{H} Q. \quad (17)$$

The diagonal part of (14) gives

$$\dot{D}_{kk} = \dot{z}_k = P_{kk}, \quad (18)$$

while the off-diagonal part

$$\dot{D}_{kl} = 0 = D_{kk} U_{kl} - U_{kl} D_{ll} + P_{kl} \quad (19)$$

yields

$$U_{kl} = -\frac{P_{kl}}{z_k - z_l}. \quad (20)$$

Using the arbitrariness of the matrix U we impose the necessary condition that its diagonal elements vanish, that is, $U_{kk} = 0$. Finally, from (16) we derive the equations

$$\dot{Q}_{ij} = \sum_{l \neq j} \frac{Q_{kl} P_{lj}}{z_l - z_i} \quad (21)$$

and

$$\dot{Q}_{ij}^{-1} = \sum_{l \neq i} \frac{P_{il} Q_{lj}^{-1}}{z_i - z_l} \quad (22)$$

for the evolution of the eigenvectors. The above equations, together with the initial conditions provided by the Hermitian matrix $H(0)$, form a complete system of first-order differential equations that can be solved numerically to obtain eigenvalues and eigenvectors along the transition.

Insofar as the dependence of H with the parameter t has not been specified, this set of differential equations is general. Assuming this dependence to be given by Eq. (3), the second derivative \ddot{H} vanishes due to the linear dependence with t and, explicitly, Eq. (17) gives

$$\dot{P}_{kk} = \sum_{m \neq k} \frac{2P_{km} P_{mk}}{z_k - z_m} \quad (23)$$

for the diagonal elements and

$$\dot{P}_{kl} = \sum_{m \neq k, m \neq l} P_{km} P_{mk} \left(\frac{1}{z_k - z_m} + \frac{1}{z_l - z_m} \right) \quad (24)$$

for the other elements. These two equations, together with Eq. (19), form a set of $2N(N+1)$ coupled differential equations to be integrated. Regarding the initial conditions, since $H(0)$ is Hermitian, its eigenvalues are real and its eigenvector matrix is unitary, that is, $Q^{-1} = Q^\dagger$. Taking this into account, we find that initially the matrix P is given by

$$P(0) = Q^\dagger(0) \frac{S - S^\dagger}{2} Q(0), \quad (25)$$

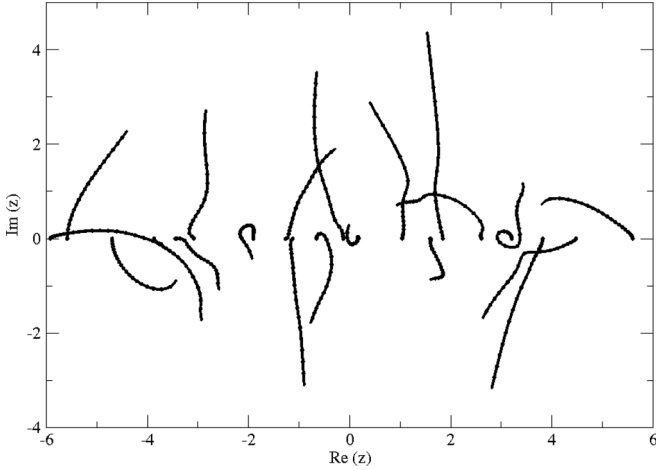


FIG. 1. The lines are $N = 20$ eigenvalue trajectories generated integrating the equations of motion, and the dots over the lines are eigenvalues obtained by diagonalizing the matrix.

such that

$$P(0) = -P^\dagger(0), \quad (26)$$

whose diagonal part

$$P_{kk}(0) = -P_{kk}^*(0) \quad (27)$$

shows that the diagonal elements of $P(0)$ are pure imaginary. Substituting these initial “velocities” in Eq. (18), we find that the eigenvalues leave the real axis at $t = 0$ perpendicularly.

We observe that integrating these equations between the initial value $t = 0$ and some final value $t = t_f$ is equivalent to diagonalizing $H(t)$ at each intermediate value of t . This is illustrated in Fig. 1, in which $N = 20$ eigenvalue trajectories obtained performing the integration from $t = 0$ to $t = 1$,

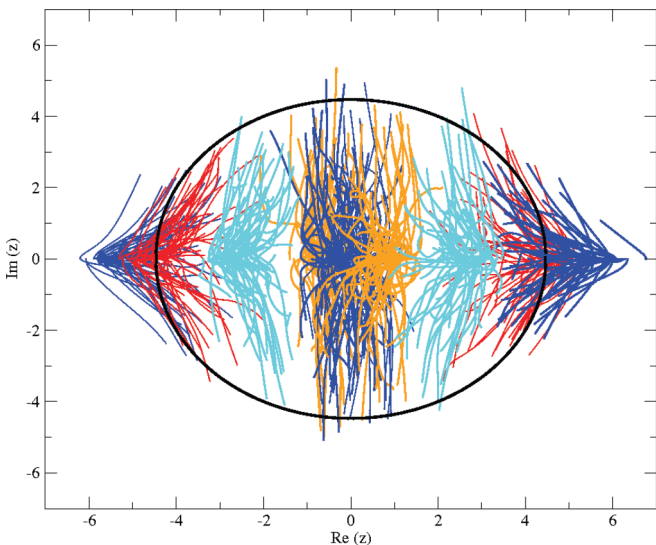


FIG. 2. (Color online) Ensemble of trajectories of 60 matrices of size $N = 20$ generated evolving eigenvalues initially at the edges, the middle parts, and the center of the spectra, together with the density circle of radius \sqrt{N} .

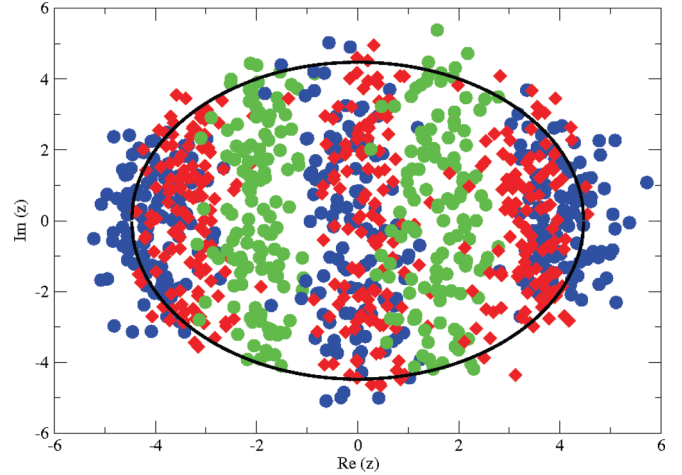


FIG. 3. (Color online) Ensemble of eigenvalues at the Ginibre regime evolved from eigenvalues at the edges, the middle parts, and the center of the spectra, together with the density circle of radius \sqrt{N} .

the equations of motion, are compared with the result of diagonalizing the matrices at several intermediate values of t . The agreement is perfect, showing that the above system is reliable and amenable to numerical integration.

Figure 1 also shows that eigenvalues may have complicated trajectories in the complex plane. Therefore, in order to reveal structures the trajectories may have, it is necessary to accumulate results of many simulations, that is, to construct an ensemble. This is done in Fig. 2, in which trajectories were obtained by evolving eigenvalues of 60 different initial Hermitian matrices, clearly exhibiting a structure. Indeed, trajectories starting on the real axis at the edge cover regions of a meniscus shape whose curvature decreases as eigenvalues move to the center are considered. Trajectories starting at the central region of the initial spectrum move inside strips. The presence of these strips is better seen in Fig. 3, where points correspond to eigenvalues at the end of the transition, that is, for $t = 1$. The structure of the distribution of points shows that, statistically, eigenvalues preserve to some extent the relative positions they had on the real axis.

III. CONCLUDING REMARKS

In conclusion, evolving under the action of the external confining harmonic potential and the repulsion force among them, eigenvalues present structures in their trajectories and in their final distribution in the complex plane that reflect the ordering they have on the real axis. We remark that the present study can be considered an instance of the so-called parametric statistics used to characterize the evolution of individual eigenvalues as a function of an external parameter [16]. In the case of complex eigenvalues, parametric evolution has been experimentally investigated by considering resonances trapped in an open microwave cavity in which the slit width can be varied [17].

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