Decay of unstable states driven by colored noise in an electromagnetic field

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The statistics of the first passage time in connection with the quasideterministic (QD) approach is used to characterize the non-Markovian decay process of the unstable state of an electrically charged Brownian particle under the influence of an electromagnetic field. We consider a constant magnetic field and a fluctuating electric field, which satisfies the properties of a Gaussian exponentially correlated noise. It is shown that at the beginning of the decay process, the magnetic field is strongly coupled to the noise correlation time and thus the requirements of the QD approach are not satisfied. Only in the approximation of a weak coupling between both parameters can the time characterization of the decay process be successfully achieved. Our theoretical approach relies on a Langevin equation for the charged particle in an arbitrary two-dimensional unstable potential and applies to a bistable potential as a particular case.

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I. INTRODUCTION

The first passage time (FPT) distribution and related topics have been widely used to characterize the random relaxation processes of a variety of out-of-equilibrium physical, chemical, and biological systems wherein the stochastic fluctuations (or noise) play a fundamental role [1–41]. The mathematical tool used for such a purpose relies basically upon stochastic differential equations like Langevin-type equations, Fokker-Planck equations, master equations, and their possible generalizations. When the considered stochastic differential equation is driven by the so-called white noise, which has no memory effects or zero correlation time, the process is said to be Markovian. On the other hand, if the stochastic term in the aforementioned equation is driven by colored noise, i.e., one with memory effects or finite correlation time, then we have a non-Markovian process. In modeling real phenomena we often idealize the underlying physical processes, and those modeled by means of white noise are examples of such an idealization. However, in real situations, correlated fluctuations are present and the corresponding physical processes must, in principle, be treated within the theory of non-Markovian processes. Nevertheless, a large number of approaches reported in the literature are focused more on the Markovian processes than on the non-Markovian ones. This is because the Markovian processes are more easily handled than the non-Markovian ones. In this latter case, it is not an easy task to extract the exact statistical information [14–40]. The study of non-Markovian processes seems to be mathematically more accessible in terms of stochastic differential equations than that given in terms of equations for the probability densities (Fokker-Planck or master equations). In this sense we can mention the studies of the dynamical characterization through the passage time distribution of the decay of an unstable state driven by an external Gaussian colored noise [Ornstein-Uhlenbeck (OU) noise] given in Refs. [24–26]. In [24] the study is given in terms of the statistics of the FPT distribution in the context of a general one-dimensional Langevin-type equation, whereas in [26] the decay process has been described through the so-called nonlinear relaxation time in terms of a Fokker-Planck-type equation. The study done in [24] was compared with the analog simulation previously reported in [21].

In the present contribution we study the statistics of the FPT distribution along with the QD approach to characterize the decay process of the unstable state of a charged Brownian particle in the presence of an electromagnetic field. The QD approach is a good approximation because it gives a precise physical picture of the mechanism responsible for the decay of the unstable state. This physical mechanism is twofold: small fluctuations change the initial condition in the neighborhood of the unstable state and then the deterministic motion drives the system out of this state. The same picture holds for a charged Brownian particle in the presence of an additional electromagnetic field. In this case the magnetic field is assumed to be a constant vector pointing along the z axis and the electric field as an external Gaussian exponentially correlated noise (OU noise). Hence, the decay process of the charged particle is accelerated by the electric force and rotationally evolved due to the action of the magnetic field. The dynamical characterization is formulated within the context of the Langevin equation for the charged particle which is localized around the unstable state of a two-dimensional (2D) bistable potential, although in reality our study is given for arbitrary 2D unstable potentials. Three mean first passage times (MFPTs) are calculated according to the manner in which the initial condition of the system is prepared. The first time scale is calculated when at time $t = 0$ the charged particle is localized in the position $\mathbf{r}_0 = 0$, corresponding to the initial unstable state of the potential. The second one is obtained when the particle's initial position has a distribution of values around

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$\mathbf{r}_0 = 0$. In this case the width of the initial distribution, which accounts for the noise intensity at $t = 0$, may not necessarily be of the same intensity than that of the noise responsible for the decay process for times $t > 0$. This initial distribution is considered to be independent of the colored noise term, and therefore both are statistically independent. The third time characterization is calculated in a more realistic situation and it corresponds to the case wherein the particle's initial position is also distributed, but now it is actually coupled to the colored noise term, and therefore both are not statistically independent. In the time characterization of the decay of unstable states, the quasideterministic (QD) approach has been shown to be a very good approximation which allows a description of the relaxation process into two time regimes [13,24,25]. The first one corresponds to early times of the decay of the unstable state where the noise and linearities are dominant. In the second regime the process is dominated by nonlinearities (deterministic force) and noise plays no fundamental role. It will be shown in the three aforementioned cases that the magnetic field is strongly coupled to the noise correlation time. As a consequence of this strong coupling effect, it is not possible to achieve the time characterization of the decay process by means of the QD approach. However, we succeed in achieving the proposed goal by considering a weak coupling between both parameters.

Our work is then structured as follows: In Sec. II we establish the Langevin equation for the charged particle in a 2D bistable potential under the action of an electromagnetic field, assuming a constant magnetic field and an OU process for the electric field. In Sec. III we introduce the QD approach and calculate the MFPT for the three aforementioned cases for linear and nonlinear 2D unstable potentials. The case of a 2D bistable potential is studied as a particular case. The three theoretical nonlinear MFPTs for the bistable potential are compared with numerical simulation. We give our concluding remarks in Sec. IV and finally, at the end of our work, we present the relevant algebraic details in an Appendix.

II. LANGEVIN EQUATION IN AN ELECTROMAGNETIC FIELD

We consider an electrically charged Brownian particle of mass m and charge q in an electromagnetic field initially located on the unstable state of a 2D bistable potential $V(x,y) = -(a_1/2)(x^2+y^2) + (b_1/4)(x^2+y^2)^2$, where $a_1, b_1 > 0$, and $r^2 = x^2 + y^2$ is the square modulus of the position vector $\mathbf{r} = (x,y)$. The force derived from this potential is clearly $\mathbf{F} = a_1\mathbf{r} - b_1 r^2\mathbf{r}$. The magnetic field is assumed to be a constant vector pointing along the z axis, that is, $\mathbf{B} = (0,0,B)$, and the electric field as the external noise which satisfies the property of an OU process. In the 2D case, the Langevin equation associated with the charged particle reads as

$$m\frac{d\mathbf{u}}{dt} = -\alpha\mathbf{u} + a_1\mathbf{r} - b_1 r^2\mathbf{r} + \frac{q}{c}\mathbf{u}\times\mathbf{B} + q\,\mathbf{E}(t)\,, \qquad (1)$$

where $\mathbf{u} = d\mathbf{r}/dt = (u_x,u_y)$ is the planar velocity vector, $\alpha > 0$ the friction coefficient, and $q\mathbf{E}(t)$ is the fluctuating electric force. In the over-damped approximation the inertial term $m\,d\mathbf{u}/dt$ can be neglected and the above Langevin equation reduces to

$$\frac{d\mathbf{r}}{dt} = \widetilde{a}\,\mathbf{r} - \widetilde{b}\,r^2\Lambda\,\mathbf{r} + \widetilde{W}\,\mathbf{r} + \alpha_e^{-1}\,\Lambda\,\boldsymbol{\mu}(t)\,, \qquad (2)$$

where $\boldsymbol{\mu}(t) = (q/\alpha)\mathbf{E}(t) \equiv (\mu_x,\mu_y)$ is the external noise satisfying the property of Gaussian colored noise with zero mean value $\langle\mu_i(t)\rangle = 0$ and correlation function

$$\langle\mu_i(t)\mu_j(t')\rangle = \frac{D}{\tau}\,\delta_{ij}\,\mathrm{e}^{-|t-t'|/\tau} \quad i,j = x,y\,, \qquad (3)$$

D being the noise intensity and τ its correlation time. The matrices $\widetilde{W}$ and Λ are defined as

$$\widetilde{W} = \begin{pmatrix} 0 & \widetilde{\Omega} \\ -\widetilde{\Omega} & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & C \\ -C & 1 \end{pmatrix}, \qquad (4)$$

and the parameters $\widetilde{a} = a/\alpha_e$, $\widetilde{b} = b/\alpha_e$, $\alpha_e = (1+C^2)$, $\widetilde{\Omega} = \widetilde{a}C$, with $C = qB/c\alpha$ a dimensionless constant, and $a = a_1/\alpha$, $b = b_1/\alpha$. The dynamical characterization will be further described in a transformed space of coordinates $\mathbf{r}' = \mathrm{e}^{-\widetilde{W}t}\mathbf{r}$, where the QD approach is better understood. In this case, the Langevin Eq. (2) transforms into

$$\frac{d\mathbf{r}'}{dt} = \widetilde{a}\,\mathbf{r}' - \widetilde{b}\,r'^2\Lambda\,\mathbf{r}' + \alpha_e^{-1}\,\Lambda\,\boldsymbol{\mu}'(t)\,, \qquad (5)$$

where $\boldsymbol{\mu}'(t) = \mathcal{R}^{-1}(t)\boldsymbol{\mu}(t)$ and $\mathcal{R}(t) = \mathrm{e}^{\widetilde{W}t}$ is an orthogonal rotation matrix such that the transpose is its inverse, that is, $\mathcal{R}^T(t) = \mathcal{R}^{-1}(t)$ and $\mathcal{R}^{-1}(t) = \mathrm{e}^{-\widetilde{W}t}$, with

$$\mathcal{R}(t) = \begin{pmatrix} \cos\widetilde{\Omega}\,t & \sin\widetilde{\Omega}\,t \\ -\sin\widetilde{\Omega}\,t & \cos\widetilde{\Omega}\,t \end{pmatrix}. \qquad (6)$$

In Eq. (5) the quantity $r'^2 = x'^2 + y'^2$ is the square modulus of vector $\mathbf{r}'$ and it satisfies $r'^2 = r^2$, which means that the modulus of vector $\mathbf{r}$ remains invariant under the transformation $\mathcal{R}^{-1}(t)$.

III. MFPT AND QD APPROACH

To characterize the decay of the unstable state of an arbitrary 2D nonlinear potential through the MFPT within the QD approach, we first study the time characterization of its linear approximation, which is given by $V(x,y) = -(\widetilde{a}/2)(x^2+y^2) = -(\widetilde{a}/2)\,r^2$ such that $-R^2 \leqslant x^2+y^2 \leqslant R^2$, where R^2 represents the absorbing barrier. We must notice that in the transformed space of coordinates the linear potential is also given by $V(x',y') = -(\widetilde{a}/2)\,r'^2$ and also $-R'^2 \leqslant x'^2 + y'^2 \leqslant R'^2$. Thus the linear approximation of the overdamped Langevin Eq. (5) is simply

$$\frac{d\mathbf{r}'}{dt} = \widetilde{a}\,\mathbf{r}' + \alpha_e^{-1}\,\Lambda\,\boldsymbol{\mu}'(t)\,, \qquad (7)$$

and its solution is easily given by $\mathbf{r}'(t) = \mathbf{h}'(t)\,\mathrm{e}^{\widetilde{a}t}$, where

$$\mathbf{h}'(t) = \mathbf{r}'_0 + \mathbf{g}'(t)\,, \qquad (8)$$

being $\mathbf{r}'(0) \equiv \mathbf{r}'_0$ and

$$\mathbf{g}'(t) = \alpha_e^{-1}\int_0^t \mathrm{e}^{-\widetilde{a}s}\,\Lambda\,\mathcal{R}^{-1}(s)\boldsymbol{\mu}(s)\,ds\,. \qquad (9)$$

In the dynamical characterization of the decay of a linear unstable state, the QD approach has shown to be a very good theory [13,24,25]. It tells us that, as time increases and for a small noise intensity, the stochastic process $\mathbf{h}'(t)$ becomes a Gaussian random variable. This is indeed the case, because in the large time limit $d\mathbf{h}'(t)/dt = d\mathbf{g}'(t)/dt \to 0$, and therefore the process $\mathbf{h}'(t)$ becomes a Gaussian random variable denoted by $\mathbf{h}'(\infty) = \mathbf{h}'$ such that $\mathbf{h}' = \mathbf{r}'_0 + \mathbf{g}'$ and

$$\mathbf{g}' = \alpha_e^{-1} \int_0^\infty \mathrm{e}^{-\widetilde{a}t}\, \Lambda\, \mathcal{R}^{-1}(t) \boldsymbol{\mu}(t)\, dt\,. \tag{10}$$

In this approximation the process $\mathbf{r}'(t)$ becomes $\mathbf{r}'(t) = \mathbf{h}'\mathrm{e}^{\widetilde{a}t}$, which is called a quasideterministic process, and $\mathbf{h}'$ plays the role of an effective initial condition. The square of the process also satisfies

$$r'^2(t) = h'^2\, \mathrm{e}^{2\widetilde{a}t}\,, \tag{11}$$

where $h'^2 \equiv |\mathbf{h}'|^2 = h_1'^2 + h_2'^2$ plays the role of an effective initial condition. Inverting expression (11), we can calculate the passage time distribution required by the charged particle to reach the absorbing barrier $r'^2 = R'^2$, that is,

$$t_{\scriptscriptstyle L} = \frac{1}{2\widetilde{a}} \ln\left(\frac{R'^2}{h'^2}\right). \tag{12}$$

This time scale is also a random variable due to the randomness of h' and its statistical properties can be determined through the marginal probability density $P(h')$. The latter can be calculated from the joint probability density given by the Gaussian distribution [41,42]

$$P(h_1',h_2') = N \exp\left[-\frac{1}{2}\sum_{i,j=1}^{2} (\sigma^{-1})_{ij}(h_i' - \langle h_i'\rangle)(h_j' - \langle h_j'\rangle)\right], \tag{13}$$

where $N = 1/2\pi(\det\sigma_{ij})^{1/2}$ is the normalization factor. The matrix $\sigma_{ij} = \sigma_{ji}$ is assumed to be positive definite; then the inverse matrix $(\sigma^{-1})_{jk} = (\sigma^{-1})_{kj}$ and its square root $(\sigma^{1/2})_{jk} = (\sigma^{1/2})_{jk}$, as well as its inverse square root $(\sigma^{-1/2})_{jk} = (\sigma^{-1/2})_{jk}$, exist. In this case $\sigma_{ij} \equiv \langle h_i' h_j'\rangle - \langle h_i'\rangle\langle h_j'\rangle$ is the correlation matrix of the effective initial conditions. Because of $\mathbf{h}' = \mathbf{r}'_0 + \mathbf{g}'$ the correlation matrix can be written as

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}^g + \sigma_{ij}^{0g} + \sigma_{ji}^{0g}\,, \tag{14}$$

where

$$\sigma_{ij}^0 = \langle x_{0i}' x_{0j}'\rangle - \langle x_{0i}'\rangle\langle x_{0j}'\rangle\,, \tag{15}$$

$$\sigma_{ij}^g = \langle g_i' g_j'\rangle - \langle g_i'\rangle\langle g_j'\rangle\,, \tag{16}$$

$$\sigma_{ij}^{0g} = \langle x_{0i}' g_j'\rangle - \langle x_{0i}'\rangle\langle g_j'\rangle\,, \tag{17}$$

$$\sigma_{ji}^{0g} = \langle x_{0j}' g_i'\rangle - \langle x_{0j}'\rangle\langle g_i'\rangle\,. \tag{18}$$

The matrix σ_{ij}^0 accounts for the fluctuations of the initial conditions at time $t = 0$, σ_{ij}^g is related to the noise-driven fluctuations for times $t > 0$, and σ_{ij}^{0g} and σ_{ij}^{g0} account for the statistical dependence between the colored noise and the initial state of the system. Accordingly, we can obtain three expressions for the correlation matrix given by Eq. (14), and therefore three MFPTs. Two of them correspond to the physical situation wherein the initial condition is statistically noise independent (decoupled case) and the other one to the physical situation wherein the initial condition of the system is coupled to the noise.

a. Decoupled case with fixed initial condition. This is the simplest case which satisfies the condition $\mathbf{r}'_0 = 0$, and therefore $\sigma_{ij}^0 = 0$, $\sigma_{ij}^{0g} = \sigma_{ji}^0 = 0$. In this case the correlation matrix is only given by $\sigma_{ij} = \sigma_{ij}^g$ and according to Eqs. (A2) and (A3) of Appendix A1, it satisfies $\sigma_{11}^g = \sigma_{22}^g$ and $\sigma_{12}^g = -\sigma_{21}^g$, where

$$\begin{aligned}\sigma_{11}^g &= \frac{D(1+\widetilde{a}\tau)}{a\,[(1+\widetilde{a}\tau)^2 + (\widetilde{\Omega}\tau)^2]},\\ \sigma_{12}^g &= -\frac{D\,\widetilde{\Omega}\tau}{a\,[(1+\widetilde{a}\tau)^2 + (\widetilde{\Omega}\tau)^2]}\,.\end{aligned} \tag{19}$$

It must be noticed that the correlation matrix does not satisfy the properties of the joint probability density given by Eq. (13). This fact is due to the coupling effect between the magnetic field and the noise correlation time through the factor $\widetilde{\Omega}\tau$. However, if we suppose that $\widetilde{\Omega}\tau \ll 1 + \widetilde{a}\tau$, then $\sigma_{12}^g \approx 0$ and the correlation matrix σ_{ij} becomes diagonal with elements $\sigma_{ii} = \sigma_{\rm fic}^2$, where

$$\sigma_{\rm fic}^2 = \frac{D}{a\,(1+\widetilde{a}\tau)} = \frac{D_e}{\widetilde{a}\,(1+\widetilde{a}\tau)}\,, \tag{20}$$

and $D_e \equiv D/(1+C^2)$. Here, the second equality has been written to see more clearly the influence of the magnetic field.

b. Decoupled case with distributed initial condition. This case may occur when the noise source for times $t \leqslant 0$ is not the same that the noise source for times $t > 0$, with the system initiating its decay process from an arbitrarily randomly selected point at time $t = 0$. In this case the initial condition is noise-independent and therefore $\langle \mathbf{r_0}\boldsymbol{\mu}\rangle = 0$, and again $\sigma_{ij}^{0g} = \sigma_{ji}^0 = 0$; thus, the correlation matrix is $\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}^g$. It is shown in Appendix A2 that $\sigma_{11}^0 = \sigma_{22}^0$ and $\sigma_{21}^0 = -\sigma_{12}^0$, where

$$\sigma_{11}^0 = \frac{D(1+\widetilde{a}_0\tau)}{a_0[(1+\widetilde{a}_0\tau)^2 + (\widetilde{\Omega}\tau)^2]}, \tag{21}$$

$$\sigma_{12}^0 = \frac{D\widetilde{\Omega}\tau}{a_0[(1+\widetilde{a}_0\tau)^2 + (\widetilde{\Omega}\tau)^2]}\,. \tag{22}$$

In a similar way as in case (i), the matrix σ_{ij} becomes diagonal if also $\widetilde{\Omega}\tau \ll 1 + \widetilde{a}_0\tau$ and $\widetilde{\Omega}\tau \ll 1 + \widetilde{a}\tau$, leading to $\sigma_{12}^0 \approx 0$. The diagonal elements of matrix σ_{ij} are defined by $\sigma_{ii} = \sigma_{\rm dic}^2$, where

$$\sigma_{\rm dic}^2 = \frac{D_e}{\widetilde{a}_0\,(1+\widetilde{a}_0\tau)} + \frac{D_e}{\widetilde{a}\,(1+\widetilde{a}\tau)}\,. \tag{23}$$

As an additional information we must comment here that, very recently, we have obtained the exact analytical solution for the stationary-state probability density of a charged Brownian harmonic oscillator in crossed electric and magnetic fields [27]. Under the same hypothesis established in this work for the electromagnetic field, we have been able to show that the marginal probability density $P_{st}(\mathbf{r}) = (1/2\pi\,\sigma_{11}^0)\exp(-|\mathbf{r}|^2/2\sigma_{11}^0)$, where σ_{11}^0 is the same as that given by Eq. (21). This is a fact which support the veracity of Eq. (21).

c. Coupled case with distributed initial condition. This case represents a more realistic physical situation because the initial condition, being not arbitrarily distributed, is shown to be coupled in a natural way to the colored noise term. In a similar way as done in Ref. [24], we also suppose that at time $t = 0$ the charged particle is localized in a steady-state (for instance, around the minimum of a harmonic potential) in such a way that, for times $t \leqslant 0$, the dynamical evolution of the system satisfies the following linear stationary equation:

$$\frac{d\mathbf{r}_0}{dt} = -\widetilde{a}_0\,\mathbf{r}_0 + \widetilde{W}\mathbf{r}_0 + \alpha_e^{-1}\,\Lambda\,\boldsymbol{\mu}(t)\,, \tag{24}$$

where $\widetilde{a}_0 = (a_0/\alpha_e)$ with $a_0 > 0$; $\widetilde{W}$ and Λ are the same as those given in Eq. (4). In this dynamics the noise term $\boldsymbol{\mu}(t)$ is also assumed to satisfy the properties of a Gaussian colored noise with zero mean value and an exponentially correlated function, similar to Eq. (3). For practical purposes we again consider the initial noise intensity D' to be $D' = D$. Hence, the physical situation described by the dynamics (24) is the following: at time $t = 0$, the system suffers a change in the control parameter $-\widetilde{a}_0 \to \widetilde{a}$, allowing the system to be localized in the initial unstable state described by the linear approximation of the dynamics (2). The solution of Eq. (24) then reads

$$\mathbf{r}_0 = \alpha_e^{-1}\int_{-\infty}^{0} e^{\widetilde{a}_0 t}\,\Lambda\,\mathcal{R}^{-1}(t)\boldsymbol{\mu}(t)\,dt\,. \tag{25}$$

In a similar way, for the transformed space of coordinates $\mathbf{r}'$ at time $t = 0$ the system is also localized in a steady state such that, for times $t < 0$, its dynamical evolution satisfies the equation

$$\frac{d\mathbf{r}'_0}{dt} = -\widetilde{a}_0\,\mathbf{r}'_0 + \alpha_e^{-1}\,\Lambda\,\boldsymbol{\mu}'(t)\,, \tag{26}$$

where $\mathbf{r}'_0 = \mathbf{r}_0$ and $\boldsymbol{\mu}'(t)$ has been defined in Sec. II. In this case, the physical situation is that at time $t = 0$, the system suffers a change in the control parameter $-\widetilde{a}_0 \to \widetilde{a}$, allowing the charged particle to be localized in the initial unstable state described by the linear dynamics given by Eq. (7). The solution of Eq. (26) is also the same as that to Eq. (25), as expected; that is,

$$\mathbf{r}'_0 = \alpha_e^{-1}\int_{-\infty}^{0} e^{\widetilde{a}_0 t}\,\Lambda\,\mathcal{R}^{-1}(t)\boldsymbol{\mu}(t)\,dt\,, \tag{27}$$

where we can clearly see that $\langle x'_{0i}\rangle = 0$. The correlation matrix σ_{ij} is now given by the complete expression of Eq. (14) and the four corresponding matrix elements are given in Eqs. (A15)–(A18) of Appendix A2. For arbitrary values of τ, the matrix σ_{ij} does not satisfy the requirements of the probability density (13). However, in the approximation $\widetilde{\Omega}\tau \ll 1 + \widetilde{a}_0\tau$ and $\widetilde{\Omega}\tau \ll 1 + \widetilde{a}\tau$, the correlation matrix σ_{ij} becomes diagonal with elements defined by $\sigma_{\rm cic} = \sigma_{ii}$, where

$$\sigma_{\rm cic}^2 = \frac{D_e}{\widetilde{a}_0(1+\widetilde{a}_0\tau)} + \frac{D_e}{\widetilde{a}(1+\widetilde{a}\tau)} + \frac{2D_e\tau}{(1+\widetilde{a}_0\tau)(1+\widetilde{a}\tau)}\,. \tag{28}$$

As can be seen, the non-Markovian contribution of the third term of this equation arises in a natural way as a consequence of the aforementioned coupling effect.

To verify our theoretical results, we evaluate them for zero magnetic field, that is, $C = 0$. In this case, $D_e = D$, $\widetilde{a}_0 = a_0$, and $\widetilde{a} = a$; thus

$$\sigma_{\rm fic}^2 = \frac{D}{a\,(1+a\tau)},\quad \sigma_{\rm dic}^2 = \frac{D}{a_0\,(1+a_0\tau)} + \frac{D}{a\,(1+a\tau)}\,, \tag{29}$$

and

$$\sigma_{\rm cic}^2 = \frac{D}{a_0(1+a_0\tau)} + \frac{D}{a(1+a\tau)} + \frac{2D\tau}{(1+a_0\tau)(1+a\tau)}\,, \tag{30}$$

consistently with the results given in Refs. [24,26]. Therefore the presence of the magnetic field with respect to the colored noise induces a renormalization in the parameters D, a_0, and a by the factor $1/(1+C^2)$, leading to D_e, $\widetilde{a}_0$, and $\widetilde{a}$, respectively, as can be seen in Eqs. (20), (23), and (28).

We thus have shown in all cases that, in the approximation of weak coupling between the magnetic field and the noise correlation time such that $\widetilde{\Omega}\tau \ll 1 + \widetilde{a}_0\tau$ and $\widetilde{\Omega}\tau \ll 1 + \widetilde{a}\tau$, the correlation matrix of the effective initial conditions is indeed a diagonal one, and therefore the joint probability density (13) reduces to its simplest expression

$$P(h'_1,h'_2) = \frac{1}{2\pi\sigma^2}\,e^{-(h_1'^2+h_2'^2)/2\sigma^2}\,, \tag{31}$$

where σ^2 represents any of Eqs. (20), (23), or (28). The marginal probability density $P(h')$ can be easily calculated by means of the transformation

$$P(h'_1,h'_2)dh'_1dh'_2 \to e^{-h'^2/2\sigma^2}\,J(h',\theta)\,dh'd\theta'\,, \tag{32}$$

where $J(h',\theta) = h'$ is the Jacobian of the transformation. After integration over the θ' variable, we obtain the normalized marginal probability density

$$P(h') = \frac{h'}{\sigma^2}\,e^{-h'^2/2\sigma^2}\,. \tag{33}$$

The statistical properties of FPT distribution can be calculated through the moment generating function defined as $G(2\widetilde{a}\nu) \equiv \langle e^{-2\widetilde{a}\nu t}\rangle$. For the linear passage time given by Eq. (12), the generating function is $G_{L}(2\widetilde{a}\nu) = \langle (R'^2/h'^2)^{-\nu}\rangle$, which, according to the marginal probability density (33), reads as

$$G_{L}(2\widetilde{a}\nu) = \left(\frac{R'^2}{2\sigma^2}\right)^{-\nu}\Gamma(\nu+1)\,, \tag{34}$$

$\Gamma(x)$ being the γ function [43]. The MFPT is then calculated from $\langle 2\widetilde{a}\,t\rangle = [-dG(2\widetilde{a}\nu)/d\nu]_{\nu=0}$, and after some algebra, we obtain the linear MFPT

$$\langle t\rangle_{L} = \frac{1}{2\widetilde{a}}\left\{\ln\left(\frac{R^2}{2\sigma^2}\right) - \psi(1)\right\}, \tag{35}$$

where $\psi(1) = -\gamma = -0.577$ is the Euler's constant. The variance of the passage time defined as $\langle(\Delta t)^2\rangle \equiv \langle t^2\rangle - \langle t\rangle^2$ can be calculated through the second moment $\langle(2\widetilde{a}\,t)^2\rangle = [d^2G(2\widetilde{a}\nu)/d\nu^2]_{\nu=0}$. Again, after some easy algebra, it can be shown that

$$\langle(\Delta t)^2\rangle = \frac{1}{4\widetilde{a}^2}\,\psi'(1)\,, \tag{36}$$

with $\psi'(1) = \pi^2/6 \approx 1.644\,5$ [43].

A. Nonlinear contributions

To deal with nonlinear contributions of the unstable potential, a general definition of a nonlinear potential is required. Hence, in terms of the variable $\mathfrak{r} \equiv r^2 = x^2 + y^2$, such a nonlinear deterministic dynamics can be defined by [13]

$$\frac{d\mathfrak{r}}{dt} = f(\mathfrak{r}), \quad f(\mathfrak{r}) = \frac{\mathfrak{r}(\mathfrak{r}_{st} - \mathfrak{r})}{C_0 + \mathfrak{r}g(\mathfrak{r})}, \tag{37}$$

where $C_0 = \mathfrak{r}_{st}/2\bar{a}$, $\mathfrak{r}_{st}$ is the steady-state value, and $g(\mathfrak{r}) > 0$ is a polynomial. The function $f(\mathfrak{r})$ has two roots: one is at $\mathfrak{r} = 0$, which corresponds to the unstable state such that $f'(\mathfrak{r})|_{\mathfrak{r}=0} > 0$, and the other one is at $\mathfrak{r} = \mathfrak{r}_{st}$, corresponding to the stable state and thus $f'(\mathfrak{r})|_{\mathfrak{r}=\mathfrak{r}_{st}} < 0$. The deterministic evolution of Eq. (2) or Eq. (5) must be compatible with Eq. (37) for a particular expression of $g(\mathfrak{r})$.

The connection between the passage time and the QD approach can be achieved by assuming that $\mathfrak{r}(0) \equiv r^2(0) = h^2$ is a random variable which plays the role of an effective initial condition responsible for the decay of the unstable state towards its steady state, characterized by the value $\mathfrak{r}(\infty) \equiv r^2(\infty) = \mathfrak{r}_{st}$. The nonlinear passage time distribution along with Eq. (37) can be defined as

$$t_{NL} = \int_{h^2}^{R^2} \frac{d\mathfrak{r}}{f(\mathfrak{r})} = \int_{h^2}^{R^2} \frac{C_0 + \mathfrak{r}g(\mathfrak{r})}{\mathfrak{r}(\mathfrak{r}_{st} - \mathfrak{r})}\, d\mathfrak{r}. \tag{38}$$

After integration it reduces to

$$t_{NL} = \frac{1}{2\widetilde{a}} \ln\left(\frac{R'^2}{h'^2}\right) + \frac{1}{2\widetilde{a}} C_{NL}, \tag{39}$$

where C_{NL} takes into account the nonlinear contributions and reads

$$C_{NL} = \lim_{h\to 0}\left[\int_{h^2}^{R^2} \frac{d\mathfrak{r}}{\mathfrak{r}_{st} - \mathfrak{r}} + 2\widetilde{a}\int_{h^2}^{R^2} \frac{g(\mathfrak{r})}{\mathfrak{r}_{st} - \mathfrak{r}}\right]. \tag{40}$$

It is very clear that the first logarithmic term of Eq. (39) accounts for the decay process in the linear regime of the nonlinear potential, wherein the stochastic fluctuations are dominant. The second one is practically a constant; it comes from the nonlinear contributions of the potential away from the initial unstable state. In this nonlinear regime, the dynamical evolution of the particle is practically deterministic and the stochastic fluctuations are not relevant. It is then calculated in the limit of $h \to 0$. For the nonlinear passage time (39) the generating function is now $G_{NL}(2\widetilde{a}\nu) = \langle(\widehat{R}'^2/h'^2)^{-\nu}\rangle$, with $\widehat{R}'^2 = e^{C_{NL}} R'^2$. Again, it is given by $G_{NL}(2\widetilde{a}\nu) = (\widehat{R}'^2/2\sigma^2)^{-\nu}\Gamma(\nu+1)$, and thus the nonlinear MFPT reads very similar to Eq. (35), that is,

$$\langle t\rangle_{NL} = \frac{1}{2\widetilde{a}}\left\{\ln\left(\frac{\widehat{R}'^2}{2\sigma^2}\right) + \gamma\right\}, \tag{41}$$

and the variance reminds the same as Eq. (36).

B. Nonlinear bistable potential

To calculate the nonlinear MFPT associated with the Langevin dynamics (2) or (5), we first construct their corresponding deterministic equations in terms of the variables $\mathfrak{r}$ or $\mathfrak{r}'$. The deterministic equations for these variables have the expected form and read

$$\frac{d\mathfrak{r}}{dt} = 2\widetilde{a}\,\mathfrak{r} - 2\widetilde{b}\,\mathfrak{r}^2 = \frac{2\widetilde{a}\,\mathfrak{r}}{\mathfrak{r}_{st}}(\mathfrak{r}_{st} - \mathfrak{r}) = 2\widetilde{a}\,\mathfrak{r}' - 2\widetilde{b}\,\mathfrak{r}'^2 = \frac{2\widetilde{a}\,\mathfrak{r}'}{\mathfrak{r}'_{st}}(\mathfrak{r}'_{st} - \mathfrak{r}'), \tag{42}$$

where $\mathfrak{r}_{st} = r_{st}^2 = a/b$ is the stationary-state value. Equation (42) is compatible with the general definition given by Eq. (37) if $g(\mathfrak{r}) = 0$. In this case the constant given by Eq. (40) is $C_{NL} = \ln[M^2/(1-M^2)]$, where $M^2 = R^2/r_{st}^2$. Finally, the nonlinear MFPT associated with the Langevin Eq. (2) or Eq. (5) will be

$$\langle t\rangle_{NL} = \frac{1}{2\widetilde{a}}\left\{\ln\left(\frac{R^2 M^2}{2\sigma^2(1-M^2)}\right) + \gamma\right\}. \tag{43}$$

Next we proceed to calculate this nonlinear MFPT for each of the aforementioned studied cases.

a. Decoupled case with fixed initial condition. Using Eq. (20), it is very easy to check that the nonlinear MFPT (43) can be written as

$$T_{\rm fic} = -\frac{1}{2\widetilde{a}} \ln\left[\frac{D_e}{\widetilde{a}(1+\widetilde{a}\tau)}\right] + \frac{1}{2\widetilde{a}} \ln\left[\frac{M^2 R^2 e^{\gamma}}{2(1-M^2)}\right]. \tag{44}$$

b. Decoupled case with distributed initial condition. Here we use Eq. (23) to show that the nonlinear MFPT (43) reads

$$T_{\rm dic} = -\frac{1}{2\widetilde{a}} \ln\left[\frac{D_e}{\widetilde{a}_0(1+\widetilde{a}_0\tau)} + \frac{D_e}{\widetilde{a}(1+\widetilde{a}\tau)}\right] + \frac{1}{2\widetilde{a}} \ln\left[\frac{M^2 R^2 e^{\gamma}}{2(1-M^2)}\right]. \tag{45}$$

c. Coupled case with distributed initial condition. According to Eq. (28) the nonlinear MFPT (43) is now

$$T_{\rm cic} = -\frac{1}{2\widetilde{a}} \ln(D_e) + \frac{1}{2\widetilde{a}} \ln\left[\frac{(1+\widetilde{a}\tau)(1+\widetilde{a}_0\tau)}{1+(\widetilde{a}+\widetilde{a}_0)\tau}\right] + \frac{1}{2\widetilde{a}} \ln\left[\frac{M^2 R^2 e^{\gamma}}{2(1-M^2)(\widetilde{a}_0^{-1}+\widetilde{a}^{-1})}\right]. \tag{46}$$

The time scales (44)–(46) show clearly the influence of the magnetic field through the parameters D_e, $\widetilde{a}_0$, and $\widetilde{a}$. The time scale in the coupled case has a similar structure to that obtained in Ref. [24] in the one-dimensional case, except by the parameters D_e, $\widetilde{a}_0$, and $\widetilde{a}$. Hence, depending on the coupling between the system's initial state and the colored noise term, the non-Markovian time characterization of the decay of the unstable state of the charged Brownian particle in an electromagnetic field is given by any of Eqs. (44)–(46).

For the decoupled cases we must notice the following points: in the Gaussian white noise (GWN) limit, even in the presence of the magnetic field, Eqs. (44) and (45) can be written respectively as

$$T_{\rm fic} = -\frac{1}{2\widetilde{a}} \ln\left(\frac{D_e}{\widetilde{a}}\right) + \frac{1}{2\widetilde{a}} \ln\left(\frac{M^2 R^2 e^{\gamma}}{2(1-M^2)}\right) \tag{47}$$

and

$$T_{\rm dic} = -\frac{1}{2\widetilde{a}} \ln\left(\frac{D_e}{\widetilde{a}_0} + \frac{D_e}{\widetilde{a}}\right) + \frac{1}{2\widetilde{a}} \ln\left(\frac{M^2 R^2 e^{\gamma}}{2(1-M^2)}\right). \tag{48}$$

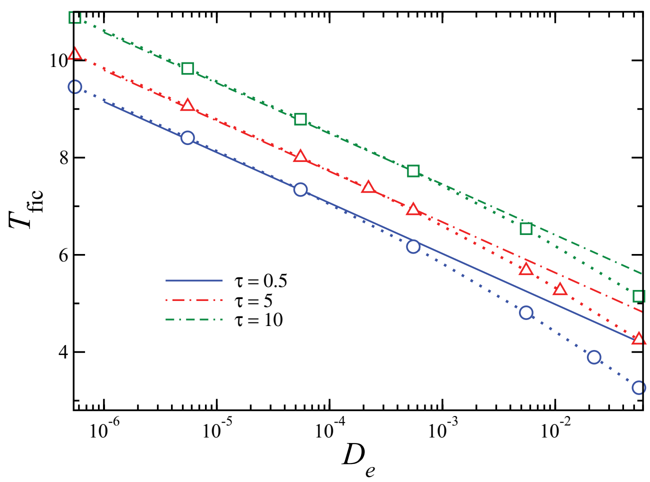

FIG. 1. (Color online) Decoupled case with fixed initial conditions $a = 2$, $b = 1$, $\alpha = 1$, $C = 1$, and $M^2 = 0.995$. Symbols correspond to simulation results. From bottom to top the depicted cases are $\tau = 0.5$ (continuous line and circles), $\tau = 5$ (dot-dashed line and triangles), and $\tau = 10$ (double-dash dot line and squares). Curves have been shifted upward for clarity.

Therefore, the nonlinear time scales (44) and (45) are the same as those calculated in the GWN case if in Eq. (47) the noise intensity D_e is simply rescaled by the factor $1/(1+\widetilde{a}\tau)$ and in Eq. (48) by the factors $1/(1+\widetilde{a}_0\tau)$ and $1/(1+\widetilde{a}\tau)$.

C. Numerical results

In this section the values derived from our theoretical expressions (44)–(46) are compared with the corresponding numerical simulation results. In Fig. 1 we present the comparison between the theoretical result given by Eq. (44) with numerical simulation for values of the parameters $a = 1$, $b = 1$, $\alpha = 1$, $C = 1$, and three values of $\tau = 0.5, 5.0, 10$. It can be seen that, as the noise correlation time increases, a better agreement between both results is obtained. A very similar behavior happens between the theoretical result given by Eq. (45) and the numerical simulation, as shown in Fig. 2. In both cases we simulate the Langevin Eq. (5) with the GWN algorithm, taking into account the corresponding renormalization of the involved parameters. As we can see, the agreement between the theory and simulation results are excellent as the noise correlation time is larger than one. This is so because the larger τ is, the better is the approximation $\widetilde{\Omega}\tau \ll 1+\widetilde{a}\tau$.

In the coupled case the situation is different due to the additional coupling effect appearing in the third term of Eq. (28). In this case, we simulate the Langevin Eq. (5) using the Gaussian colored noise (GCN) algorithm for an OU process. Both the theory given by Eq. (46) and numerical simulation are displayed in Fig. 3. Again, both results are in very good agreement, with a similar behavior as that shown in the previous figures. We notice that the agreement for the larger τ value considered in the GCN case is better than in the GWN case. However, for the given values of the parameters, the values of the noise correlation time cannot be greater than $\tau = 0.6$ for the GCN case.

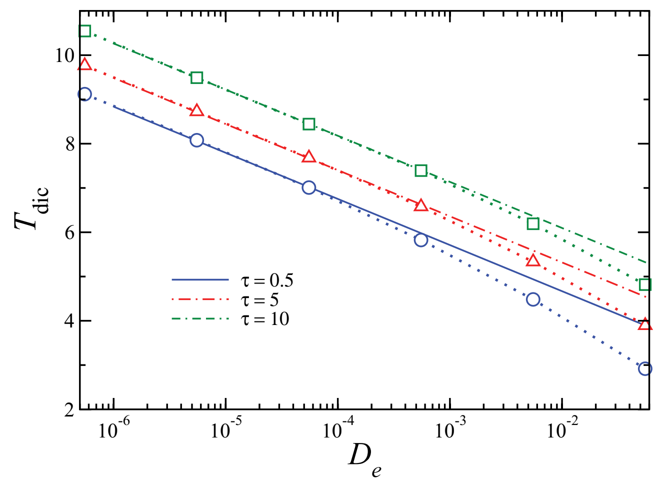

FIG. 2. (Color online) Decoupled case with distributed initial conditions. Same parameter values and symbols as in Fig. 1. Curves shifted upward for clarity.

IV. CONCLUDING REMARKS

In this work we have shown that the presence of the magnetic field in the non-Markovian time characterization of the decay of the unstable states of a charged Brownian particle driven by OU noise is strongly coupled to the noise correlation time. The coupling effect is actually quantified by the term $(\widetilde{\Omega}\tau)^2$ appearing in Eqs. (A6), (A7), and (A15)–(A18), when the correlation matrix of effective initial conditions is calculated. It is shown in the three herein studied cases that such a correlation matrix given by Eq. (14) does not satisfy the requirements demanded by the joint probability density (13) and then also by the QD approach. Under these circumstances, the non-Markovian time characterization of the decay process cannot be achieved. However, by assuming a weak coupling effect between the magnetic field and the noise correlation time

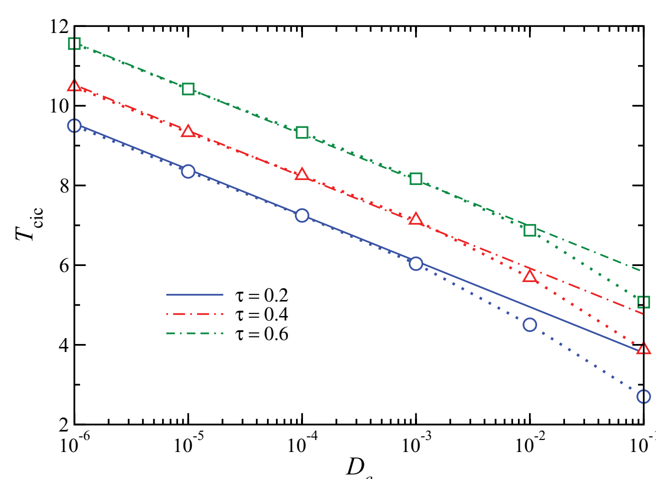

FIG. 3. (Color online) Coupled case with distributed initial conditions. Same a, b, α, and C values as in the previous figures, but now with $M^2 = 0.9975$. Same symbols as in previous figures now corresponding, in the upward direction, to $\tau = 0.2$ (continuous line and circles), $\tau = 0.4$ (dot-dashed line and triangles), and $\tau = 0.6$ (double-dash dot line and squares). Curves shifted upward for clarity.

such that $\tau \ll 1+\widetilde{a}_0\tau$ and $\widetilde{\Omega}\tau \ll 1+\widetilde{a}\tau$, then the correlation matrix σ_{ij} becomes a diagonal one, as shown in Eqs. (20), (23), and (28). The weak coupling condition allows the QD approach to successfully achieve the dynamical characterization of the present problem, as shown in Eqs. (44), (45), and (46).

The interesting result appearing in the weak coupling limiting case is the following: The two decoupled cases can be considered as a GWN problem, wherein the influence of the colored noise, in the presence of a magnetic field, induces a renormalization of the rescaled noise intensity $D_e = D/(1+C^2)$ by a factor $1/(1+\widetilde{a}\tau)$, as shown in Eqs. (44), (45), (47), and (48). For the coupled case, the comparison is given between the influence of the magnetic field with respect to the ordinary Brownian motion colored noise case. Here, as can be corroborated in Eqs. (28) and (30), the magnetic field induces a rescaling in the noise intensity D and in the control parameters a_0 and a by the factor $1/(1+C^2)$, leading, respectively, to D_e, $\widetilde{a}_0$, and $\widetilde{a}$. Our theoretical results given by Eqs. (44), (45), and (46) have been compared with the numerical simulation results, showing very good agreement for the given values of the involved parameters.

APPENDIX A: EFFECTIVE INITIAL CONDITIONS CORRELATION MATRIX σ_{ij}

1. Decoupled case

For a fixed initial condition $\mathbf{r}'_0 = 0$, we have from Eq. (8) that $\mathbf{h}' = \mathbf{g}'$, and thus $\langle g'_i\rangle = 0$ and the correlation matrix is $\sigma^g_{ij} = \langle g'_i g'_j\rangle$, where

$$\langle g'_i g'_j\rangle = \frac{1}{\alpha_e^2}\int_0^\infty\int_0^\infty \mathrm{e}^{-\widetilde{a}(t+t')}\Lambda_{ik}\Lambda_{jl}\mathcal{R}^{-1}_{km}(t)\,\mathcal{R}^{-1}_{ln}(t') \times \langle \mu_m(t)\mu_n(t')\rangle\, dt\,dt'. \tag{A1}$$

By substituting Eq. (3), the matrix elements of $\mathcal{R}^{-1}(t)$, as well as those of Λ, it can be shown, after a long but straightforward algebra, that

$$\sigma^g_{11} = \langle g'_1 g'_1\rangle = \langle g'_2 g'_2\rangle = \sigma^g_{22}, \tag{A2}$$

$$\sigma^g_{12} = \langle g'_1 g'_2\rangle = -\langle g'_2 g'_1\rangle = -\sigma^g_{21}\,, \tag{A3}$$

where

$$\langle g'_1 g'_1\rangle = 2K\left(\int_0^\infty dt\, \mathrm{e}^{-A_1 t}\sin\widetilde{\Omega}t\int_0^t dt'\,\mathrm{e}^{-A_2 t'}\sin\widetilde{\Omega}t' + \int_0^\infty dt\,\mathrm{e}^{-A_1 t}\cos\widetilde{\Omega}t\int_0^t dt'\,\mathrm{e}^{-A_2 t'}\cos\widetilde{\Omega}t'\right), \tag{A4}$$

$$\langle g'_1 g'_2\rangle = 2K\left(\int_0^\infty dt\, \mathrm{e}^{-A_1 t}\cos\widetilde{\Omega}t\int_0^t dt'\,\mathrm{e}^{-A_2 t'}\sin\widetilde{\Omega}t' - \int_0^\infty dt\,\mathrm{e}^{-A_1 t}\sin\widetilde{\Omega}t\int_0^t dt'\,\mathrm{e}^{-A_2 t'}\cos\widetilde{\Omega}t'\right), \tag{A5}$$

with $K = D(1+C^2)/\tau\,\alpha_e^2$, $A_1 = (\widetilde{a}+\tau^{-1})$, and $A_2 = (\widetilde{a}-\tau^{-1})$. Performing explicitly the integrals we arrive at

$$\sigma^g_{11} = \frac{D(1+\widetilde{a}\tau)}{a\,[(1+\widetilde{a}\tau)^2+(\widetilde{\Omega}\tau)^2]}\,, \tag{A6}$$

$$\sigma^g_{12} = -\frac{D\,\widetilde{\Omega}\tau}{a\,[(1+\widetilde{a}\tau)^2+(\widetilde{\Omega}\tau)^2]}\,. \tag{A7}$$

In this case, the correlation matrix σ_{ij} does not satisfy the requirements established in the Gaussian probability density given by Eq. (13).

2. Coupled case

For this physical situation we must calculate the additional terms of Eq. (14), that is, σ^0_{ij}, σ^{0g}_{ij}, and σ^{0g}_{ji}. It is clear from Eqs. (10) and (27) that we have the mean values $\langle g'_i\rangle = 0$ and $\langle x'_{0i}\rangle = 0$; thus the following correlation functions are defined:

$$\sigma^0_{ij} = \langle x'_{0i}x'_{0j}\rangle,\quad \sigma^{0g}_{ij} = \langle x'_{0i}g'_j\rangle,\quad \sigma^{0g}_{ji} = \langle x'_{0j}g'_i\rangle\,, \tag{A8}$$

where

$$\langle x'_{0i}x'_{0j}\rangle = \frac{1}{\alpha_e^2}\int_{-\infty}^0\int_{-\infty}^0 \mathrm{e}^{\widetilde{a}_0(t+t')}\Lambda_{ik}\Lambda_{jl}\mathcal{R}^{-1}_{km}(t)\,\mathcal{R}^{-1}_{ln}(t'), \times\langle\mu_m(t)\mu_n(t')\rangle\,dt\,dt', \tag{A9}$$

$$\langle x'_{0i}g'_j\rangle = \frac{1}{\alpha_e^2}\int_0^\infty\int_{-\infty}^0 \mathrm{e}^{-\widetilde{a}t+\widetilde{a}_0 t'}\Lambda_{ik}\Lambda_{jl}\mathcal{R}^{-1}_{km}(t)\,\mathcal{R}^{-1}_{ln}(t') \times\langle\mu_m(t)\mu_n(t')\rangle\,dt\,dt'. \tag{A10}$$

We substitute again Eq. (3) and the matrix elements of $\mathcal{R}^{-1}(t)$ and Λ to show that $\sigma^0_{11} = \langle x'_{01}x'_{01}\rangle = \langle x'_{02}x'_{02}\rangle = \sigma^0_{22}$, $\sigma^0_{12} = \langle x'_{01}x'_{02}\rangle = -\langle x'_{02}x'_{01}\rangle = -\sigma^0_{21}$, $\sigma^{0g}_{11} = \langle x'_{01}g'_1\rangle = \langle x'_{02}g'_2\rangle = \sigma^{0g}_{22}$, and $\sigma^{0g}_{12} = \langle x'_{01}g'_2\rangle = -\langle x'_{02}g'_1\rangle = -\sigma^{0g}_{21}$, where

$$\langle x'_{01}x'_{01}\rangle = 2K\left(\int_{-\infty}^0 dt\,\mathrm{e}^{A_1 t}\sin\widetilde{\Omega}t\int_{-\infty}^t dt'\,\mathrm{e}^{A_2 t'}\sin\widetilde{\Omega}t' + \int_{-\infty}^0 dt\,\mathrm{e}^{A_1 t}\cos\widetilde{\Omega}t\int_{-\infty}^t dt'\,\mathrm{e}^{A_2 t'}\cos\widetilde{\Omega}t'\right), \tag{A11}$$

$$\langle x'_{01}x'_{02}\rangle = 2K\left(\int_{-\infty}^0 dt\,\mathrm{e}^{A_1 t}\cos\widetilde{\Omega}t\int_{-\infty}^t dt'\,\mathrm{e}^{A_2 t'}\sin\widetilde{\Omega}t' - \int_{-\infty}^0 dt\,\mathrm{e}^{A_1 t}\sin\widetilde{\Omega}t\int_{-\infty}^t dt'\,\mathrm{e}^{A_2 t'}\cos\widetilde{\Omega}t'\right), \tag{A12}$$

$$\langle x'_{01}g'_1\rangle = K\left(\int_0^\infty dt\,\mathrm{e}^{-B_1 t}\sin\widetilde{\Omega}t\int_{-\infty}^0 dt'\,\mathrm{e}^{A_2 t'}\sin\widetilde{\Omega}t' + \int_0^\infty dt\,\mathrm{e}^{-B_1 t}\cos\widetilde{\Omega}t\int_{-\infty}^0 dt'\,\mathrm{e}^{A_2 t'}\cos\widetilde{\Omega}t'\right), \tag{A13}$$

$$\langle x'_{01}g'_2\rangle = K\left(\int_0^\infty dt\,\mathrm{e}^{-B_1 t}\cos\widetilde{\Omega}t\int_{-\infty}^0 dt'\,\mathrm{e}^{A_2 t'}\sin\widetilde{\Omega}t' - \int_0^\infty dt\,\mathrm{e}^{-B_1 t}\sin\widetilde{\Omega}t\int_{-\infty}^0 dt'\,\mathrm{e}^{A_2 t'}\cos\widetilde{\Omega}t'\right), \tag{A14}$$

where now $B_1 = (\widetilde{a}_0 - \tau^{-1})$ and $B_2 = (\widetilde{a}_0 + \tau^{-1})$. After evaluating the integrals we get

$$\sigma_{11}^{0} = \frac{D(1+\widetilde{a}_0\tau)}{a_0[(1+\widetilde{a}_0\tau)^2 + (\widetilde{\Omega}\tau)^2]}, \quad \text{(A15)}$$

$$\sigma_{12}^{0} = \frac{D\widetilde{\Omega}\tau}{a_0[(1+\widetilde{a}_0\tau)^2 + (\widetilde{\Omega}\tau)^2]}, \quad \text{(A16)}$$

$$\sigma_{11}^{0g} = \frac{D_e\tau[(1+\widetilde{a}_0\tau)(1+\widetilde{a}\tau) - (\widetilde{\Omega}\tau)^2]}{[(1+\widetilde{a}_0\tau)^2 + (\widetilde{\Omega}\tau)^2][(1+\widetilde{a}\tau)^2 + (\widetilde{\Omega}\tau)^2]}, \quad \text{(A17)}$$

$$\sigma_{12}^{0g} = -\frac{D_e\,\widetilde{\Omega}\tau^2[2 + (\widetilde{a}_0 + \widetilde{a})\tau)]}{[(1+\widetilde{a}_0\tau)^2 + (\widetilde{\Omega}\tau)^2][(1+\widetilde{a}\tau)^2 + (\widetilde{\Omega}\tau)^2]}, \quad \text{(A18)}$$

with $D_e = D/(1+C^2)$. Now, the matrix elements σ_{ij} of Eq. (14) are then

$$\sigma_{11} = \sigma_{11}^{0} + \sigma_{11}^{g} + 2\sigma_{11}^{0g}, \quad \text{(A19)}$$

$$\sigma_{22} = \sigma_{22}^{0} + \sigma_{22}^{g} + 2\sigma_{22}^{0g}, \quad \text{(A20)}$$

$$\sigma_{12} = \sigma_{12}^{0} + \sigma_{12}^{g} + \sigma_{12}^{0g} + \sigma_{21}^{0g}, \quad \text{(A21)}$$

$$\sigma_{21} = \sigma_{21}^{0} + \sigma_{21}^{g} + \sigma_{21}^{0g} + \sigma_{12}^{0g}. \quad \text{(A22)}$$

However, due to the results obtained in this Appendix we see that $\sigma_{11} = \sigma_{22}$ and $\sigma_{12} = -\sigma_{21}$, where $\sigma_{12} = \sigma_{12}^{0} + \sigma_{12}^{g}$ because of $\sigma_{12}^{0g} + \sigma_{21}^{0g} = 0$. We conclude that the correlation matrix σ_{ij} does not actually satisfy the properties established in the joint probability density (13). If, however, we suppose that $\widetilde{\Omega}\tau \ll 1 + \widetilde{a}\tau$ and $\widetilde{\Omega}\tau \ll 1 + \widetilde{a}_0\tau$, then it can be shown approximately that $\sigma_{12}^{0} = 0$, $\sigma_{12}^{g} = 0$, and therefore $\sigma_{12} = -\sigma_{21} = 0$, as well as

$$\sigma_{11}^{0} = \sigma_{22}^{0} = \frac{D_e}{\widetilde{a}_0(1+\widetilde{a}_0\tau)}, \quad \text{(A23)}$$

$$\sigma_{11}^{g} = \sigma_{22}^{g} = \frac{D_e}{\widetilde{a}(1+\widetilde{a}\tau)}, \quad \text{(A24)}$$

$$\sigma_{11}^{0g} = \sigma_{22}^{og} = \frac{D_e\tau}{(1+\widetilde{a}_0\tau)(1+\widetilde{a}\tau)}. \quad \text{(A25)}$$

Therefore, the correlation matrix of effective initial conditions σ_{ij} becomes diagonal with elements defined by $\sigma^2 = \sigma_{ii}$, where $\sigma^2 = \sigma_{11}^{0} + \sigma_{11}^{g} + 2\sigma_{11}^{0g}$.

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