Asymptotic response of observables from divergent weak-coupling expansions: A fractional-calculus-assisted Padé technique

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Appropriate constructions of Padé approximants are believed to provide reasonable estimates of the asymptotic (large-coupling) amplitude and exponent of an observable, given its weak-coupling expansion to some desired order. In many instances, however, sequences of such approximants are seen to converge very poorly. We outline here a strategy that exploits the idea of fractional calculus to considerably improve the convergence behavior. Pilot calculations on the ground-state perturbative energy series of quartic, sextic, and octic anharmonic oscillators reveal clearly the worth of our endeavor.

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I. INTRODUCTION

Given a power series expansion in a variable x of some observable $F(x)$ that has the form

$$F(x) = \sum_j f_j x^j, \quad x \to 0, \tag{1}$$

and given that $F(x)$ is an entire function of x over the whole positive real axis, it is often of interest to extract the asymptotic ($x \to \infty$) behavior of $F(x)$ from a knowledge of (1) up to some order. In essence, we like to estimate parameters α_0 and β_0, defined by

$$F(x) \sim \alpha_0 x^{\beta_0}, \quad x \to \infty, \tag{2}$$

from (1). Here, α_0 is the amplitude and β_0 is the exponent. While the problem possesses a very general character, we shall illustrate its various facets by confining attention to quantum-mechanical perturbation theory. Usually, form (1) is obtained from the Rayleigh-Schrödinger perturbation theory and (2) is intuitively obvious. In a few situations, however, form (2) stands for the leading behavior; in actuality, it is replaced by a strong convergent expansion at very large x. In cases of external perturbations, the coupling parameter x can be varied. Then, the problem becomes of real interest. Ready examples include the variations of energy of an atom under high electric and/or magnetic fields [1]. In dealing with some problems, there exist alternative routes of arriving at (2) [e.g., variational], but not always. For instance, the variation method does not apply if the corresponding system Hamiltonian is not bounded from below. Severity of the problem of getting (2) from (1) intensifies if the parent expansion (1) diverges for any $x > 0$. Such is the case with anharmonic oscillator perturbations. It is also known that this divergence becomes more prominent as the degree of anharmonicity M (see later) increases [2–5].

A major impetus in envisaging whether (2) can be obtained from information of (1) is provided by Symanzik's scaling argument [3]. Briefly, the argument goes as follows. Choose the Hamiltonian (with $m = \frac{1}{2}$ and $\hbar = 1$)

$$H(\lambda) = -d^2/dx^2 + x^2 + \lambda x^{2M} = H_0 + \lambda V, \tag{3}$$

and transform $x \to \mu x = y$. Putting $\mu = \lambda^{-1/[2(M+1)]}$, one obtains from (3)

$$H(\lambda) = \lambda^{1/(M+1)}[-d^2/dy^2 + \lambda^{-2/(M+1)} y^2 + y^{2M}]. \tag{4}$$

From (4), it is clear that the energy perturbation series can be written as

$$E^M(\lambda) = \lambda^{1/(M+1)} \sum_{j=0}^{\infty} \varepsilon_j^{\infty}(M) \lambda^{-2j/(M+1)}. \tag{5}$$

However, on the basis of the natural starting point of perturbation theory, we may rewrite (1) in the form

$$E^M(\lambda) = \sum_{j=0}^{\infty} \varepsilon_j^0(M) \lambda^j. \tag{6}$$

In (5) and (6), we have specifically incorporated "M" in the symbols. Most widely studied problems involve $M = 2$, 3, and 4. An increase in M actually worsens drastically the divergent character of (6). This is evident from the known [2–5] asymptotic growth of the coefficients in (6) for the ground stationary state as

$$\lim_{j \to \infty} \varepsilon_j^0(M) \sim [(M-1)j]! A_M^j, \tag{7}$$

where A_M is some M-dependent constant. Thus, with energy as an observable, here we notice that, if the system Hamiltonian is given by (3), one has ready results for β_0 in (2) from (5), viz.,

$$\beta_0 = \lambda^{1/(M+1)}. \tag{8}$$

The remaining problem is to determine α_0 in (2), which in this context follows the association

$$\alpha_0 = \varepsilon_0^{\infty}. \tag{9}$$

However, if one encounters a series like (1) that does not have any reference to some Hamiltonian, the scaling argument leading to (8) would not follow. In that case, both the exponent and the amplitude are to be estimated.

Another problem is to calculate $F(x)$ at some given x value. While a number of sequence accelerative transforms are available [4,6–8], and each of them works with varying degrees of success to evaluate $F(x)$ from a series like (1), at least up to moderate values of x, the most popular one is the

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construction of suitable Padé approximants (PA) [8]. An early attempt was made by Simon [3]. A few effective variants of the PA in estimating values of $F(x)$ at large x have recently been put forward [9] where major references to earlier works may be found. We also tried to get reasonable estimates of β_0 [10] and α_0 [11] earlier by using specific variants of the PA. A different sort of approach is to employ multivalued algebraic approximants [12] that are constructed in the spirit of the PA, but take due care of multivaluedness of the function $F(x)$ in the complex x plane. Needless to mention, such complexities are reflected in the observed rapid divergence of concerned series expansions. Yet another strategy involves the self-similar approximation [13] that is particularly effective with only a few known coefficients.

Let us note that quite a few very successful methods of deriving strong-coupling expansions from a given weak-coupling one for the anharmonic oscillator perturbation under study have come up from time to time [14–18]. However, they commonly rest on the use of different kinds of quantum-mechanical techniques, e.g., path integral or related methods, nonlinearization technique, renormalized perturbation series or scale transformations, or other *Ansätze*. In such contexts, often the knowledge of large-j behavior of f_j (e.g., [7]) is also implicitly employed, along with the scaling relation (5).

Our purpose here is to view (1) as a purely numerical series, with no reference to any Hamiltonian origin. We also do not use (8). Rather, we like to see how closely one can reach such a relation from (1). In calculating α_0, however, we employ the known β_0 value, though it is not mandatory. The plan is to check the efficacy of the endeavor, and, it goes without saying that a rougher input β_0 would only worsen the target value sought. The basic idea has its origin elsewhere. Bender and Boettcher [19] had chosen a number of examples, with $\beta_0 = 0$, to find that the sequences of diagonal PAs, appropriate here, tend to the true α_0 very slowly as $x\to\infty$. Indeed, they have numerically found an inverse relationship between the error and logarithm of the sequence number. This endeavor prompted us to explore how far one can go in arriving at result (2) from (1) by employing some modified scheme involving the PA that may improve the convergence. To achieve this end, we couple the appropriate PA strategy with fractional calculus (FC).

The FC, first conceived by Leibniz, has received a lot of attention over the last few decades [20] in a variety of situations. For a series like (1) with a finite radius of convergence, we have once seen [21] how improved estimates of critical indices can be had by importing this notion. In the current context, the problem is much more involved. Therefore, we need to choose systems for which benchmark results are available, so that the efficiency can be judged transparently. Hence, we have opted to study the ground-state energy perturbation series for the three anharmonic oscillator problems, defined by the Hamiltonian (3) with $M = 2$, 3, and 4. For calculational purposes, we have taken the coefficients $\varepsilon_j^0(M)$ in (6) up to $j = 50$ [22], scaled suitably to suit (3). The problem concerned has one more advantage. A successful interpolation of (1) and (2) [13] has been found to quickly furnish nice measures of $F(x)$ over the entire range of x. This provides another motivation for the task undertaken, apart from demonstrating an application of FC in an untouched area.

II. THE STRATEGY

We first briefly consider the standard scheme of employing the PA in estimation of β_0 and α_0. From (2), one obtains in the limit $x\to\infty$,

$$x\, d \ln F(x)/dx = x\, F'(x)/F(x) = \beta_0. \tag{10}$$

This forms the basis of the so-called "d log Padé" method, widely employed in estimating critical exponents [23]. The left side of (10) can be expressed as a power series in x on the basis of (1). We call

$$T_1(x) = x d \ln F(x)/dx. \tag{11}$$

Therefore, sequences of diagonal PAs to $T_1(x)$ can be evaluated in the $x\to\infty$ limit. These sequences are of the form

$$S_1^N = \{[N/N]T_1(x)\}_{x=\infty}. \tag{12}$$

The limit point of such a sequence will hopefully converge to β_0. The convergence, however, turned out to be quite slow [10] and hence the procedure is not effective. Even for the simplest case of $M = 2$, we have noted that variants of this scheme [10] improve results only marginally.

In evaluating α_0, on the other hand, we assume that β_0 is somehow known *a priori*. Then, one can construct [19] the function, by virtue of (2),

$$[F(x)]^{1/\beta_0} = (\alpha_0)^{1/\beta_0} x, \tag{13}$$

that is valid as $x\to\infty$. We designate the left side of (13) by $T_2(x)$. Using expansion (1), we then form the $[N/(N–1)]$ PA to $T_2(x)$. A sequence of values is next obtained as

$$S_2^N = (\{[N/(N-1)]T_2(x)/x\}_{x=\infty})^{\beta_0} \tag{14}$$

that is likely to approach α_0 as $N\to\infty$.

To improve convergences of sequences $\{S_1^N\}$ and $\{S_2^N\}$, we exploit FC in the following way. Sticking to the Riemann-Liouville convention [20], we define

$$D^g y^n = \frac{\Gamma(n+1)}{\Gamma(n+1-g)} y^{n-g}, \tag{15}$$

where g may be a noninteger. Prescription (15) now allows us to construct a function $G(x)$ where

$$G(x) = x^g D^g F(x), \tag{16}$$

that also has a power-series form like (1). Indeed, we have

$$G(x) = \sum_j f_j \frac{\Gamma(j+1)}{\Gamma(j+1-g)} x^j, \quad x \to 0, \tag{17}$$

and, in addition,

$$G(x) \sim \alpha_0 \frac{\Gamma(\beta_0+1)}{\Gamma(\beta_0+1-g)} x^{\beta_0}, \quad x \to \infty. \tag{18}$$

Note that (17) and (18), in effect, replace (1) and (2). In other words, given a series (1) and its asymptotic behavior (2), one can generate an infinitude of possibilities of such pairs simply by varying the parameter g, provided terms in (16) and (17) remain finite. Only the asymptotic amplitude differs. In fact, the identification can be extended to include form (5) as well, if such a form exists, or is known, and if the asymptotic expansion involves fractional powers of x. The advantages of generating $G(x)$ would be immediately apparent below.

First, in estimating β_0, we no longer need to proceed via (10). Simply, we take the series for the ratio $T_3(x) = G(x)/F(x)$ and construct its diagonal PA. One then has

$$S_3^N = \{[N/N]T_3(x)\}_{x=\infty} = \frac{\Gamma(\beta_0+1)}{\Gamma(\beta_0+1-g)}, \quad (19)$$

that can be solved for approximate β_0 at a given N and g. Thus, sequences of values for the asymptotic exponent are obtained at different input g values. Note that an integral differentiation process is involved in (10) and this reduces the input information of f_j by one unit. No such reduction occurs if one adheres to (19) with nonintegral g. This is particularly advantageous.

Second, in determining α_0, we proceed as before; however, $F(x)$ is replaced by $G(x)$. We note that

$$T_4(x) = [G(x)]^{1/\beta_0} = \left[\alpha_0 \frac{\Gamma(\beta_0+1)}{\Gamma(\beta_0+1-g)}\right]^{1/\beta_0} x, \quad \lim x \to \infty. \quad (20)$$

Therefore, at each g value, one can have a sequence

$$S_4^N = [([N/(N-1)]T_4(x)/x)_{x=\infty}]^{\beta_0} \frac{\Gamma(\beta_0+1-g)}{\Gamma(\beta_0+1)}. \quad (21)$$

This sequence should gradually approach α_0.

One final point remains, anyway. How should we choose the right value of g? The simplest possibility is to define some kind of error and minimize it. The error should be such that, for a faster converging sequence, it would reduce. Note that, if we go up to $j = K$ in (6), the maximum number of diagonal PAs that we can construct is $[L/L]$ where $L = K/2$. Therefore, it is legitimate to define the error by

$$\Delta = \frac{1}{L-1} \sum_{N=1}^{L-1} \left(\frac{[N/N]}{[L/L]} - 1 \right)^2. \quad (22)$$

Of course, a few other measures, close to (22) in spirit, may be constructed. For example, one may use (i) $[(N+1)/(N+1)]$ in place of $[L/L]$ in the parentheses of (22), or (ii) employ some sequence accelerator like the ε algorithm or Aitken transform [4,8] and use any one extrapolated value in place of $[L/L]$, extending the sum in (22) to L terms and dividing the result by L, instead of $(L–1)$. We have checked all these variants of the error and their minima; the final outcomes differ only marginally. Therefore, we adhere to the simplest choice, (22).

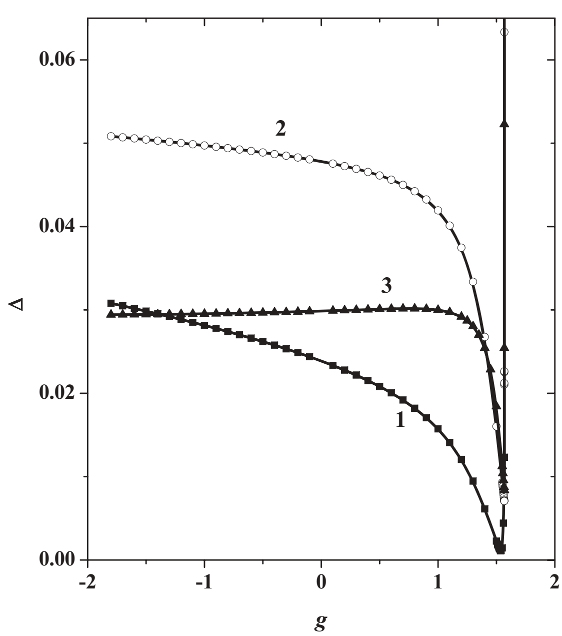

FIG. 1. Plot of the error Δ [see Eq. (22)] as a function of the fractional derivative g in Eq. (19) for the cases of $M = 2$, 3, and 4, respectively denoted by the lines 1, 2, and 3. All data refer to $K = 50$. A common deep minimum is found for all M.

III. RESULTS AND DISCUSSION

Let us first concentrate on β_0. Then, sequences S_1^N and S_3^N are to be studied, i.e., the outcomes of (12) need to be compared with those of (19). We show in Fig. 1 the nature of variation of Δ as a function of g in all three cases of $M = 2$, 3, and 4, denoted respectively by lines 1, 2, and 3 in the figure, with $L = 25$. One notes a deep minimum around $g \approx 1.6$ for all these cases. Thus, preliminary investigation shows some promise of the endeavor.

Table I displays results for the $M = 2$ case. Both the plain PA results and the FC-assisted ones are shown, the latter being taken at the g value for which Δ becomes a minimum at $L = 25$ [see (22)]. Note that the exact value ($\frac{1}{3}$) is approached in a much better way via (19). This is transparently reflected in the reduction of Δ by an order of at least 10. One may, however, argue that the gradual approach to the true value is lost in the sequence $\{S_3^N\}$, as it is seen to exceed the result sought. To settle this issue, we show in Table II another set of results for $K = 10$ (equivalently, $L = 5$). Here, one finds that the minimum in Δ is achieved at somewhat larger g. At the same time, the final value is still larger. This observation may lead us to conclude that the approach to the true result is followed more closely at higher K, with a concomitant reduction in the optimum g value.

In Table III, we display such results, viz., $\{S_1^N\}$ and $\{S_3^N\}$, for cases $M = 3$ and 4, with $K = 50$. A notable point is that, in both the cases actual values ($\frac{1}{4}$ and $\frac{1}{5}$, respectively) are gradually approached from below. While for $M = 3$, the FC-assisted scheme performs quite nicely compared to the parent version, the $M = 4$ case does not show that much promise. This is particularly due to the strongest divergence [see (7)] of the series concerned among all those studied here, and a clear reflection of it is the extremely poor result of the parent sequence, as found in Table III.

We summarize all our findings about β_0 in Table IV for the three cases of M with varying K values, viz., $K = 10$, 30, and 50. A few points of interest are the following: (i) There is a significant reduction in Δ as we switch over to $\{S_3^N\}$ from $\{S_1^N\}$; (ii) both the absolute percentage errors $|\varepsilon|$ and Δ (subscript zero now refers to the parent strategy) gradually decrease with increasing K, though the extent of reduction

TABLE I. Results of employing the parent scheme [see (12)] and the FC-assisted scheme [see (19)] for the case of quartic anharmonic oscillator ($M = 2$) with $K = 50$. The error Δ attains a minimum at the quoted g value.

N	$[N/N]$ $g = 0.0$	$[N/N]$ $g = 1.5386$
1	0.176 470 6	0.285 755 8
2	0.231 216 1	0.317 015 4
3	0.257 004 4	0.326 473 5
4	0.271 931 8	0.330 256 6
5	0.281 669 8	0.332 022 9
6	0.288 535 5	0.332 931 4
7	0.293 646 3	0.333 429 1
8	0.297 606 0	0.333 712 9
9	0.300 769 5	0.333 878 4
10	0.303 358 7	0.333 975 4
11	0.305 520 0	0.334 031 5
12	0.307 353 4	0.334 062 9
13	0.308 929 9	0.334 079 3
14	0.310 301 2	0.334 086 7
15	0.311 506 0	0.334 089 1
16	0.312 573 6	0.334 089 4
17	0.313 526 9	0.334 089 6
18	0.314 383 8	0.334 091 4
19	0.315 158 6	0.334 096 6
20	0.315 863 0	0.334 107 1
21	0.316 506 5	0.334 125 8
22	0.317 096 8	0.334 156 9
23	0.317 640 6	0.334 208 1
24	0.318 143 3	0.334 295 2
25	0.318 609 6	0.334 456 0
Δ	0.015 728 0	0.001 031 5

is larger for smaller M; (iii) a gradual fall-off in g is seen only for the $M = 2$ case where sequences $\{S_3^N\}$ start from a lower value and finally exceed the actual value at different K; in all other situations, g virtually remains fixed; (iv) $|\varepsilon|$ gets diminished by almost an order of magnitude at some fixed K by adopting our prescription, except for the $M = 4$ case where the lowering is not that prominent; (v) what we achieve through (19) by using $K = 10$ is also much better (error reduced by about a factor of 2–3) than even the $K = 50$ case of the parent strategy (12). This last point is the most striking one.

Our next attempt will be to measure the asymptotic *amplitude*. Hence, here we explore how (21) succeeds over (14) in estimating α_0. Again, we first examine whether there exists an optimum g value for which the error Δ has a minimum at some fixed M and K. For the sake of exactness, here we take the right asymptotic exponent [see (8)] as input, though approximate β_0 could be used in the absence of any kind of scaling relation. Figure 2 shows the variation of Δ as a function of g for sequences $\{S_4^N\}$ in the three cases at $K = 50$. Unlike

TABLE II. Results same as those of Table I, but with $K = 10$. Note the shift in g value.

N	$[N/N]$ $g = 0.0$	$[N/N]$ $g = 1.5674$
1	0.176 470 6	0.305 069 5
2	0.231 216 1	0.331 567 3
3	0.257 004 4	0.338 322 4
4	0.271 931 8	0.340 397 1
5	0.281 669 8	0.341 021 1
Δ	0.045 109 8	0.002 987 2

TABLE III. Results of $\{S_1^N\}$ and the FC-assisted $\{S_3^N\}$ for the evaluation of β_0 in cases $M = 3$ and 4, with $K = 50$. The error Δ attains a minimum at the quoted g values for $\{S_3^N\}$.

	$V = x^6$		$V = x^8$	
N	$[N/N]$ $g = 0.0$	$[N/N]$ $g = 1.5674$	$[N/N]$ $g = 0.0$	$[N/N]$ $g = 1.5674$
1	0.060 483 9	0.169 578 9	0.020 003 8	0.095 112 1
2	0.084 505 4	0.196 319 8	0.027 156 2	0.110 426 1
3	0.098 188 7	0.208 577 2	0.030 994 8	0.117 642 4
4	0.107 328 5	0.215 732 6	0.033 452 3	0.121 956 1
5	0.114 007 5	0.220 474 6	0.035 189 1	0.124 872 2
6	0.119 177 9	0.223 875 2	0.036 496 6	0.126 998 8
7	0.123 343 7	0.226 448 7	0.037 525 0	0.128 631 3
8	0.126 800 3	0.228 474 0	0.038 360 4	0.129 932 1
9	0.129 733 5	0.230 115 8	0.039 055 8	0.130 998 0
10	0.132 267 1	0.231 478 1	0.039 646 2	0.131 890 9
11	0.134 487 1	0.232 629 9	0.040 155 2	0.132 652 2
12	0.136 455 5	0.233 618 7	0.040 599 9	0.133 310 8
13	0.138 218 1	0.234 478 6	0.040 992 7	0.133 887 4
14	0.139 809 7	0.235 234 5	0.041 342 7	0.134 397 6
15	0.141 257 6	0.235 905 3	0.041 657 3	0.134 852 8
16	0.142 582 9	0.236 505 5	0.041 942 0	0.135 262 3
17	0.143 802 7	0.237 046 2	0.042 201 1	0.135 632 9
18	0.144 931 0	0.237 536 5	0.042 438 3	0.135 970 5
19	0.145 979 2	0.237 983 5	0.042 656 5	0.136 279 5
20	0.146 956 8	0.238 393 1	0.042 858 1	0.136 563 7
21	0.147 871 7	0.238 770 1	0.043 045 0	0.136 826 3
22	0.148 730 7	0.239 118 5	0.043 218 9	0.137 069 8
23	0.149 539 5	0.239 441 6	0.043 381 3	0.137 296 3
24	0.150 303 2	0.239 742 4	0.043 533 4	0.137 507 7
25	0.151 025 9	0.240 023 1	0.043 676 2	0.137 705 6
Δ	0.041 940 9	0.007 077 8	0.030 003 2	0.008 363 7

TABLE IV. A comparative survey of the performance level of $\{S_1^N\}$ and the FC-assisted $\{S_3^N\}$ for estimation of β_0 in the cases $M = 2$, 3, and 4, at varying K. The error Δ_0 refers to (22) with the parent sequence $\{S_1^N\}$. Absolute values of percentage errors (ε) are also displayed.

V	N	Δ_0	$\lvert\varepsilon_0\rvert$	g	Δ	β_0	$\lvert\varepsilon\rvert$
x^4	5	4.51×10^{-2}	15.50	1.5674	2.99×10^{-3}	0.3410	2.31
	15	2.31×10^{-2}	6.55	1.5512	1.35×10^{-3}	0.3374	1.22
	25	1.57×10^{-2}	4.42	1.5386	1.03×10^{-3}	0.3345	0.34
x^6	5	7.75×10^{-2}	54.40	1.5674	1.72×10^{-2}	0.2205	11.81
	15	5.30×10^{-2}	43.50	1.5674	9.91×10^{-3}	0.2359	5.64
	25	4.19×10^{-2}	39.59	1.5674	7.08×10^{-3}	0.2400	3.99
x^8	5	6.37×10^{-2}	82.41	1.5674	1.85×10^{-2}	0.1249	37.55
	15	4.00×10^{-2}	79.17	1.5674	1.14×10^{-2}	0.1349	32.55
	25	3.00×10^{-2}	78.16	1.5674	8.36×10^{-3}	0.1377	31.15

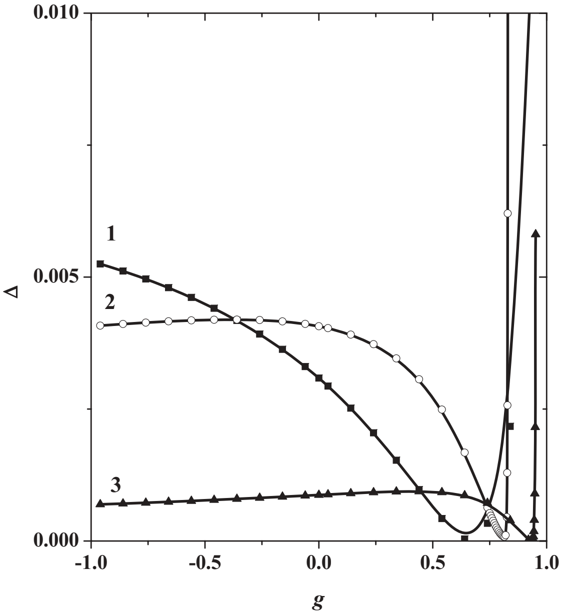

FIG. 2. Plot of the error Δ [see Eq. (22)] as a function of the fractional derivative g in Eq. (21) for the cases of $M = 2, 3,$ and 4, respectively denoted by the lines 1, 2, and 3. All data refer to $K = 50$. Each M shows its own minimum.

the earlier figure, here we note a rather wider variation in optimum g; but, in all cases, minima exist within $g = 1$. This reveals again the suitability of our prescription.

Table V shows how the parent sequence, obtained via (14), competes with the FC-assisted one, given by (21), in the three situations ($M = 2, 3, 4$) at $K = 50$. One may remember that the exact variational values [24] for the three cases, respectively for $M = 2$, 3, and 4, are 1.060 36, 1.144 80, and 1.225 82. However, it is also true that the $M = 4$ case is the most difficult one to obtain perturbatively (see, e.g., Weniger *et al.* [16] and Fernandez [18]). The gradual difficulty of approaching the right answers with rise in M is also evident from the table. We observe a progressively slower approach to the actual values with increasing M. Thus, the parent sequence for the $M = 4$ case is farthest from the right answer. However, the FC-adapted sequence (21), with the same given input information performs always much better, the more so for smaller M.

A detailed comparative survey of the improvement in going from (14) to (21) is presented in Table VI. Notable points here are (i) considerable reduction in Δ is seen as we switch over to $\{S_4^N\}$ from $\{S_2^N\}$; (ii) the absolute percentage errors $|\varepsilon|$ gradually decrease with increasing K, though Δ does not always reduce, except for $M = 2$; (iii) a gradual fall-off in g is seen for any M; (iv) at a given K, $|\varepsilon|$ is lowered by 10–40 times for $M = 2$, 2–5 times for $M = 3$, and 1.5–2.0

TABLE V. Results of $\{S_2^N\}$ and the FC-assisted $\{S_4^N\}$ for the evaluation of α_0 in cases $M = 2, 3,$ and 4, with $K = 50$. The error Δ attains a minimum at the quoted g values for $\{S_4^N\}$.

N	$V = x^4$		$V = x^6$		$V = x^8$	
	$[N/N]$ $g = 0.0$	$[N/N]$ $g = 0.6408$	$[N/N]$ $g = 0.0$	$[N/N]$ $g = 0.8145$	$[N/N]$ $g = 0.0$	$[N/N]$ $g = 0.9320$
1	1.310 371	1.091 128	1.654 875	1.138 498	2.010 055	0.889 394
2	1.205 071	1.067 166	1.543 421	1.107 718	1.927 631	0.880 077
3	1.165 281	1.063 308	1.496 376	1.100 366	1.895 134	0.878 748
4	1.144 038	1.062 273	1.468 904	1.097 759	1.877 047	0.878 504
5	1.130 684	1.061 910	1.450 307	1.096 673	1.865 257	0.878 469
6	1.121 445	1.061 756	1.436 605	1.096 193	1.856 839	0.878 469
7	1.114 635	1.061 680	1.425 938	1.095 981	1.850 460	0.878 462
8	1.109 385	1.061 635	1.417 309	1.095 896	1.845 422	0.878 442
9	1.105 201	1.061 603	1.410 128	1.095 869	1.841 318	0.878 408
10	1.101 778	1.061 577	1.404 019	1.095 865	1.837 893	0.878 363
11	1.098 920	1.061 551	1.398 733	1.095 864	1.834 983	0.878 310
12	1.096 493	1.061 521	1.394 093	1.095 858	1.832 470	0.878 251
13	1.094 403	1.061 484	1.389 975	1.095 838	1.830 272	0.878 187
14	1.092 582	1.061 436	1.386 282	1.095 803	1.828 331	0.878 121
15	1.090 979	1.061 375	1.382 945	1.095 749	1.826 599	0.878 053
16	1.089 555	1.061 298	1.379 906	1.095 675	1.825 043	0.877 984
17	1.088 282	1.061 210	1.377 123	1.095 580	1.823 634	0.877 914
18	1.087 135	1.061 118	1.374 560	1.095 462	1.822 351	0.877 845
19	1.086 095	1.061 032	1.372 187	1.095 322	1.821 177	0.877 775
20	1.085 148	1.060 962	1.369 982	1.095 159	1.820 097	0.877 707
21	1.084 282	1.060 912	1.367 925	1.094 971	1.819 099	0.877 639
22	1.083 485	1.060 879	1.365 998	1.094 758	1.818 174	0.877 572
23	1.082 750	1.060 860	1.364 189	1.094 519	1.817 313	0.877 506
24	1.082 069	1.060 850	1.362 485	1.094 252	1.816 508	0.877 441
25	1.081 436	1.060 847	1.360 875	1.093 956	1.815 756	0.877 378
Δ	0.003 087	0.000 036	0.004 069	0.000 080	0.000 873	0.000 009

TABLE VI. A comparative survey of the performance level of $\{S_2^N\}$ and the FC-assisted $\{S_4^N\}$ for estimation of α_0 in the cases $M = 2$, 3, and 4, at varying K. The error Δ_0 refers to (22) with the parent sequence $\{S_2^N\}$. Absolute values of percentage errors (ε) are also displayed.

V	N	Δ_0	$\lvert\varepsilon_0\rvert$	g	Δ	α_0	$\lvert\varepsilon\rvert$
x^4	5	7.66×10^{-3}	6.63	0.6623	8.53×10^{-5}	1.0542	0.58
	15	4.40×10^{-3}	2.89	0.6418	5.61×10^{-5}	1.0612	0.08
	25	3.53×10^{-3}	1.99	0.6408	3.60×10^{-5}	1.0608	0.05
x^6	5	6.30×10^{-3}	26.69	0.8550	5.88×10^{-5}	1.0161	11.24
	15	4.93×10^{-3}	20.80	0.8244	9.00×10^{-5}	1.0785	5.79
	25	4.07×10^{-3}	18.87	0.8145	7.96×10^{-5}	1.0940	4.44
x^8	5	1.86×10^{-3}	52.16	0.9447	8.00×10^{-6}	0.7847	35.98
	15	1.18×10^{-3}	49.01	0.9351	1.08×10^{-5}	0.8568	30.11
	25	8.73×10^{-4}	48.13	0.9320	8.99×10^{-6}	0.8774	28.42

times for $M = 4$; (v) what we achieve through $\{S_4^N\}$ by using $K = 10$ performs, however, much better than even the $K = 50$ case of the parent $\{S_2^N\}$, but the reduction in error is most pronounced for $M = 2$. This last point, being common to both the asymptotic parameters, deserves premier attention. In effect, it justifies the worth of the whole endeavor.

IV. CONCLUDING REMARKS

To summarize, one may note that, although the convergence behavior of the auxiliary series (16), obtained via FC, virtually remains the same as that of the parent series, at least for large j, one can significantly improve the information about the asymptotic exponent and amplitude by employing a given number of terms of its Taylor expansion through the use of properly constructed PAs. The gain is sometimes spectacular. We have noticed that the error often reduces by an order of magnitude at a given K. More striking is that, what the standard procedure can offer with $K = 50$ is bettered in the modified FC-assisted scheme merely with $K = 10$. This last statement is true even for the $M = 4$ case, well known for its notoriously divergent character. Here lies the final success of the strategy, apart from providing another domain of applicability of FC.

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