Periodic orbits of inelastic particles on a ring

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We consider the dynamics of N rigid particles of arbitrary mass that are constrained to move on a frictionless ring. Collisions between particles are inelastic with a constant coefficient of restitution e, and between collisions the particles move with constant velocity. We study sequences of collisions that are self-similar in the sense that the relative positions return to their original relative positions after the collision sequence while the relative velocities are reduced by a constant factor. For a given collision sequence, we develop the analytic machinery to determine the particle velocities and the locations of collisions, and we show that the problem of determining self-similar orbits reduces to solving an eigenvalue problem to obtain the particle velocities and solving a linear system to obtain the locations of interparticle collisions. For inelastic systems, we show that the collision locations can always be uniquely determined. We also show that this is in sharp contrast to the case of elastic systems in which infinite families of self-similar orbits can coexist.

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I. INTRODUCTION

In this paper, we study the motion of N rigid particles of arbitrary mass constrained to move on a frictionless ring (see Fig. 1). We will consider both elastic and inelastic collisions. For elastic particles, the system may exhibit periodic behavior in which, after a sequence of interparticle collisions, particle velocities and relative positions return to the same values. Inelastic particles lose energy when they collide with other particles, and so the only genuinely periodic state is a trivial one in which all particles move with the same velocity and hence no collisions occur. However, inelastic particles can experience nontrivial motion similar to periodic motion, with the relative positions returning to the same values, but after a sequence of interparticle collisions the particle velocities are reduced by a fixed factor. Such motion repeats infinitely and we refer to such orbits as self-similar.

We develop an analytic machinery to construct such self-similar orbits and formulate the problem mathematically as an eigenvalue problem. For inelastic particles, for each eigenvalue, if a self-similar orbit exists, then it must be unique. This is in sharp contrast to the case of elastic particles in which we show that infinite families of periodic orbits can exist.

There are a wide range of industrial applications in which discrete masses experience almost instantaneous collisions that dissipate energy. Examples include vibration hammers, pneumatic drills, and many other similar devices [1–7]. Obtaining stable operating conditions, in which the masses execute some kind of periodic motion, is important in avoiding chattering and excessive wear on machine components. Systems of this type that contain small numbers of particles appear to be extremely simple. However, a number of authors have shown that such systems can give rise to a number of surprising phenomena. One of the simplest examples is the case of a single particle that experiences gravity interacting with a vibrating plate. Mehta and Luck [8] and Luck and Mehta [9] showed that such systems can exhibit complicated period doubling behavior. Of particular interest is the periodic behavior of this system, which has been studied by Gilet *et al*. [10]. An extension of this problem that contains two particles has been studied by Whelan *et al*. [11]. Problems of this type are simple enough to be able to prove mathematical results, but still extremely rich. They therefore represent an ideal setting to pin down numerous qualitative features. Elastic systems are relatively well-understood, whereas inelastic systems are much less so, and there are a number of ways in which inelastic systems differ fundamentally from elastic systems. By studying simple systems of this type, we can gain a significant understanding of the underlying reasons for the fundamental differences between elastic and inelastic systems.

The problem of elastic particles interacting on a ring has a long history [12–16]. One motivation for these studies is to test equilibrium theories of fluids. It was pointed out in Ref. [17] that the Tonks gas naturally leads to the "simplest, yet nontrivial, continuum model accounting for excluded-volume effects in dense fluids." Although the one-dimensional geometry introduces some peculiarities, these studies provide a wealth of information and significant insights into phenomenology. In the case of inelastic particles, the interaction of particles moving on an infinite line has been extremely widely studied. Murphy [18] studied the maximum number of collisions that can occur in such a system. For sufficiently inelastic particles, an infinite number of collisions can occur in a finite time, and this phenomenon, known as "inelastic collapse," has been studied extensively [19–22]. There has also been a significant interest in particles on a line interacting with walls [23–33]. Another related topic of significant interest is the propagation of impulses through one-dimensional particle systems first considered by Nesterenko [34]. There is a vast

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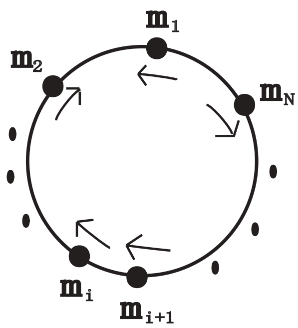

FIG. 1. N particles of different masses moving on a ring. The particles stay ordered and each particle can only collide with one of its adjacent neighbors.

literature on this subject, and we refer the reader to the review article by Sen *et al.* [35] for further details. Many of these studies have been concerned with the development of continuum equations, but the periodic motion of such systems has also received significant attention [30,32,33]. In the development of continuous equations, understanding the differences between elastic and inelastic systems is crucial. One feature of inelastic systems is that boundaries can cause very strong inhomogeneities in the system [23]. A natural problem that avoids boundary effects is the evolution of a granular gas in a periodic domain [36]. The problem we consider in this paper is equivalent to the motion of a one-dimensional granular gas in a periodic domain.

In the above-mentioned studies, the development of continuum theories requires the application of the molecular chaos hypothesis, which demands that the positions and velocities of particles eventually decorrelate. Self-similar orbits, by their nature, remain correlated indefinitely. Hence the existence of stable self-similar orbits would represent a problem for continuum theories, especially if the basins of attraction of the orbit occupied a significant amount of phase space. On the other hand, unstable periodic orbits play an important role in understanding chaos and entropy in dynamical systems. For example, the stable and unstable manifolds of a periodic orbit have important implications for the existence of chaos. In particular, an intersection of the stable and unstable manifolds of the periodic orbit implies the existence of a homoclinic tangle that provides a basis for chaotic behavior [37]. Moreover, quantities such as entropy can be efficiently expressed in terms of sums over unstable periodic orbits [38].

The motion of three equal mass inelastic particles on a ring has been studied by Grossman and Mungan [39]. They determined criteria for inelastic collapse and numerically found that quasiperiodic behavior could occur. In their study, all of the three masses are equal, and so in the elastic limit the dynamics is trivial because collisions between equal mass elastic particles are equivalent to the particles passing through one another. In our paper, we will consider the self-similar orbits of an arbitrary number of particles with arbitrary mass. In this case, the elastic limit is nontrivial, and we will study this limit in detail. Grossman and Mungan showed that the confinement of the particles to a ring greatly enhances the likelihood of inelastic collapse with almost all initial conditions leading to inelastic collapse for restitution coefficients below a critical value. Self-similar orbits, if they exist, represent a set of special orbits that never experience inelastic collapse.

Cooley and Newton [40] considered how collisions between particles affect the velocities in periodic orbits by studying the eigenvalues of matrix products. However, to completely solve the problem, one must consider both velocities and positions. In Ref. [40], the authors focused on the velocities, whereas in this paper, we analyze both velocities and positions and show that the periodic orbits with negative and complex eigenvalues obtained by Ref. [40] are unphysical since they lead to inconsistencies in the positions of the particles. Moreover, [40] implemented a numerical shooting technique to obtain the periodic orbits, whereas we will derive closed-form analytical expressions. We will determine conditions for the existence of self-similar orbits, and we will show that the inelastic case differs fundamentally from the elastic case. Furthermore, we will establish a relation between the spectra of the velocity maps and those of the position maps. We will show that this result has important consequences for the existence and uniqueness of periodic orbits.

The paper is organized as follows. In Sec. II, we explain the general system of N particles and give a mathematical formulation for determining periodic and self-similar orbits. In Sec. III, we provide a detailed analysis of the three-particle system, and we give a number of concrete examples of these types of orbits. In Sec. IV, we present the conclusion.

II. THE SYSTEM

We denote the mass and position of the ith particle as m_i and x_i, and we define vectors $m = [m_1,m_2,\dots,m_N]^T$ and $x = [x_1,x_2,\dots,x_N]^T$ (where T denotes transpose). Since the physical size of the particles does not affect the motion, we only consider point particles. We choose a length scale such that the circumference of the ring is unity. The particles interact via inelastic collisions which conserve momentum but dissipate kinetic energy. Each particle can only collide with its adjacent neighbors, and we assume the coefficients of restitution between each pair of particles are the same and denoted by e.

Let $\dot{x}$ be a vector whose elements contain the velocities of the particles. If the ith particle collides with the $(i+1)$th particle and the velocities before the collision are $\dot{x}$, then the velocities after the collision are given by $C_i\dot{x}$, where C_i is a collision matrix

$$C_i = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & c_{i,i} & c_{i,i+1} & \cdots & 0 \\ 0 & 0 & \cdots & c_{i+1,i} & c_{i+1,i+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}, \quad \begin{matrix} \leftarrow i\text{th row} \\ \leftarrow (i+1)\text{th row} \end{matrix} \tag{1}$$

and

$$c_{i,i} = \frac{m_i - em_{i+1}}{m_i + m_{i+1}}, \quad c_{i,i+1} = \frac{(1+e)m_{i+1}}{m_i + m_{i+1}},$$

$$c_{i+1,i}=\frac{(1+e)m_i}{m_i+m_{i+1}},\quad c_{i+1,i+1}=\frac{m_{i+1}-em_i}{m_i+m_{i+1}}.$$

Since the particles are on a ring, the Nth particle can also collide with the first particle, in which case the collision matrix is defined as

$$C_N=\begin{bmatrix} c_{1,1} & 0 & 0 & \cdots & 0 & 0 & c_{1,N}\\ 0 & 1 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & 1 & \cdots & \cdots & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \vdots & 1 & 0\\ c_{N,1} & 0 & 0 & \cdots & \cdots & 0 & c_{N,N}\end{bmatrix} \tag{2}$$

and

$$c_{1,1}=\frac{m_1-em_N}{m_1+m_N},\quad c_{1,N}=\frac{(1+e)m_N}{m_1+m_N},$$
$$c_{N,1}=\frac{(1+e)m_1}{m_1+m_N},\quad c_{N,N}=\frac{m_N-em_1}{m_1+m_N}.$$

Throughout the paper, we will refer to collisions between the ith particle and the $(i+1)$th particle. When i refers to the Nth particle, the collisions refer to the Nth particle colliding with the first particle.

Between particle collisions, the particles do not experience any net force in the direction of motion and so they move with constant velocities. Given the particle locations and velocities after a collision, one can easily determine which of the possible N collisions occurs next.

As we will see, it is more convenient to work with the vector

$$\widehat{p}=\begin{bmatrix} x_2-x_1\\ x_3-x_2\\ \vdots\\ x_N-x_{N-1}\\ m_1x_1+m_2x_2+\cdots+m_Nx_N\end{bmatrix} \tag{3}$$

rather than the vector x. This is because, when the ith and $(i+1)$th particles collide, the element $\widehat{p}_i=0$. When the Nth and the first particles collide, $x_N=x_1+1$, which implies that $\sum_{i=1}^{N-1}\widehat{p}_i=1$. The last element of $\widehat{p}$ represents the center of mass of particles along the ring.

The vectors $\widehat{p}$ and x are related by

$$\widehat{p}=Ax, \tag{4}$$

where

$$A=\begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0\\ 0 & -1 & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & -1 & 1 & 0\\ 0 & 0 & \cdots & 0 & -1 & 1\\ m_1 & m_2 & \cdots & m_{N-2} & m_{N-1} & m_N\end{bmatrix}. \tag{5}$$

One can show that $\det(A)=(-1)^{N+1}\sum_{i=1}^{N}m_i$. Since $m_i>0$, $\det(A)\neq 0$ and hence the matrix A is invertible, and so we can write $x=A^{-1}\widehat{p}$.

Consider a collision sequence given by $G=C_{i_n}\cdots C_{i_2}C_{i_1}$ (where the sequence is read from right to left). We denote the velocities immediately after the kth collision, C_{i_k}, as $\dot{x}^{(k)}$ and define $\dot{\widehat{p}}^{(k)}=A\dot{x}^{(k)}$. We also denote the positions at the time of the kth collision as $x^{(k)}$ and define $\widehat{p}^{(k)}=Ax^{(k)}$. Since $\dot{x}^{(n)}=G\dot{x}^{(0)}$, from Eq. (4) we obtain

$$\dot{\widehat{p}}^{(n)}=AGA^{-1}\dot{\widehat{p}}^{(0)}. \tag{6}$$

When $\dot{\widehat{p}}^{(n)}=\dot{\widehat{p}}^{(0)}$ and $\widehat{p}^{(n)}=\widehat{p}^{(0)}$, we call the orbit associated with this collision sequence periodic. Periodic orbits are only possible for elastic systems ($e=1$). Obviously, for inelastic systems, no collision sequence can be periodic, since particles lose energy during collisions and particle velocities will never be able to return to the same values after a collision sequence. However, it is possible that, after a sequence of collisions, all particle velocities have been reduced by a constant factor λ and relative positions of particles still return to the values in $\widehat{p}^{(0)}$, namely,

$$\dot{\widehat{p}}^{(n)}=\lambda\dot{\widehat{p}}^{(0)}\quad\text{and}\quad\widehat{p}^{(n)}=\widehat{p}^{(0)}. \tag{7}$$

We call an orbit satisfying Eq. (7) after a sequence of collisions a self-similar orbit. We now show that $|\lambda|<1$ for $e<1$ and $|\lambda|=1$ for $e=1$. The kinetic energy of the system at the beginning of a periodic collision sequence is given by $E^{(0)}=\frac{1}{2}(\dot{x}^{(0)})^TB\dot{x}^{(0)}$, where $B=\mathrm{diag}(m_1,m_2,\ldots,m_N)$. This can be rewritten in terms of $\dot{\widehat{p}}$ as $E^{(0)}=\frac{1}{2}(\dot{\widehat{p}}^{(0)})^T(A^{-1})^TBA^{-1}\dot{\widehat{p}}^{(0)}$. After the completion of a self-similar collision sequence of length n, we have $\dot{\widehat{p}}^{(n)}=\lambda\dot{\widehat{p}}^{(0)}$, and the kinetic energy is $E^{(n)}=\frac{1}{2}(\dot{\widehat{p}}^{(n)})^T(A^{-1})^TBA^{-1}\dot{\widehat{p}}^{(n)}=\frac{1}{2}\lambda^2(\dot{\widehat{p}}^{(0)})^T(A^{-1})^TBA^{-1}\dot{\widehat{p}}^{(0)}$. Due to inelastic collisions, $E^{(n)}<E^{(0)}$, and so we have $|\lambda|<1$ for $e<1$ (we note that this result also holds for complex λ). For elastic collisions, $e=1$, energy must be conserved and so $|\lambda|=1$. Moreover, for elastic collisions, a periodic sequence must have a collision matrix for velocities, G, that is diagonalizable. This can be proven by contradiction. If G is not diagonalizable, then it can be written in Jordan normal form $G=UJU^{-1}$. After α periods, the velocities will be given by $UJ^{\alpha}U^{-1}\dot{x}^{(0)}$. Since $|\lambda|=1$, $\|J^{\alpha}\|\longrightarrow\infty$ as $\alpha\longrightarrow\infty$. Generically, this implies that the energy will tend to infinity, which leads to a contradiction. We hence conclude that G must be diagonalizable.

In the following section, we will develop a procedure to analytically construct periodic and self-similar orbits and to determine for what parameter values, if any, a given collision sequence leads to a self-similar orbit. In typical mechanics problems, the determination of periodic and self-similar orbits requires one to solve for the position vector $\widehat{p}$ and velocity vector $\dot{\widehat{p}}$ simultaneously. However, in the current problem, for a given collision sequence, the determination of the velocity vector $\dot{\widehat{p}}$ can be decoupled from the position vector $\widehat{p}$. Therefore, we will construct the velocity vector $\dot{\widehat{p}}$ first, then use $\dot{\widehat{p}}$ to construct $\widehat{p}$. Obviously, the self-similar condition

$$\dot{\widehat{p}}^{(n)}=AGA^{-1}\dot{\widehat{p}}^{(0)}=\lambda\dot{\widehat{p}}^{(0)} \tag{8}$$

forms an eigenvalue problem with λ being the eigenvalue and $\dot{\widehat{p}}^{(0)}$ being the eigenvector.

Note that the last element of $\dot{\widehat{p}}^{(k)}$, $k=0,1,\ldots,n$, is the total momentum of the particles, which is a conserved quantity due to the absence of any net force in the direction of motion of the particles. If the kth collision is C_{i_k}, then $\dot{\widehat{p}}^{(k+1)}=AC_{i_k}A^{-1}\dot{\widehat{p}}^{(k)}$. Since the last elements of $\dot{\widehat{p}}^{(k)}$ and $\dot{\widehat{p}}^{(k+1)}$ both represent the total momentum, which remains the same before and after the kth collision, the last row of AC_iA^{-1} must have the form $[0,0,\ldots,0,1]$. The postcollision relative velocities only depend on the precollision relative velocities and are independent of the total momentum, $\sum_{i=1}^{N} m_i\dot{x}_i$, which is the last element of $\hat{\dot{p}}^{(k)}$. It follows that the last column of AC_iA^{-1} must have the form $[0,0,\ldots,0,1]^T$, and thus AC_iA^{-1} must have a block-diagonal form,

$$AC_iA^{-1} = \begin{bmatrix} & & & 0 \\ & M_i & & \vdots \\ & & & 0 \\ 0 & 0 & \cdots\ 0 & 1 \end{bmatrix}. \tag{9}$$

The block-diagonal property of AC_iA^{-1} leads to the block-diagonal property of AGA^{-1}, namely

$$\begin{aligned} AGA^{-1} &= AC_{i_n}A^{-1}AC_{i_{n-1}}A^{-1}\cdots AC_{i_2}A^{-1}AC_{i_1}A^{-1} \\ &= \begin{bmatrix} & & & 0 \\ & H & & \vdots \\ & & & 0 \\ 0 & 0 & \cdots\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{10}$$

where

$$H = M_{i_n}M_{i_{n-1}}\cdots M_{i_2}M_{i_1}. \tag{11}$$

It is therefore clear that $\lambda = 1$ and $\hat{\dot{p}}^{(0)} = [0,0,\ldots,0,1]^T$ is an eigenpair for the eigenvalue problem Eq. (8). This corresponds to the mode in which all particles moving with the same velocity along the ring without any collisions. Since this always represents trivial dynamics, we remove this eigenpair from Eqs. (8) and (10) to obtain the remaining $(N-1)$ eigenpairs that can be determined by solving

$$H\dot{p}^{(0)} = \lambda\dot{p}^{(0)},$$

where p is a vector formed by dropping the last element of $\hat{p}$. In terms of p, after a collision between the ith and $(i+1)$th particles, $\dot{p}$ is updated to $M_i\dot{p}$. We comment that M_i and H are $(N-1)$ by $(N-1)$ matrices and p is an $(N-1)$-dimensional vector. Periodicity implies

$$\dot{p}^{(n)} = H\dot{p}^{(0)} = \dot{p}^{(0)} \quad \text{and} \quad p^{(n)} = p^{(0)}, \tag{12}$$

and self-similarity of the orbit implies

$$\dot{p}^{(n)} = H\dot{p}^{(0)} = \lambda\dot{p}^{(0)} \quad \text{and} \quad p^{(n)} = p^{(0)}. \tag{13}$$

We note that previous authors [22] considered the problem of inelastic collapse on a line. They considered orbits that are self-similar in velocities, but on a line the periodicity condition for the positions clearly cannot be satisfied. Therefore, the problem we study in this paper is fundamentally different from that studied in Ref. [22]. Here we consider both velocities and positions.

To satisfy the periodic condition in particle positions given by Eqs. (12) and (13), we need to determine $p^{(n)}$ from $p^{(0)}$. This is achieved by the following procedure for each eigenpair λ and $\dot{p}^{(0)}$. First, we construct a map from the position vector after the kth collision $p^{(k)}$ to the position vector after the $(k+1)$th collision $p^{(k+1)}$. Second, we combine the maps for $k = 0,1,2,\ldots,n-1$ to obtain the desired final map from $p^{(0)}$ to $p^{(n)}$. Now we construct the map from $p^{(k)}$ to $p^{(k+1)}$. Using the collision matrices, the velocity vector immediately after the kth collision is given by $\dot{p}^{(k)} = M_{i_k}M_{i_{k-1}}\cdots M_{i_2}M_{i_1}\dot{p}^{(0)}$. Now we show how to determine the relative positions of the particles at every collision point. Since the particles all move with constant velocity between collisions, the positions of the particles at any time between the kth and $(k+1)$th collisions are given by

$$p(t) = p^{(k)} + \dot{p}^{(k)}(t - t^{(k)}), \tag{14}$$

where $t^{(k)}$ is the time of the kth collision. The $(k+1)$th collision will occur between the i_{k+1}th and $(i_{k+1}+1)$th particles. For $i_{k+1} < N$, the relative distance between these two particles is given by the i_{k+1}th element of $p(t)$, namely $\xi_{i_{k+1}}p(t)$, where

$$\xi_j = [0,\ldots,0,\underset{\substack{\uparrow \\ j\text{th}}}{1},0,\ldots,0] \quad \text{for} \quad j = 1,2,\ldots,N-1, \tag{15}$$

is an $(N-1)$-dimensional projection vector which projects out the jth element of a vector. For $i_{k+1} = N$, the projection vector will be

$$\xi_N = [1,1,\ldots,1] = \sum_{j=1}^{N-1}\xi_j$$

due to the fact that the distance between the Nth and the first particles is given by $\sum_{i=1}^{N-1} p_i = 1$. Then at the time of the $(k+1)$th collision, $t^{(k+1)}$, we have

$$\xi_{i_{k+1}}p(t^{(k+1)}) = \xi_{i_{k+1}}[p^{(k)} + \dot{p}^{(k)}(t^{(k+1)} - t^{(k)})] = \delta_{i_{k+1}N}. \tag{16}$$

Solving $t^{(k+1)}$ from the last equality of Eq. (16), we obtain

$$t^{(k+1)} = t^{(k)} + \frac{\delta_{i_{k+1}N} - \xi_{i_{k+1}}p^{(k)}}{\xi_{i_{k+1}}\dot{p}^{(k)}}. \tag{17}$$

We note that when $\xi_{i_{k+1}}\dot{p}^{(k)} = 0$, the time $t^{(k+1)}$ cannot be determined. This means the velocities of the particles are inconsistent with the desired sequence of collisions. In the following calculation, we only consider velocities with $\xi_{i_{k+1}}\dot{p}^{(k)} \neq 0$, which is a necessary condition for the consistency in velocities.

Substituting Eq. (17) into Eq. (14), we obtain

$$p^{(k+1)} = p(t^{(k+1)}) = p^{(k)} + \frac{\delta_{i_{k+1}N} - \xi_{i_{k+1}}p^{(k)}}{\xi_{i_{k+1}}\dot{p}^{(k)}}\dot{p}^{(k)}.$$

This can be rewritten in the form

$$p^{(k+1)} = \hat{F}^{(k)}p^{(k)} + \hat{b}^{(k)}, \tag{18}$$

where $\hat{F}^{(k)}$ is an $(N-1)$ by $(N-1)$ matrix given by

$$\hat{F}^{(k)} = I - \frac{\dot{p}^{(k)}\xi_{i_{k+1}}}{\xi_{i_{k+1}}\dot{p}^{(k)}}, \tag{19}$$

$b^{(k)}$ is an $(N-1)$-dimensional vector given by

$$\hat{b}^{(k)} = \frac{\delta_{i_{k+1}N}}{\xi_{i_{k+1}}\dot{p}^{(k)}}\dot{p}^{(k)},$$

and I is the $(N-1)$ by $(N-1)$ identity matrix.

Equation (18) is a map from the $(N-1)$-dimensional vector $p^{(k)}$ to the $(N-1)$-dimensional vector $p^{(k+1)}$. We now show that this map can be further reduced to a map from an $(N-1)$-dimensional vector to an $(N-2)$-dimensional vector, since $p^{(k)}$ and $p^{(k+1)}$ contain trivial information which can be removed. This is due to the fact that $p^{(l)}$, for $l = 1,2,\ldots,n$, represents the relative particle locations at the time of the lth collision. Thus, either one of the elements of $p^{(l)}$ is zero or $\sum_{j=1}^{j=N-1} p_j^{(l)} = 1$. If the lth collision is between particle i_l and i_{l+1} with $i_n \neq N$, then $p_{i_l}^{(l)} = 0$. This implies that this entry of the vector contains redundant information and can therefore be removed. If the lth collision is between the Nth particle and the first particle, i.e., $i_l = N$, we have $\sum_{j=1}^{j=N-1} p_j^{(l)} = 1$. Therefore, there is also a redundancy in the vector and we can remove any element from $p^{(l)}$. We choose to remove the $(N-1)$th element of $p^{(l)}$. To do this, we introduce the $(N-2)$ by $(N-1)$ matrices,

$$R_i = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{i-1} \\ \xi_{i+1} \\ \vdots \\ \xi_{N-1} \end{bmatrix} \quad \text{for} \quad i = 1,2,\ldots,N-2, \quad \text{and}$$
$$R_{N-1} = R_N = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \vdots \\ \xi_{N-3} \\ \xi_{N-2} \end{bmatrix}, \tag{20}$$

where ξ_j is a row vector given by Eq. (15). R_i has the property that $R_i p^{(l)}$ removes the ith element of $p^{(l)}$ when $i \neq N$, and it removes the $(N-1)$th element of $p^{(l)}$ when $i = N$. Multiplying both sides of Eq. (18) by $R_{i_{k+1}}$ to remove the redundant information from $p^{(k+1)}$, we obtain

$$R_{i_{k+1}} p^{(k+1)} = R_{i_{k+1}} \widehat{F}^{(k)} p^{(k)} + R_{i_{k+1}} \widehat{b}^{(k)}. \tag{21}$$

It is easy to check that R_i has the properties

$$R_i^T R_i + \xi_i^T \xi_i = I \quad \text{for} \quad i \neq N \tag{22}$$

and

$$R_N^T R_N + \xi_{N-1}^T \xi_{N-1} = I. \tag{23}$$

If $i_k \neq N$, the kth collision is between the i_kth particle and the (i_k+1)th particle, and we have $p_{i_k}^{(k)} = 0$. It follows that $\xi_{i_k} p^{(k)} = p_{i_k}^{(k)} = 0$. From this property and Eq. (22), we obtain the identity

$$p^{(k)} = \left(R_i^T R_i + \xi_i^T \xi_i\right) p^{(k)} = R_i^T R_i p^{(k)}.$$

Substituting this into Eq. (21), we obtain

$$R_{i_{k+1}} p^{(k+1)} = R_{i_{k+1}} \widehat{F}^{(k)} R_{i_k}^T R_{i_k} p^{(k)} + R_{i_{k+1}} \widehat{b}^{(k)} \quad \text{for} \quad i_k \neq N. \tag{24}$$

If $i_k = N$, the collision is between the Nth particle and the first particle, and

$$\xi_{N-1} p^{(k)} = p_{N-1}^{(k)} = 1 - \sum_{l=1}^{N-2} p_l^{(k)} = 1 - \xi_N R_N^T R_N p^{(k)}. \tag{25}$$

Then, from Eqs. (23) and (25), we obtain

$$\begin{aligned} p^{(k)} &= \left(R_N^T R_N + \xi_{N-1}^T \xi_{N-1}\right) p^{(k)} \\ &= R_N^T R_N p^{(k)} - \xi_{N-1}^T \xi_N R_N^T R_N p^{(k)} + \xi_{N-1}^T. \end{aligned} \tag{26}$$

Therefore, Eq. (21) can be written as

$$\begin{aligned} R_{i_{k+1}} p^{(k+1)} &= R_{i_{k+1}} \widehat{F}^{(k)} \left(I - \xi_{N-1}^T \xi_N\right) R_{i_k}^T R_{i_k} p^{(k)} \\ &\quad + R_{i_{k+1}} \left(\widehat{F}^{(k)} \xi_{N-1}^T + \widehat{b}^{(k)}\right) \quad \text{for} \quad i_k = N. \end{aligned} \tag{27}$$

Combining Eqs. (24) and (27), we obtain

$$\begin{aligned} \left[R_{i_{k+1}} p^{(k+1)}\right] &= \left[R_{i_{k+1}} \widehat{F}^{(k)} \left(I - \xi_{N-1}^T \xi_N \delta_{i_k N}\right) R_N^T\right] \left[R_N p^{(k)}\right] \\ &\quad + R_{i_{k+1}} \left(\widehat{F}^{(k)} \xi_{N-1}^T \delta_{i_k N} + \widehat{b}^{(k)}\right). \end{aligned} \tag{28}$$

We comment that i_k and i_{k+1} are unique and determined by $M_{i_{k+1}} M_{i_k}$, which are the kth and $(k+1)$th collision matrix in H. We define the $(N-1)$-dimensional vectors

$$\begin{aligned} \widetilde{p}^{(k+1)} &= R_{i_{k+1}} p^{(k+1)}, \quad \widetilde{p}^{(k)} = R_{i_k} p^{(k)}, \\ b^{(k)} &= R_{i_{k+1}} \left(\widehat{F}^{(k)} \xi_{N-1}^T \delta_{i_k N} + \widehat{b}^{(k)}\right). \end{aligned} \tag{29}$$

We also define an $(N-2)$ by $(N-2)$ matrix,

$$F^{(k)} = R_{i_{k+1}} \widehat{F}^{(k)} \left(I - \xi_{N-1}^T \xi_N \delta_{i_k N}\right) R_{i_k}^T. \tag{30}$$

We note that Eqs. (29) are a map from an $(N-1)$-dimensional vector to an $(N-2)$-dimensional vector. Using Eqs. (29) and (30), Eq. (28) can be written as

$$\widetilde{p}^{(k+1)} = F^{(k)} \widetilde{p}^{(k)} + b^{(k)}. \tag{31}$$

Equation (31) is a map from $\widetilde{p}^{(k)}$ to $\widetilde{p}^{(k+1)}$.

After composing the n maps, we obtain a map from $\widetilde{p}^{(0)}$ to $\widetilde{p}^{(n)}$,

$$\widetilde{p}^{(n)} = F \widetilde{p}^{(0)} + b, \tag{32}$$

where

$$F = \prod_{k=0}^{n-1} F^{(k)} \quad \text{and} \quad b = \sum_{k=0}^{n-1} \left(\prod_{i=k+1}^{n-1} F^{(i)}\right) b^{(k)}. \tag{33}$$

For a periodic orbit, $\widetilde{p}^{(n)} = \widetilde{p}^{(0)}$, we can then determine the initial position $\widetilde{p}^{(0)}$ by solving the linear system

$$(I - F)\widetilde{p}^{(0)} = b. \tag{34}$$

Without loss of generality, we will consider orbits that start with a collision between the $(N-1)$th and Nth particles. Thus, we have $p_{N-1}^{(0)} = 0$. Then Eq. (34) becomes

$$(I - F) \begin{bmatrix} \widetilde{p}_1^{(0)} \\ \widetilde{p}_2^{(0)} \\ \vdots \\ \widetilde{p}_{N-2}^{(0)} \end{bmatrix} = b. \tag{35}$$

After obtaining the initial position $\widetilde{p}^{(0)}$, using Eq. (31) we can then calculate the positions of all particles at each collision $\widetilde{p}^{(k)}$

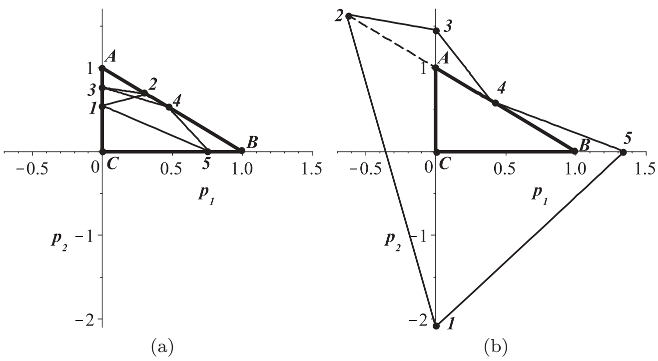

FIG. 2. The trajectories associated with eigenvalue λ_1 in the (p_1,p_2) phase plane for the sequence $M_2M_3M_1M_3M_1$ and $e=1/2$. A consistent trajectory must be confined inside the triangle ABC. (a) For $m_1=1$, $m_2=2$, and $m_3=10$, the orbit is consistent. (b) For $m_1=10$, $m_2=1$, and $m_3=5$, the trajectory is inconsistent.

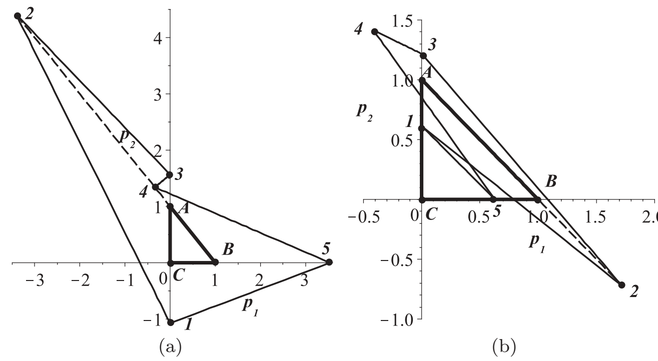

FIG. 3. The trajectories associated with eigenvalue λ_2 in the (p_1,p_2) phase plane for the sequence $M_2M_3M_1M_3M_1$ and $e=1/2$. A consistent trajectory must be confined inside the triangle ABC. (a) For $m_1=1$, $m_2=2$, and $m_3=10$, the trajectory is inconsistent. (b) For $m_1=10$, $m_2=1$, and $m_3=5$, the trajectory is also inconsistent.

for $k=1,2,\ldots,n$. However, obtaining a solution to Eq. (35) does not necessarily imply that a periodic orbit exists. This is because the solution of Eq. (35) does not necessarily guarantee that all of the particles remain in the correct spatial order, namely

$$\widetilde{p}_i^{(k)} \geqslant 0 \quad \text{and} \quad \sum_{i=1}^{N-2} \widetilde{p}_i^{(k)} \leqslant 1 \tag{36}$$

for $i=1,2,\ldots,N-2$ and $k=0,1,\ldots,n-1$. We refer to an orbit as consistent if all of these conditions hold.

In Fig. 2, we plot the results of a three-particle system in (p_1,p_2) space for two different parameter values. Collisions between the first and second particles occur on the line $p_1=0$, collisions between the second and third particles occur on the line $p_2=0$, and collisions between the third and first particles occur on the line $p_1+p_2=1$. The trajectories must be confined within the triangle ABC enclosed by the three lines. Since there are three particles on the ring in a three-particle system, any collision sequence Eq. (11) has two eigenpairs. We label them as λ_1 and λ_2 according to the condition $\lambda_1>\lambda_2$ when both of them are real. We can label them in either way when they are complex. Figure 2 shows the trajectories for solutions of Eq. (13) associated with eigenvalue λ_1 in the (p_1,p_2) phase space for the sequence $M_2M_3M_1M_3M_1$ and $e=1/2$. Figure 2(a) is the result for masses: $m_1=1$, $m_2=2$, and $m_3=10$. Since the whole trajectory remains inside of the triangle, the orbit is consistent. In Fig. 2(b), $m_1=10$, $m_2=1$, and $m_3=5$. Notice that a portion of the trajectory in Fig. 2(b) goes to the outside of the triangle. This means that the orbit is unphysical and inconsistent. Figure 3 is for the trajectories associated with eigenvalue λ_2 and for the same collision sequence and the same parameters given in Fig. 2. In this case, the trajectories shown in Figs. 3(a) and 3(b) are both inconsistent. In Fig. 4, we show how the relative velocities $\dot{p}_1$ and $\dot{p}_2$ vary over the course of a periodic sequence for the consistent orbit shown in Fig. 2(a).

Figures 2 and 3 demonstrate that for a given collision sequence and a given eigenpair, the orbit can be consistent for certain parameter values and inconsistent for others. In Fig. 5, we show the domains in which consistent orbits exist for each eigenpair for the collision sequence $M_2M_3M_1M_3M_1$ and $e=1/2$. Figure 5(a) is for the eigenpair associated with λ_1, and Fig. 5(b) is for the eigenpair associated with λ_2. It shows that for the collision sequence $M_2M_3M_1M_3M_1$ and parameter $e=1/2$, the orbit associated with λ_1 is consistent over certain parameter regions and the orbit associated with λ_2 is inconsistent for all values of m_2/m_1 and m_3/m_1.

For a given collision sequence, the corresponding matrix may have eigenvalues λ that are positive, negative, or pairs of complex conjugates. We now show that only orbits corresponding to positive eigenvalues can be consistent. Suppose that the eigenvalue λ is negative. Without loss of generality, we assume that the first collision is between the $(N-1)$th and Nth particles so that $p_{N-1}^{(0)}=0$. If the orbit is consistent, the velocity after the collision must have the property $\dot{p}_{N-1}^{(0)}>0$, since the particles must bounce back from each other. After a periodic sequence, the position will be $p_{N-1}^{(n)}=0$ and the velocity becomes $\dot{p}_{N-1}^{(n)}=\lambda\dot{p}_{N-1}^{(0)}$, but if $\lambda<0$, $\dot{p}_{N-1}^{(n)}$ will be negative and so when the next collision occurs, the distance between the $(N-1)$th and Nth particles, $p_{N-1}^{(n+1)}$, will be negative. This is unphysical, and therefore any orbit with $\lambda<0$ is inconsistent. A similar argument shows that complex eigenvalues must also give inconsistent orbits.

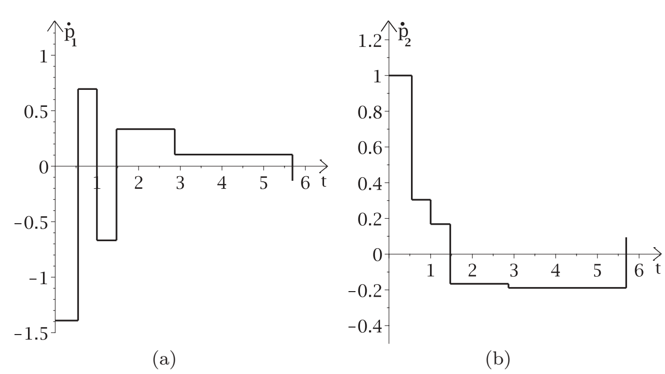

FIG. 4. The relative velocities of the particles are plotted against time for the orbit shown in Fig. 2(a). The orbit is shown over one period over which the velocities are reduced by a factor λ_1.

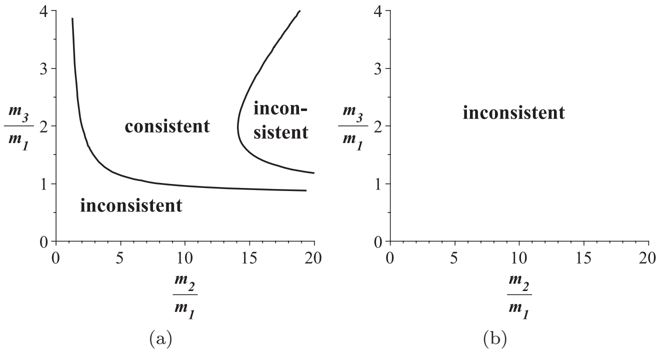

FIG. 5. Domain of consistent orbits and that of inconsistent orbits for the collision sequence $M_2M_3M_1M_3M_1$ and $e = 1/2$. (a) For the eigenpair associated with λ_1, the orbits are consistent over certain parameter regions. (b) For the eigenpair associated with λ_2, the orbits are inconsistent in the whole domain m_2/m_1 and m_3/m_1.

III. THREE-PARTICLE SYSTEM

In this section, we provide a detailed analysis for a three-particle system. In this case, there are only three possible interparticle collisions given by

$$M_1 = \begin{bmatrix} -e & 0 \\ \frac{(1+e)m_1}{m_1+m_2} & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & \frac{(1+e)m_3}{m_2+m_3} \\ 0 & -e \end{bmatrix},$$
$$M_3 = \frac{1}{m_1+m_3}\begin{bmatrix} m_1 - em_3 & -(1+e)m_3 \\ -(1+e)m_1 & m_3 - em_1 \end{bmatrix}. \tag{37}$$

It can readily be verified that M_i has the property that $\det(M_i) = -e$. We now consider a general sequence consisting of n collisions given by Eq. (11). The eigenvalues of H are determined by the quadratic equation

$$\lambda^2 - \operatorname{tr}(H)\lambda + \det(H) = 0. \tag{38}$$

Since

$$\det(H) = \det\left[\prod_{k=1}^{n} M_{i_k}\right] = \prod_{k=1}^{n} \det\left(M_{i_k}\right) = \prod_{k=1}^{n}(-e) = (-e)^n,$$

Eq. (38) becomes

$$\lambda^2 - \operatorname{tr}(H)\lambda + (-e)^n = 0, \tag{39}$$

with solutions given by

$$\lambda_\pm = \tfrac{1}{2}\left[\operatorname{tr}(H) \pm \sqrt{[\operatorname{tr}(H)]^2 - 4(-e)^n}\right]. \tag{40}$$

When n is odd, then $(-e)^n < 0$ and both eigenvalues of H are real with one being positive and the other negative. Only the orbit corresponding to the positive eigenvalue can be consistent. In particular, for the elastic case, $e = 1$, $|\lambda| = 1$ and so $\lambda_+ = 1$ and $\lambda_- = -1$.

For a collision sequence with even collisions, the only case in which one can obtain real positive roots is if $\operatorname{tr}(H) \geqslant 2e^{n/2}$. In this case, both eigenvalues are real and positive. If $\operatorname{tr}(H) < 2e^{n/2}$, both eigenvalues are negative or are complex conjugates and so the corresponding orbit must be inconsistent.

Now we state the result that underlies the dramatic difference between elastic and inelastic systems. Let $\dot{p}^{(0)}$ be one of the eigenvectors of H, with associated eigenvalue λ_+. We will show that the position map will have the form $\widetilde{p}^{(n)} = \lambda_-\widetilde{p}^{(0)} + b$. Comparing this expression with Eq. (32), we see that $F = \lambda_-$. Similarly, if $\dot{p}^{(0)}$ is associated with λ_-, the position map will have the form $\widetilde{p}^{(n)} = \lambda_+\widetilde{p}^{(0)} + b$. Therefore, the eigenvalues of H are also the eigenvalues of F. The proof of this statement is given in Appendix A.

This result allows us to show the sharp contrast between inelastic and elastic systems. For a given collision sequence in an inelastic system, a consistent self-similar orbit associated with one of the eigenvectors, if it exists, is always unique. For a given collision sequence in an elastic system, it is possible to have a unique periodic orbit, no periodic orbit, or an infinite family of periodic orbits. Below we derive this result. For an inelastic system, when a consistent self-similar orbit exists, we have

$$p^{(0)} = p^{(n)} = \lambda_b p^{(0)} + b.$$

Since $|\lambda_b| < 1$ for an inelastic particle system ($e < 1$), this gives a unique solution

$$p^{(0)} = b/(1 - \lambda_b). \tag{41}$$

For an elastic system ($e = 1$), when n is odd, $\lambda_+ = 1$ and $\lambda_- = -1$. If $p^{(0)}$ is associated with the eigenvalue $\lambda_- = -1$, no consistent orbit exists. If $p^{(0)}$ is associated with the eigenvalue $\lambda_+ = 1$, then

$$p^{(0)} = p^{(n)} = \lambda_- p^{(0)} + b = -p^{(0)} + b,$$

which gives a unique solution $p^{(0)} = b/2$. However, when n is even, the only possible consistent orbits must have $\lambda_+ = \lambda_- = 1$, and as shown in Sec. II, H must be diagonalizable and hence must be the identity matrix. Then, any vector can be an eigenvector of H. In this case, any vector is an eigenvector, and so the velocity map is periodic for an infinite family of velocities. However, for a periodic orbit the position map must also be periodic. The position map is given by $p^{(n)} = p^{(0)} + b$, where b depends on the particular eigenvector under consideration. For a general eigenvector, $b \neq 0$, and so the position map will not have periodic solutions. However, generically, one can find a particular eigenvector such that $b = 0$ and in this case the position map will be periodic for any position $p^{(0)}$. Hence, there is a particular eigenvector for

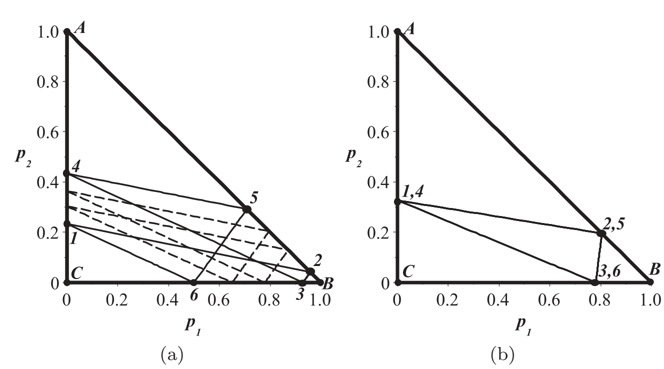

FIG. 6. The trajectories in the (p_1,p_2) phase plane for the sequence $M_2M_3M_1M_2M_3M_1$ and $m_1 = 1$, $m_2 = 2$, and $m_3 = 5$. (a) Elastic system with $e = 1$. In this case, there are an infinite family of periodic orbits and we plot two of them, one as a solid line and the other as a dashed line. (b) Inelastic system with $e = 1/2$. In this case, there is only one periodic orbit.

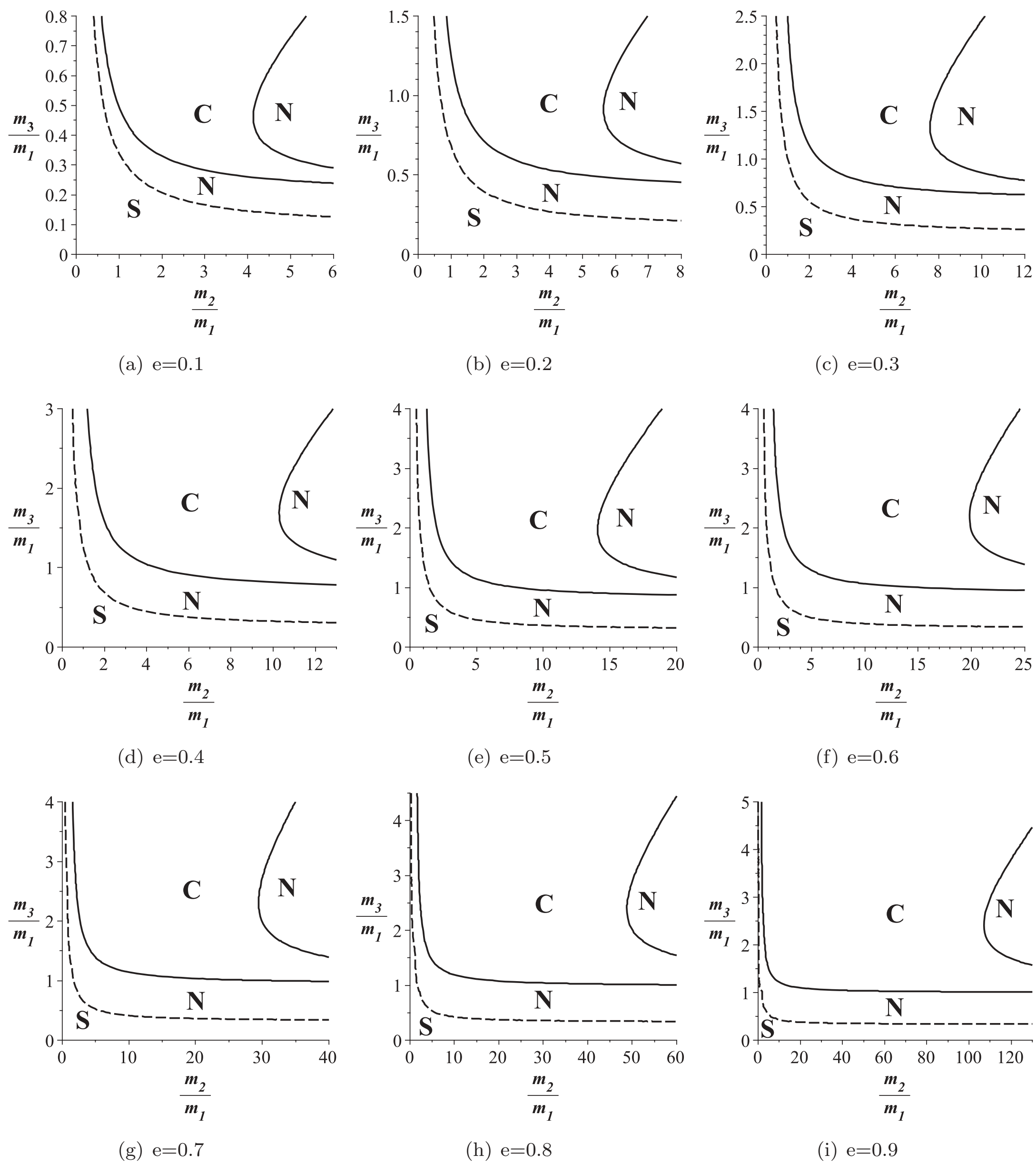

FIG. 7. Domain of consistency and stability for collision sequence $M_2M_3M_1M_3M_1$ with the eigenvalue λ_+ for various values of e. The dashed curve is the phase boundary for stability, and the solid curves are the phase boundary for consistency. The region labeled by **S** is stable but inconsistent, the region labeled by **C** is consistent but unstable, and the regions labeled by **N** are neither consistent nor stable. There is no consistent and stable region.

which there is an infinite family of periodic orbits. In Fig. 6(a), we plot two members of such an infinite family of periodic orbits for the collision sequence $M_2M_3M_1M_2M_3M_1$. This is in sharp contrast to the unique orbit shown in Fig. 6(b) for an equivalent inelastic system ($e < 1$).

Above, we have derived a number of results regarding the existence of periodic orbits. We now turn our attention to their stability. For an orbit to be stable, both the velocity and position maps must be stable. We begin by considering the velocity map. We consider a periodic sequence, whose collision matrix for velocities is given by H. If H is diagonalizable, we write $H = UDU^{-1}$, where D is a diagonal matrix that contains the eigenvalues. Starting with an arbitrary set of relative velocities $\dot{p}^{(0)}$, after α periodic cycles, the relative velocities are given by

$$\dot{p}^{(\alpha n)} = UD^{\alpha}U^{-1}\dot{p}^{(0)}.$$

As $\alpha \to \infty$, D^{α} will be dominated by the eigenvalue with the largest magnitude. If the eigenvalue with the largest magnitude is unique, then the associated eigenvector will be stable for the velocity map. The orbits associated with the other eigenvalues will be unstable since small perturbations from the orbit will ultimately dominate. If there is more than one eigenvalue having the largest magnitude, then the associated eigenvector will be neutrally stable if H is diagonalizable. If H is nondiagonalizable, similar criteria can be derived by considering the nature of the associated Jordan normal form (see Appendix B for details). We also require that the position

TABLE I. Elastic system ($e = 1$). All orbits are unstable. For odd n, there is always only one positive λ. For even n, either both λ are positive or there is no positive λ. For each positive λ, we report the number of solutions and whether it is consistent. The entry "in certain regions" means that, in the phase space of $(m_2/m_1, m_3/m_1)$, it is consistent in certain regions and inconsistent in the other.

n	H	No. of $\lambda > 0$	No. of solution for each λ	Consistent	Consistent and stable
3	$M_2M_3M_1$	1	1	always	never
			∞	in certain regions	never
4	$M_2M_3M_1M_3$	2	0		never
5	$M_2M_3M_1M_3M_1$	1	1	in certain regions	never
			∞	in certain regions	never
	$M_2M_3M_1M_2M_3M_1$	2	0		never
			∞	in certain regions	never
6	$M_2M_1M_3M_1M_3M_1$	2	0		never
	$M_2M_1M_3M_2M_3M_1$	0	0		never
	$M_2M_3M_1M_2M_3M_1M_3$	1	0		never
7	$M_2M_1M_3M_2M_3M_1M_3$	1	0		never
	$M_2M_1M_3M_1M_3M_1M_3$	1	1	in certain regions	never

map Eq. (A4) be stable. For $e < 1$, Eq. (A14) implies that this is automatically satisfied since the eigenvalues of F are also eigenvalues of the velocity map and hence must have magnitude less than unity due to energy loss. For $e = 1$, the stability is more subtle. The velocity map is always neutrally stable since H is diagonalizable with $|\lambda| = 1$. For $e = 1$ and n even, the position map Eq. (A4) only has a solution for an eigenvector in a particular direction (corresponding to the case of an infinite family of periodic orbits). However, if one adds a perturbation to the velocity, the position map will generically have $b \neq 0$ and so the positions will be given by $p^{(\alpha n)} = p^{(0)} + \alpha b$. Thus the size of the perturbation $|\alpha b|$ increases with α and so the orbit is unstable.

For $e = 1$, we now show that all orbits with odd n are unstable. We prove this by the method of the contradiction. Assume that H with an odd number of collisions has a stable orbit. We consider two adjacent periods of this orbit, which is equivalent to considering the matrix H^2. Then the same orbit is also stable for the repeated sequence H^2. However, H^2 contains an even number of collisions and therefore cannot have stable orbits. Therefore, H must have no stable orbits. Combining the results for both even and odd n, we have proven that no stable periodic orbits can exist for elastic systems.

Now we summarize our procedure to determine the stability and consistency of a collision sequence. For the purpose of brevity, we only refer to the most fundamental equations. For a given collision sequence, we do the following:

Step 1. We use Eqs. (9)–(11) to calculate the matrix H, which gives the velocity map. For three-particle systems, M_i is given by Eq. (37).

Step 2. Based on Eq. (13), we calculate the eigenpairs of H. The purpose is to compare the magnitudes of eigenvalues and determine the stability of the orbits. Generically, only the eigenvector with the largest magnitude eigenvalue is stable. For three-particle systems, the eigenvalues are given by Eq. (39). A necessary condition for consistency is that $\lambda > 0$. Therefore, we will only consider the eigenpairs with positive eigenvalues. Thus the remaining steps listed below are only carried out for eigenpairs with positive eigenvalues.

Step 3. Based on Eqs. (29), (30), and (33), we calculate F and b. Then based on Eq. (35), we determine the initial position required to make the position vector periodic. For three-particle systems, the initial position is given by Eq. (41).

Step 4. Based on Eq. (31), we determine the position vector after the kth collision for $k = 0,1,2,n-1$. If all these position vectors satisfy Eq. (36), then the collision sequence is consistent. Otherwise it is inconsistent.

As we have discussed, if $\dot{p}^{(0)}$ is an eigenvector of H with eigenvalue λ, a necessary condition for consistency is that $\lambda > 0$ and the sufficient condition for the stability of orbit is that $|\lambda|$ is larger than the magnitude of all other eigenvalues of H. Both conditions are functions of n and tr(H) for three-particle systems. Based on Eq. (40), it is easy to determine the sign and stability of λ as a function of n and tr(H).

Note that all these quantities are functions of $e,m_1,m_2,\ldots,m_n$. Since the value of the total mass of the system does not affect the consistency and stability of a collision sequence, the eigenpairs are functions of $e,m_2/m_1,m_3/m_1,\ldots,m_n/m_1$ only. Therefore, for a given e, in the phase space parametrized by the mass ratios, an eigenvector can be either always consistent, always inconsistent, or consistent in certain regions and inconsistent in others. We demonstrate this for a sequence with five collisions $M_2M_3M_1M_3M_1$ in a three-particle system. Since n is odd, Eq. (40) shows that there is one positive and one negative eigenvalue given by $\lambda_\pm = \frac{1}{2}[\mathrm{tr}(H) \pm \sqrt{[\mathrm{tr}(H)]^2 - 4(-e)^n}]$. In the parameter regions where tr$(H) > 0$, $|\lambda_+| > |\lambda_-|$, and so the solution associated with λ_+ is stable. In the parameter regions where tr$(H) < 0$, $|\lambda_-| > |\lambda_+|$, and so the solution associated with λ_- is stable, but it is inconsistent since $\lambda_- < 0$. Therefore, the phase boundary which separates the stable region from the unstable region in phase space is determined by the condition tr$(H) = 0$. In Fig. 7, we show the domains of stability and instability and the domains of consistency and inconsistency only for the positive eigenvalue λ_+ for various values of e. The dashed curve is the phase boundary for stability, and the solid curves are the phase boundary for consistency. The region labeled by **S** is stable but

TABLE II. Consistency and stability of orbits in inelastic systems ($e < 1$). The eigenvalues λ_+ and λ_- are given by Eqs. (40). The entry "in certain parameter regions" means that, in the phase space of $(e,m_2/m_1,m_3/m_1)$, it is consistent in certain regions and inconsistent in others. The table shows that no orbit reported here is consistent and stable.

n	H	λ	Consistent	Consistent and stable
		λ_+	always	never
3	$M_2M_3M_1$	λ_-	never	never
		λ_+	never	never
4	$M_2M_3M_1M_3$	λ_-	in certain regions	never
		λ_+	never	never
5	$M_2M_3M_1M_3M_1$	λ_-	in certain regions	never
		λ_+	never	never
	$M_2M_3M_1M_2M_3M_1$	λ_-	in certain regions	never
		λ_+	never	never
6	$M_2M_1M_3M_1M_3M_1$	λ_-	in certain regions	never
		λ_+	never	never
	$M_2M_1M_3M_2M_3M_1$	λ_-	in certain regions	never
		λ_+	never	never
	$M_2M_3M_1M_2M_3M_1M_3$	λ_-	in certain regions	never
		λ_+	never	never
7	$M_2M_1M_3M_2M_3M_1M_3$	λ_-	in certain regions	never
		λ_+	never	never
	$M_2M_1M_3M_1M_3M_1M_3$	λ_-	in certain regions	never

inconsistent, the region labeled by **C** is consistent but unstable, and the regions labeled by **N** are neither consistent nor stable. Figure 7 shows that there is no consistent and stable region for the collision sequence $M_2M_3M_1M_3M_1$. We have performed further extensive tests of the parameter space and have found that stable and consistent self-similar periodic orbits do not exist for this sequence.

We carried out this type of study for all sequences up to seven collisions for the three-particle system. The results are shown in Table I for elastic particle systems and in Table II for inelastic particle systems. Since all particles are on a ring, certain collision sequences, such as $M_3M_2M_1$ and $M_1M_2M_3$, are equivalent under particle relabeling. Only nonequivalent sequences are shown in Tables I and II. In Table I, we report a number of eigenvalues with positive λ (we note that if λ is positive, we must have $\lambda = 1$ since the system is elastic). For odd n, there is one and only one positive λ. For even n, either both λ are positive or there is no positive λ. For each positive λ, we report the number of solutions and whether it is consistent. The entry "in certain parameter regions" means that, in the phase space of $(m_2/m_1,m_3/m_1)$, it is consistent in certain regions and inconsistent in others. Table I clearly shows that, for elastic systems, all three possibilities (no periodic orbits, unique periodic orbits, or an infinite family of periodic orbits) can exist.

For inelastic systems, the self-similar orbits are always unique if the velocity is consistent. We need to check the stability and consistency of velocity vectors and positions. We report the results in Table II. Tables I and II show that for both elastic and inelastic systems, depending on the collision sequence, the orbits can be always consistent, always inconsistent, or consistent only in certain parameter ranges. Tables I and II show that, for all sequences up to seven collisions, there is no stable consistent self-similar periodic orbit for three inelastic particles on a ring.

We have also considered systems with four and five particles. At first glance, there appear to be a large number of possible orbits. However, using various symmetries of sequences, one can dramatically reduce the number of orbits one must consider. For example, for a four-particle system with four collisions, one only needs to check two cases: $M_1M_2M_3M_4$ and $M_1M_2M_4M_3$. For a four-particle system with five collisions, one only needs to check two cases: $M_1M_2M_1M_3M_4$ and $M_1M_3M_2M_1M_4$. For a five-particle system with five collisions, one only needs to check two cases: $M_1M_2M_3M_4M_5$ and $M_1M_2M_3M_5M_4$. We have performed extensive parameter studies for all of the above sequences, and we were also unable to find any stable consistent orbits. We therefore conjecture that there is no stable consistent self-similar periodic orbit for any sequence.

IV. CONCLUSION

In this paper, we have carried out a detailed study of the motion of N particles of different masses on a ring. For elastic systems, we have shown that periodic orbits can exist. A periodic orbit has the properties that, after a sequence of interparticle collisions, all particle velocities and the relative particle positions return to the same values before the collision sequence. For inelastic systems, we have shown that self-similar orbits can exist. A self-similar orbit has the same properties as a periodic orbit except that all particle velocities are reduced by a constant factor after a collision sequence. We developed a procedure to determine the periodic and self-similar orbits for any given collision sequence. It consists of solving an eigenvalue problem for the particle velocities, solving a system of linear equations for the particle positions, and checking the consistency of the solutions. We have shown that the orbits of inelastic particles on a ring are dramatically different from those of elastic particles on a ring. In inelastic systems, a self-similar orbit, if it exists, is always unique for a given eigenvalue of the particle velocity. However, for elastic systems, an infinite family of periodic orbits can exist. This means that the detailed knowledge that has been developed for elastic systems cannot even provide a qualitative understanding of inelastic systems. In this paper, we examined many periodic and self-similar orbits for systems with three particles on a ring. We found that all examples of periodic or self-similar orbits we have constructed are unstable. This means any small disturbance to the periodic or self-similar orbits will destroy the orbit. This leads us to conjecture that there are no stable periodic or self-similar orbits for either elastic or inelastic particles on a ring.

ACKNOWLEDGMENTS

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APPENDIX A

In this Appendix, we prove that if $\dot{p}^{(0)}$ is one of the eigenvectors of H associated with the eigenvalue λ_a, then the position map must have the form $p^{(n)} = \lambda_b p^{(0)} + b$, where λ_b is the other eigenvalue of H.

For a three-particle system, Eqs. (15) and (20) become

$$\begin{aligned} \xi_1 &= [1,0], \quad \xi_2 = [0,1], \quad \xi_3 = [1,1], \\ R_1 &= [0,1], \quad R_2 = R_3 = [1,0]. \end{aligned} \tag{A1}$$

We consider a general collision sequence given by Eq. (11). For any two adjacent collisions $M_{i_{k+1}} M_{i_k}$ in the sequence, it is easy to check, from Eqs. (19), (29), and (30), that

$$F^{(k)} = s^{(k)} \frac{\xi_{i_k} \dot{p}^{(k)}}{\xi_{i_{k+1}} \dot{p}^{(k)}} \quad \text{and}$$
$$b^{(k)} = \begin{cases} \frac{\xi_1 \dot{p}^{(k)}}{\xi_3 \dot{p}^{(k)}} & \text{if} \quad i_{k+1} = 3, \\ 0 & \text{if} \quad i_{k+1} \neq 3. \end{cases} \tag{A2}$$

Here $s^{(k)}$ can be either 1 or -1. The rule for determining the value of $s^{(k)}$ is as follows: there are only three particles, and so one of the particles must be involved in both of the collisions $M_{i_{k+1}}$ and M_{i_k} (since adjacent collisions cannot be the same). If the third particle is involved in both collisions, then $s^{(k)} = 1$, otherwise $s^{(k)} = -1$. We comment that since H is an orbit with period n, it follows that $M_{i_0} = M_{i_n}$, $M_{i_{n+1}} = M_{i_1}$, $\xi_{i_0} = \xi_{i_n}$, and $s_0 = s_n$, which is determined by $M_{i_1} M_{i_n}$.

One can readily check that ξ_{i_k} is a left eigenvector of M_{i_k} with eigenvalue e, that is, $\xi_{i_k} M_{i_k} = -e\xi_{i_k}$. Using this result and $\dot{p}^{(k)} = M_{i_k} \dot{p}^{(k-1)}$ in Eq. (A2), we have

$$F^{(k)} = s^{(k)} \frac{-e\xi_{i_k} \dot{p}^{(k-1)}}{\xi_{i_{k+1}} \dot{p}^{(k)}}. \tag{A3}$$

From the position map Eq. (34) with the collision sequence given by Eq. (11), we have

$$p^{(n)} = F p^{(0)} + b, \tag{A4}$$

where

$$F = F^{(n-1)} F^{(n-2)} \cdots F^{(1)} F^{(0)} \tag{A5}$$

and

$$b = \sum_{k=0}^{n-1} \left[\prod_{i=k+1}^{n-1} F^{(i)} \right] b^{(k)}. \tag{A6}$$

In Eq. (A5), we replace the terms $F^{(n-1)}, F^{(n-2)}, \ldots, F^{(1)}$ by the expression in Eq. (A3) to obtain

$$F = s^{(n-1)} \frac{-e\xi_{i_{n-1}} \dot{p}^{(n-2)}}{\xi_{i_n} \dot{p}^{(n-1)}} s^{(n-2)} \frac{-e\xi_{i_{n-2}} \dot{p}^{(n-3)}}{\xi_{i_{n-1}} \dot{p}^{(n-2)}} \cdots s^{(1)} \frac{-e\xi_{i_1} \dot{p}^{(0)}}{\xi_{i_2} \dot{p}^{(1)}} F^{(0)}.$$

Since the sequence is periodic, we have $\xi_{i_0} = \xi_{i_n}$, and so using Eq. (A2) to replace $F^{(0)}$, we obtain

$$\begin{aligned} F &= s^{(n-1)} \frac{-e\xi_{i_{n-1}} \dot{p}^{(n-2)}}{\xi_{i_n} \dot{p}^{(n-1)}} s^{(n-2)} \frac{-e\xi_{i_{n-2}} \dot{p}^{(n-3)}}{\xi_{i_{n-1}} \dot{p}^{(n-2)}} \\ &\quad \cdots s^{(1)} \frac{-e\xi_{i_1} \dot{p}^{(0)}}{\xi_{i_2} \dot{p}^{(1)}} s^{(0)} \frac{\xi_{i_n} \dot{p}^{(0)}}{\xi_{i_1} \dot{p}^{(0)}} \\ &= (-e)^{n-1} \frac{\xi_{i_n} \dot{p}^{(0)}}{\xi_{i_n} \dot{p}^{(n-1)}} \prod_{k=0}^{n-1} s^{(k)}. \end{aligned} \tag{A7}$$

Since there are three particles on the ring, any periodic or self-similar orbits must have at least three collisions. We now prove that for $n \geqslant 3$, $\prod_{k=0}^{n-1} s^{(k)}$ always equals 1. For $n = 3$, one can easily check that the factor $s^{(2)} s^{(1)} s^{(0)}$ in $M_{i_3} M_{i_2} M_{i_1}$ is always 1. We now proceed by the method of induction. We assume that $\prod_{k=0}^{n-1} s^{(k)} = 1$ and we examine a sequence of length $(n+1)$. Any sequence of length $(n+1)$ can be constructed by attaching $M_{i_{n+1}}$ to the end of a sequence of length n. Suppose

$$S_n = M_{i_n} M_{i_{n-1}} \cdots M_{i_1} \tag{A8}$$

and

$$S_{n+1} = M_{i_{n+1}} M_{i_n} M_{i_{n-1}} \cdots M_{i_1}. \tag{A9}$$

Using the induction assumption, we have $s_{n-1} \cdots s_1 s_0 = 1$. Comparing the sequences in Eqs. (A8) and (A9) with adjacent pairs of collisions, one can easily see that the sequence S_{n+1} does not have the pair $M_{i_1} M_{i_n}$, but has two additional pairs $M_{i_{n+1}} M_{i_n}$ and $M_{i_1} M_{i_{n+1}}$. There are two possibilities: collisions $M_{i_1} M_{i_n}$ involve the third particle either once or twice. If the pair of collisions $M_{i_1} M_{i_n}$ involves the third particle once, then $s^{(0)} = -1$ in S_n. In this case, the third particle only appears once in one of the collision pairs $M_{i_{n+1}} M_{i_n}$ and $M_{i_1} M_{i_{n+1}}$, and twice in the other pair. This implies that $s^{(n)} s^{(0)} = -1$ in S_{n+1}. This means S_n and S_{n+1} have the same sign. If the pair of collisions $M_{i_1} M_{i_n}$ involves the third particle twice, then $s^{(0)} = 1$ in S_n. In this case, the third particle must appear once in each of the collision pairs $M_{i_{n+1}} M_{i_n}$ and $M_{i_1} M_{i_{n+1}}$. This implies that $s^{(n)} = s^{(0)} = -1$ in S_{n+1}. Again, this means S_n and S_{n+1} have the same sign. This gives $\prod_{k=0}^{n} s_k = 1$ for all $n \geqslant 3$. Thus, Eq. (A7) becomes

$$F = (-e)^{n-1} \frac{\xi_{i_n} \dot{p}^{(0)}}{\xi_{i_n} \dot{p}^{(n-1)}}. \tag{A10}$$

Let λ_a be one of the eigenvalues, either λ_+ or λ_-, and let λ_b be the other eigenvalue. Let $\dot{p}^{(0)}$ be an eigenvector of H

associated with eigenvalue λ_a, namely

$$\dot{p}^{(n)} = H\dot{p}^{(0)} = M_{i_n}\cdots M_{i_1}\dot{p}^{(0)} = \lambda_a\dot{p}^{(0)}, \quad \text{(A11)}$$

which gives

$$\dot{p}^{(0)} = \frac{\dot{p}^{(n)}}{\lambda_a} = \frac{M_{i_n}\dot{p}^{(n-1)}}{\lambda_a}. \quad \text{(A12)}$$

After substituting Eq. (A12) into Eq. (A10), we have

$$F = (-e)^{n-1}\frac{\xi_{i_n}M_{i_n}\dot{p}^{(n-1)}}{\lambda_a\xi_{i_n}\dot{p}^{(n-1)}}. \quad \text{(A13)}$$

Since $\xi_{i_n}M_{i_n} = -e\xi_{i_n}$, we have

$$F = (-e)^{n-1}\frac{-e\xi_{i_n}\dot{p}^{(n-1)}}{\lambda_a\xi_{i_n}\dot{p}^{(n-1)}} = \frac{(-e)^n}{\lambda_a} = \lambda_b. \quad \text{(A14)}$$

The last equality in the above equation comes from the relation $\det(H) = \lambda_a\lambda_b = (-e)^n$. This completes the proof.

APPENDIX B

The velocity map Eq. (13) is characterized by the collision matrix H. If H has p distinct eigenvalues, then it can be written in Jordan normal form $H = UJU^{-1}$, where J can be expressed as a block-diagonal matrix,

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{bmatrix}, \quad \text{(B1)}$$

and each block J_i is given by

$$J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{bmatrix}, \quad \text{(B2)}$$

where N_i is the multiplicity of the eigenvalue of λ_i.

Starting with an arbitrary set of relative velocities $\dot{p}^{(0)}$, after α periodic cycles, the relative velocities are given by

$$\dot{p}^{(\alpha n)} = UJ^{\alpha}U^{-1}\dot{p}^{(0)}.$$

Since the matrix J is a block-diagonal matrix, J^{α} can be written in block-diagonal form as

$$J^{\alpha} = \begin{bmatrix} J_1^{\alpha} & & & \\ & J_2^{\alpha} & & \\ & & \ddots & \\ & & & J_p^{\alpha} \end{bmatrix}, \quad \text{(B3)}$$

where J_i^{α} are upper triangle matrices given by

$$J_i^{\alpha} = \begin{bmatrix} \lambda_i^{\alpha} & B_{1,\alpha}\lambda_i^{\alpha-1} & B_{2,\alpha}\lambda_i^{\alpha-2} & B_{3,\alpha}\lambda_i^{\alpha-3} & \cdots & \cdots \\ & \lambda_i^{\alpha} & B_{1,\alpha}\lambda_i^{\alpha-1} & B_{2,\alpha}\lambda_i^{\alpha-2} & \ddots & \vdots \\ & & \lambda_i^{\alpha} & B_{1,\alpha}\lambda_i^{\alpha-1} & \ddots & B_{3,\alpha}\lambda_i^{\alpha-3} \\ & & & \lambda_i^{\alpha} & \ddots & B_{2,\alpha}\lambda_i^{\alpha-2} \\ & & & & \ddots & B_{1,\alpha}\lambda_i^{\alpha-1} \\ & & & & & \lambda_i^{\alpha} \end{bmatrix}, \quad \text{(B4)}$$

and $B_{i,\alpha}$ is the binomial coefficient given by

$$B_{i,\alpha} = \frac{\alpha!}{i!(\alpha-i)!}.$$

For the inelastic case ($e < 1$), all the eigenvalues of H must have $|\lambda_i| < 1$. As $\alpha \to \infty$, according to Eq. (B4), each J_i^{α} will be dominated by the element in the top right-hand corner. This element is associated with the generalized eigenvector at the end of the Jordan chain. If the Jordan block associated with the largest magnitude eigenvalue is unique, then the associated generalized eigenvector at the end of that Jordan chain will be stable for the velocity map. The orbits associated with all other generalized eigenvectors will be unstable. If there is more than one Jordan block having the largest magnitude eigenvalue (which we denote by $J_{\max_1}, J_{\max_2}, \ldots, J_{\max_i}$), then we need to consider the dimension of these Jordan blocks. From Eq. (B4), we immediately see that, if there is a unique $J_{\max_j}$ whose dimension is larger than all the others, then only the generalized eigenvector at the end of the Jordan chain associated with that block will be stable and all other generalized eigenvectors will be unstable. If there are multiple $J_{\max_j}$ with largest dimension, then the generalized eigenvectors at the ends of the Jordan chains associated with those blocks will be neutrally stable and the other generalized eigenvectors will be unstable.

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