Comment on "Time needed to board an airplane: A power law and the structure behind it"

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Frette and Hemmer [Phys. Rev. E **85**, 011130 (2012)] recently showed that for a simple model for the boarding of an airplane, the mean time to board scales as a power law with the number of passengers N and the exponent is less than 1. They note that this scaling leads to the prediction that the "back-to-front" strategy, where passengers are divided into groups from contiguous ranges of rows and each group is allowed to board in turn from back to front once the previous group has found their seats, has a *longer* boarding time than would a single group. Here I extend their results to a larger number of passengers using a sampling approach and explore a scenario where the queue is presorted into groups from back to front, but allowed to enter the plane as soon as they can. I show that the power law dependence on passenger numbers is different for large N and that there is a boarding time reduction for presorted groups, with a power law dependence on the number of presorted groups.

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Frette and Hemmer's recent work [1] showed that for a simple model of the airliner boarding process, the dependence of the mean time to board $\langle T(N)\rangle$ on the number of passengers $N \leqslant 16$ is well fit by a power law with an exponent $\alpha = 0.69 \pm 0.01$. The model consists of one passenger per row, where passengers enter a single aisle at the front of the plane, advance instantaneously until they reach their row or run into another passenger, and take a fixed time to sit in their seat and clear the aisle once they have reached it. They developed a sophisticated analysis based on an explicit enumeration of all possible permutations to calculate the time exactly for all $N \leqslant 14$ and sampled the set of possible permutations for $N = 15$ and 16. The power law was computed by fitting the resulting mean boarding time as a function of N. As noted by the authors, this scaling has an interesting implication for what they call the "back-to-front" strategy. They assume that the passengers are presorted by row into several groups, so that the passengers in the last M rows are in the first group, the next M rows in the next, etc., but the order of the passengers within each group is random. The groups are allowed to board the plane one at a time from back to front, but each group does not begin until the previous one is finished. The result is that the total mean time to board is longer than the time for boarding as a single random group because, for example, with two groups,

$$N^{\alpha} < 2\,(N/2)^{\alpha} = N^{\alpha} 2^{(1-\alpha)} \qquad (1)$$

if $\alpha < 1$. However, requiring that each group finishes sitting before the next group is allowed to begin is not efficient and is in fact not the way airlines structure their boarding process. A much closer approximation (neglecting variants such as letting frequent travelers or higher paying passengers board first) is that passengers are presorted by row into groups within the queue, but allowed to board as soon as they can.

I have implemented Frette and Hemmer's model and simulated the boarding time for randomly sampled permutations. The sampling algorithm randomly selects from a uniform distribution of permutations by generating permutations of the initially sorted queue using Durstenfeld's algorithm [2] with the random number generator built into GFORTRAN 4.5.1 [3]. I use 10^7 samples for each value of N up to 4096 and 10^6 samples for N up to 16 384 and evaluate the means and variances of the boarding times. The variance divided by the number of samples is the square of the one standard deviation uncertainty of the estimate of the true mean. To confirm that my sampling approach reproduces the exact calculation of Ref. [1], I plot $\langle T(N)\rangle$ as a function of N in Fig. 1. I compute power law fits

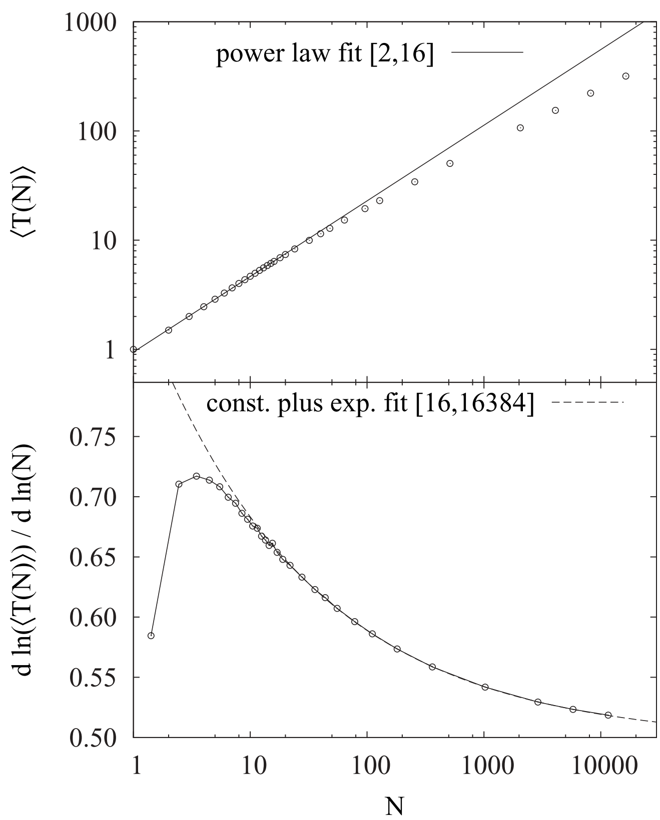

FIG. 1. Top: mean time to board a plane $\langle T(N)\rangle$ as a function of the number of passengers N on a log-log scale. Symbols show simulation results and the solid line is a power law fit for small N, as in Ref. [1]. Bottom: logarithmic derivative of $\langle T(N)\rangle$ as a function of N on a lin-log scale. Symbols show simulation results, the solid line is a guide to the eye connecting the points, and the dashed line is a fit of $\frac{d\ln(\langle T(N)\rangle)}{d\ln(N)}$ to a constant plus a decaying exponential of $\ln(N)$ for $N \geqslant 16$.

of my results by fitting the logarithm of $\langle T(N)\rangle$ to a linear function of the logarithm of N. The errors on $\langle T(N)\rangle$ are negligible on this scale (less than 7×10^{-4} for $N \leqslant 4096$ and less than 3×10^{-3} for larger N). For $2 \leqslant N \leqslant 16$, a power law fit

$$\langle T(N)\rangle = cN^{\alpha} \tag{2}$$

gives

$$c = 0.945(1 \pm 0.01), \quad \alpha = 0.692 \pm 0.004, \tag{3}$$

in excellent agreement with the published results. I also find that the standard deviation of the distribution fits a power law with a best fit exponent of 0.343 ± 0.002, within the error bars of the published result of 0.32 ± 0.02. A comparison of my sampled $\langle T(N)\rangle$ to Frette and Hemmer's exact results for $10 \leqslant N \leqslant 14$ [4] (data not shown) shows excellent agreement and no sign of systematic error.

It appears from the plot that the data for $N > 20$ deviate significantly from the fit. A plot of the logarithmic derivative (Fig. 1), which would be constant for a power law, shows an effective exponent that decreases for increasing N. The logarithmic derivative appears to approach 0.5 and is indeed well fit by a constant plus a decaying exponential in $\ln(N)$,

$$\frac{d\ln(\langle T(N)\rangle)}{d\ln(N)} = a + b\exp[-c\ln(N)], \tag{4}$$

with a best fit value for the constant a of 0.4966 ± 0.0007. This may not be the true functional form for the logarithmic derivative, but it does appear that the limiting behavior for $\langle T(N)\rangle$ in the large N limit is a power law with exponent 0.5.

I checked for systematic error in the sampling algorithm by comparing its results to several alternatives for $N = 48$, where the sample mean of 12.82 is significantly different from the extrapolated low N power law prediction of 13.77. Using a different permutation algorithm (randomizing an initially sorted queue by swapping $100N$ randomly selected pairs), a different, linear-congruential random number generator [5], or increasing the number of samples to 10^8 or 10^9 gives results for $\langle T(N)\rangle$ that agree within 0.001. This range of variation is comparable to the error estimate of 0.0004, confirming the reliability of the sampling results and the statistical significance of the deviation between the sampling results and the low N power law. I do not have an explanation for the observed dependence of the effective exponent on N. While the sampling algorithm has been tested in several ways, it is possible that some unidentified systematic error is affecting the results.

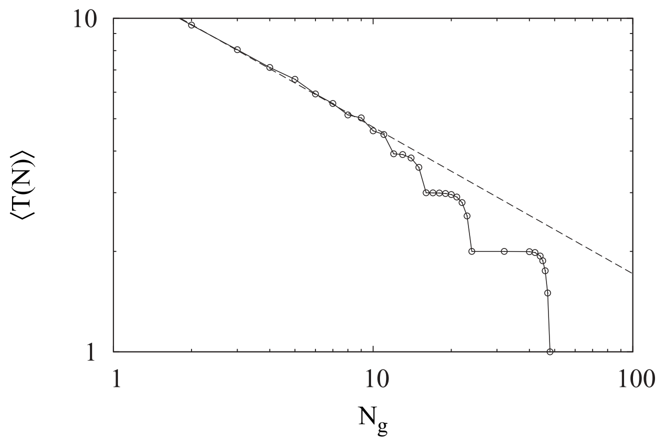

FIG. 2. Mean time to board plane $\langle T(N)\rangle$ as a function of number of presorted groups N_g for $N = 48$ on a log-log scale. Symbols show simulation results and the dashed line is a power law fit for small N_g.

To measure the effect of partially sorting the incoming passengers into groups but allowing each one to board as soon as possible, I presorted the passengers into N_g nearly equal size groups within the boarding queue (the first $N \bmod N_g$ groups are larger by one passenger than the remaining groups). I then sampled the distribution of boarding times for these partially sorted queues for $N = 48$. The mean boarding time for a partially sorted queue $\langle T_s(N,N_g)\rangle$ as a function of group size N_g is plotted in Fig. 2. For $N_g = 1$ I recover the original result and for $N_g = N$, i.e., a fully sorted queue, the boarding time is exactly 1. For small numbers of groups the reduction in boarding time is approximately a power law in N_g, with variation at large N_g because of the discreteness of small groups. The best fit exponent of the power law for $N_g \leqslant 11$ is -0.45 ± 0.01. This scaling means that presorting the full queue into 5 groups reduces the mean boarding time by about a factor of 2. The overall efficiency of the boarding process, however, would also be influenced by any additional time required to presort into groups, which could offset some or all of this simple model's predicted boarding time decrease.

I thank K. E. Ross for pointing out the difference between original boarding strategy and the one actually used by airlines.

[1] V. Frette and P. C. Hemmer, Phys. Rev. E **85**, 011130 (2012).
[2] R. Durstenfeld, Commun. ACM **7**, 420 (1964).
[3] http://gcc.gnu.org/fortran/ and http://gcc.gnu.org/gcc-4.5/
[4] V. Frette and P. C. Hemmer (private communication).
[5] P. A. W. Lewis, A. S. Goodman, and J. M. Miller, IBM Syst. J. **8**, 136 (1969).