Performance of a multilevel quantum heat engine of an ideal N-particle Fermi system

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We generalize the quantum heat engine (QHE) model which was first proposed by Bender *et al.* [J. Phys. A **33**, 4427 (2000)] to the case in which an ideal Fermi gas with an arbitrary number N of particles in a box trap is used as the working substance. Besides two quantum adiabatic processes, the engine model contains two isoenergetic processes, during which the particles are coupled to energy baths at a high constant energy E_h and a low constant energy E_c, respectively. Directly employing the finite-time thermodynamics, we find that the power output is enhanced by increasing particle number N (or decreasing minimum trap size L_A) for given L_A (or N), without reduction in the efficiency. By use of global optimization, the efficiency at possible maximum power output (EPMP) is found to be universal and independent of any parameter contained in the engine model. For an engine model with any particle-number N, the efficiency at maximum power output (EMP) can be determined under the condition that it should be closest to the EPMP. Moreover, we extend the heat engine to a more general multilevel engine model with an arbitrary 1D power-law potential. Comparison between our engine model and the Carnot cycle shows that, under the same conditions, the efficiency $\eta = 1 - \frac{E_c}{E_h}$ of the engine cycle is bounded from above the Carnot value $\eta_c = 1 - \frac{T_c}{T_h}$.

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I. INTRODUCTION

A quantum engine cycle consists of several basic quantum thermodynamic processes, including quantum adiabatic process, quantum isothermal process, quantum isobaric process, quantum isochoric process, and isoenergetic process [1–5]. Based on these thermodynamic processes, various quantum engine models have been studied in a series of publications [1–27]. One of the important attributes shared by quantum engine cycles is that the working substance is composed of quantum matter. Using a single particle confined in a 1D box potential as the working substance, Bender *et al.* [2] constructed a two-level engine model by changing both the potential wall and the quantum state in a specific manner.

For a two-level Bender engine model of a single particle in various potentials, the performance has been studied intensively [2,19–22]. It has been found that the efficiency at maximum power output (EMP) is a constant for a given potential [19,21] but sensitively dependent on the form of the potential [21]. Two of the authors of the present work generalized the Bender model [2] and gave a performance analysis, extending a single two-state particle into two three-state particles [23]. These engine models were well established and analyzed, but under the certain approximations that the working substance contains only one (or two) particle(s) and that only two (or three) energy levels are taken into account. It is natural to generalize these previous models by establishing an arbitrary M-level engine model that works with an arbitrary finite number N of particles (with $N \leqslant M$). Moreover, so far there has been no universal and complete discussion on the performance of such a quantum heat engine (QHE) model in the literature [2,19–21,23] and, in particular, the problem of EMP [26] of the generalized engine model has not been addressed adequately and clearly.

In the present paper, we study the performance of an engine model which consists of two quantum adiabatic and two isoenergetic processes, by extending the previous models of a (two) two-state (three-state) particle(s) in a potential to a multilevel engine cycle with an arbitrary number N of noninteracting fermions confined in a trap. Usually, efficiency is sacrificed to maximize the power in finite-time thermodynamics [17,26]. By contrast, the finite-time process analysis finds that, for given minimum trap size or particle number, the power output can be enhanced by increasing particle number or decreasing minimum trap size, without decrease in efficiency. We prove by using global optimization that the efficiency at possible maximum power output (EPMP) is intrinsic and independent of the model parameters, such as the potential size, and the particle number as well as the number of energy levels. The EMP for any engine cycle can be determined, under the certain condition that the efficiency of this model is closest to the EPMP. The general case, in which a M-level engine model uses the N-particle Fermi gas trapped in an arbitrary 1D power-law potential as the working substance, is discussed. We compare the engine cycle with the Carnot cycle, describing the quantum Carnot cycle which works with N noninteracting fermions in an arbitrary power-law potential. As expected, the value of the efficiency of our engine model is found to be bounded from above the Carnot value.

II. QUANTUM VERSION OF THE FIRST THERMODYNAMIC LAW AND OF ADIABATIC THEOREM

The first law of thermodynamics in quantum-mechanical systems can be expressed as a function of eigenenergies ε_n and probability distributions p_n [5,26],

$$dE = đQ + đW = \sum_n \varepsilon_n dp_n + \sum_n p_n d\varepsilon_n, \qquad (1)$$

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where $đQ = \sum_n \varepsilon_n dp_n$ and $đW = \sum_n p_n d\varepsilon_n$ depict the heat exchange and work done, respectively. Here we have used $đQ$ ($đW$) instead of dQ (dW), since the heat exchange Q (work W) is a process variable. As in a classical system where the generalized force Y_n, conjugate to the generalized coordinate y_n, is defined by $Y_n = -\frac{đW}{dy_n}$, the force for a quantum system reads

$$F = -\frac{đW}{dL} = -\sum_n p_n \frac{d\varepsilon_n}{dL}, \tag{2}$$

where L is the generalized coordinate corresponding to the force F.

Because the von Neumann entropy $S^{\mathrm{VN}} = k_B \mathrm{Tr}(\rho \ln \rho)$, with k_B being Boltzmann constant, identically vanishes when the density matrix ρ is of a pure state [19,21], the Shanon entropy instead of the von Neumann entropy will be considered. The Shanon entropy can be expressed in terms of the probability distributions as

$$S = -k_B \sum_n p_n \ln p_n. \tag{3}$$

The quantum adiabatic theorem [28] predicts that an isolated system would remain in its initial state during an adiabat. It is clear from Eq. (3) that the entropy in an adiabatic process remains constant through no change in the probability distributions p_n, with $n = 1,2,3,\ldots$.

III. A MULTILEVEL ENGINE MODEL OF FINITE N NONINTERACTING FERMIONS TRAPPED IN A 1D BOX

An arbitrary state $|\psi\rangle$ can be expanded in terms of the eigenstates $|u_n\rangle$ as $|\psi\rangle = \sum_n a_n |u_n\rangle$, with the expansion coefficients satisfying $\sum_{n=1}^{\infty} |a_n|^2 = 1$. For an ideal Fermi gas with a finite number of particles confined in a 1D box with width L, the Schrödinger equation of the system can be written as

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - E\psi(x) = 0, \tag{4}$$

where $\psi(x)$ is required to satisfy the boundary conditions $\psi(0) = 0$ and $\psi(L) = 0$. The expectation value of the system Hamiltonian is thus given by $E = \langle\psi|H|\psi\rangle = \sum_n \varepsilon_n |a_n|^2$, where the single-particle energy spectrum is $\varepsilon_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$, with m being the mass of a particle, and $n = 1,2,3,\ldots$.

A multilevel QHE model of N noninteracting fermions in a 1D box potential, consisting of two adiabatic and two isoenergetic processes, is illustrated in Fig. 1(a). The isoenergetic and quantum adiabatic processes are, respectively, identified as:

(i) Isoenergetic process: The potential width L changes as the potential wall moves, and the system is coupled to an energy bath in the isoenergetic process where the energy of the system $E(L)$ remains constant. So the energy exchange is a form of heat exchange by definition, and the external fields play the role of heat baths in traditional quantum heat engines. From a physical point of view, an isoenergetic process can be achieved when the working substance interacts with an external field [2,22,27,29,30], as shown in Fig. 1(b). For instance, when the atom system interacts with a radiation field [2,22,27], photons are converted from high or low to low or high states during the operation of the engine, leading to a change in energetic value. In such a case, the particles undergo transitions among the energy levels under the action of the radiation fields if the width of the trapping potential varies very slowly, thereby indicating that the isoenergetic process could be realized [31].

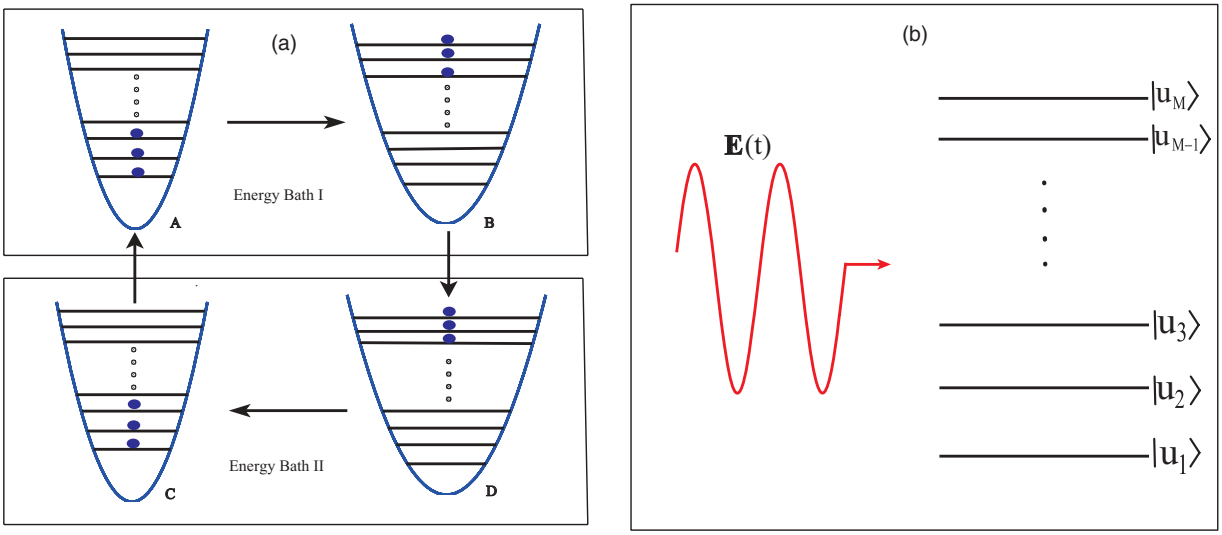

FIG. 1. (Color online) (a) Interaction of a M-level system with an external field $\mathbf{E}(t)$. (b) Graphic sketch of a multilevel QHE model of an ideal Fermi gas with a finite number N of particles in a 1D box trap. At instants A and D, the N particles stay in the N energy levels with $n = 1,2,\ldots,N-1,N$, while at instants B and C, the particles stay in highest energy levels with $n = M, M-1,\ldots,M-N+2, M-N+1$. In the process $A \to B (C \to D)$ the system absorbs (releases) energy from (to) energy bath I (II) and the energy of the system is kept unchanged at a constant energy E_h (E_c). In the adiabatic branches $B \to C$ and $D \to A$ the N-particle system is decoupled from the energy bath and stays in fixed states.

(ii) Adiabatic process: During the adiabatic process, the system is decoupled from the energy bath and remains a fixed state through no change of the probability distributions p_n.

The four processes that our engine model operates are described as follows:

(1) At the start of the isoenergetic expansion $A \to B$, the finite N noninteracting fermions are assumed to occupy the lowest energy levels and the system is brought into contact with an energy bath (labeled I). As a result, the energy of the system is given by

$$E_{AB} = E_h = \frac{\pi^2\hbar^2}{2mL_A^2}\sum_{i=1}^{N} i^2 = \frac{\pi^2\hbar^2}{2mL_A^2}G_1, \tag{5}$$

where $G_1 \equiv \sum_{i=1}^{N} i^2 = \frac{1}{6}N(N+1)(2N+1)$. In this isoenergetic expansion, the state of a single particle is a linear combination of the M energy eigenstates. Using the condition $\sum_n |a_n^{(i)}|^2 = 1$ with $a_n^{(i)}$ being the expansion coefficients of ith particle occupying the nth eigenstate, the energy of the system as a function of potential width L is

$$E_h = \frac{\hbar^2\pi^2}{2mL^2}\sum_{i=1}^{N}\sum_{n=1}^{M}\left|a_n^{(i)}\right|^2 n^2. \tag{6}$$

Combination of Eq. (5) with Eq. (6) gives rise to the following relation:

$$L^2 = \frac{L_A^2}{G_1}\sum_{i=1}^{N}\sum_{n=1}^{M}\left|a_n^{(i)}\right|^2 n^2. \tag{7}$$

When these fermions occupy from the Mth to the $(M-N+1)$th level, respectively, the maximum value of L in the isoenergetic expansion is obtained as $L_B = \sqrt{\frac{G_2}{G_1}}L_A$, where $G_2 \equiv \sum_{i=M-N+1}^{M} i^2$ and, thus, $G_2 = \frac{1}{6}M(M+1)(2M+1) - \frac{1}{6}(M-N)(M-N+1)(2M-2N+1)$. From Eq. (6), the force F is determined by $F_{AB}(L) = \frac{\hbar^2\pi^2}{mL^2}\sum_{i=1}^{N}\sum_{n=1}^{M}|a_n^{(i)}|^2$, which, according to Eq. (7), can be written as

$$F_{AB}(L) = \frac{\hbar^2\pi^2}{mL_A^2 L}G_1. \tag{8}$$

The heat quantity Q_h absorbed from the energy bath I is given by

$$Q_h = \int_{L_A}^{L_B} F_{AB}(L)dL = \frac{\pi^2\hbar^2}{2mL_A^2}G_1\ln\left(\frac{G_2}{G_1}\right). \tag{9}$$

(2) In the adiabat $B \to C$, the system adiabatically expands, outputing work while isolated from any energy bath. During this process the system remains in the fixed state and no heat exchange occurs.

(3) This isoenergetic compression $C \to D$, where the Fermi system is coupled to the other energy bath (labeled II), is almost an inverse process of the first process. At the end of the compression $L_D = \sqrt{\frac{G_1}{G_2}}L_C$, the system is back in the initial state in which all particles occupy the lowest energy levels. During this isoenergetic compression, the expectation value of the Hamiltonian is kept constant as

$$E_{CD} = E_c = \frac{\pi^2\hbar^2}{2mL_C^2}G_2. \tag{10}$$

Similar to the isoenergetic expansion, the force as a function of L can be obtained as

$$F_{CD}(L) = \frac{\hbar^2\pi^2}{mL_C^2 L}G_2, \tag{11}$$

and the heat quantity Q_c released to the energy bath II is determined according to

$$Q_c = \left|\int_{L_C}^{L_D} F_{CD}(L)dL\right| = \frac{\pi^2\hbar^2}{2mL_C^2}G_2\ln\left(\frac{G_2}{G_1}\right). \tag{12}$$

(4) In the adiabatic compression $D \to A$ the system is decoupled from the energy baths. As in the adiabat $B \to C$, the system remains in the fixed state with no heat exchange in this compression process.

The engine cycle $A \to B \to C \to D \to A$ is drawn in the (F,L) plane [see Fig. 2]. By repeatedly performing the above sequence of processes, we have a scenario in which energy is systematically extracted from the energy bath I, some of which released to the energy bath II and the rest of which delivered as work. In discussing the engine model, we assume unitary transformation from a pure state to a pure state. Although the system may be in a mixed state during any engine process, the system is assumed to remain in a pure state at the special instants A, B, C, and D [of course, it is possible for the system to be in a mixed sate [20] even at these special instants].

The total work per cycle and the efficiency are, respectively, given by

$$W = Q_h - Q_c = \frac{\pi^2\hbar^2}{2m}\left(\frac{G_1}{L_A^2} - \frac{G_2}{L_C^2}\right)\ln\left(\frac{G_2}{G_1}\right), \tag{13}$$

$$\eta = \frac{W}{Q_h} = 1 - \frac{G_2 L_A^2}{G_1 L_C^2} = 1 - \frac{E_c}{E_h}. \tag{14}$$

Before ending this section, we would like to point out that, although no equations of motions and thus no real dynamics are used to describe the evolution of the quantum system, the time required for completing a thermodynamic process should satisfy the condition of quantum speed limit. That is,

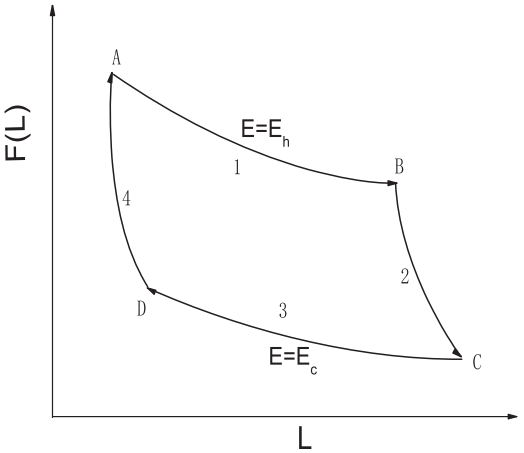

FIG. 2. Schematic diagram of a QHE cycle in the plane of the width L and force $F(L)$. The quantum states of the N particles and the values of the potential width at the four special instants are as follows: $|u_1\rangle, |u_2\rangle, \ldots, |u_{N-1}\rangle$, and $|u_N\rangle$ at instants A and D, $|u_M\rangle, |u_{M-1}\rangle, \ldots, |u_{M-N+2}\rangle$, and $|u_{M-N+1}\rangle$ at instants B and C.

in order for the initial state of the system to evolve through a unitary evolution to a state, the finite time $\sim\hbar/E$ is required for completing that transition [32].

IV. OPTIMIZATION ON THE QHE CYCLE

We are now in a position to discuss the power output and efficiency for the QHE cycle. As emphasized, in order for the adiabatic theorem to apply, the time scale associated with the variation of the state must be much lager than that of the dynamical one, $\sim\hbar/E$ [21,23,26,28]. Let $\bar{v}(t)$ and τ_0 be the average speed of the variation of L and the total cycle time, respectively. The speed $\bar{v}(t)$ should be slow enough that the change of L is much slower than the dynamical time scale, $\sim\hbar/E$, and thus the adiabatic as well as isonergetic scheme remains valid. In other words, the speed $\bar{v}(t)$ must satisfy the condition that $\bar{v}(t) \ll \frac{L}{\hbar/E}$, which, together with Eqs. (5) and (14), gives an estimate:

$$\bar{v}(t) \ll \frac{\pi^2\hbar G_2 L}{2mL_A^2 R^2}. \tag{15}$$

The total change in the potential with L per cycle, L_T, is given by

$$\begin{aligned} L_T &= (L_B - L_A) + (L_C - L_B) + (L_C - L_D) + (L_D - L_A) \\ &= 2(L_C - L_A). \end{aligned} \tag{16}$$

The total cycle time τ_0 as a function of the average speed is

$$\tau_0 = L_T/\bar{v} = 2(L_C - L_A)/\bar{v}. \tag{17}$$

Defining $R = L_C/L_A$ and $\Gamma = G_2/G_1$, we find the power output by using Eqs. (13) and (17),

$$P = \frac{W}{\tau_0} = \frac{\pi^2\hbar^2\bar{v}G_1}{4mL_A^3}\frac{R^2-\Gamma}{R^3-R^2}\ln(\Gamma), \tag{18}$$

which is a monotonous increasing (decreasing) function of G_1 (or L_A), indicating that the power output can be enhanced by increasing particle number N (or decreasing minimum potential width L_A) for given L_A (or N). Note that, according to inequality Eq. (15), the power output should satisfy the condition that $P \ll \frac{\pi^4\hbar^3 G_1 G_2 L}{8mL_A^3 R^2}\frac{R^2-\Gamma}{R^3-R^2}\ln(\Gamma)$. The positive work ($P > 0$ or $W > 0$) condition can be obtained from Eq. (18) as

$$R > \sqrt{\Gamma}, \tag{19}$$

which depends on the ratio Γ. Only when this positive work condition is satisfied can the positive work be extracted.

The efficiency in Eq. (14) as a function of Γ and R becomes

$$\eta = 1 - \frac{\Gamma}{R^2}, \tag{20}$$

increasing monotonously as Γ decreases.

In Fig. 3, we plot the three-dimensional diagram of (P^*,Γ,R), with $P^* \equiv P/(\frac{\pi^2\hbar^2\bar{v}G_1}{4mL_A^3})$ being the dimensionless power output. Figure 3 shows that there exists a maximum dimensionless power output for given values of Γ and R. When the ratio Γ is kept unchanged, we have the following condition by setting the derivative of the power P^* with respect to the ratios R equal to zero:

$$R_m^3 + \Gamma(-3R_m + 2) = 0. \tag{21}$$

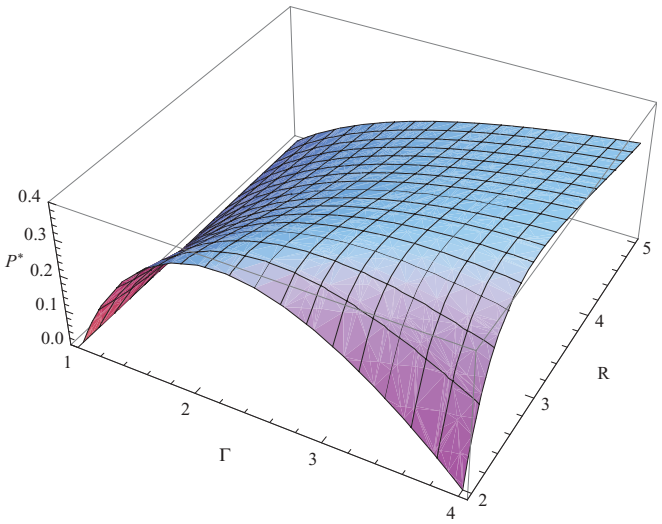

FIG. 3. (Color online) Dimensionless power output P^* as a function of Γ and R.

Note that, for a two-level (or three-level) engine with a single (or two) particle(s) in a box trap, $\Gamma = 2$ ($\Gamma = 13/5$) and Eq. (21) reduces to the corresponding one in Ref. [19] (or Ref. [23]). Mathematically, the maximum power can be found by using global optimization that sets the derivatives of the power P^* with respect to the ratios R and Γ equal to zero. In such a case, the maximization conditions $\frac{\partial P^*}{\partial R}|_{R=R_{\rm opt}} = 0$ and $\frac{\partial P^*}{\partial \Gamma}|_{\Gamma=\Gamma_{\rm opt}} = 0$ lead to the solution at $R_{\rm opt} \simeq 2.9525$ and $\Gamma_{\rm opt} \simeq 3.7532$, yielding the universal value of the EPMP,

$$\eta_{\rm opt} \simeq 0.5697. \tag{22}$$

However, the values of $\Gamma_{\rm opt}$, $R_{\rm opt}$, and $\eta_{\rm opt}$ are determined at the possible maximum power output. Physically, contrary to the continuous variation of R (the ratio of the maximum potential width L_C to minimum potential width L_A), the ratio Γ (the ratio of the energy-level number M to the particle number N) for a small particle-number N must be discrete since both N and M are positive integers. As a consequence, the maximum dimensionless power output $P^*_{\rm max}$ and the corresponding efficiency η_m change discretely, as shown in both Table I and Fig. 4. For a model with a fixed particle-number N, the optimal energy-level number M can be determined, based on the principle that the ratio Γ (this value is nothing but the optimal value of Γ_m in physical meaning) should be the one closest to the value of $\Gamma_{\rm opt}$. As

TABLE I. The optimal values of M, Γ_m, R_m, and η_m for different values of N.

N	M	Γ_m	R	η_m
2	4	5.0000	3.4360	0.5765
3	5	3.5714	2.8898	0.5723
5	8	3.4546	2.8586	0.5773
10	16	3.6494	2.9103	0.5691
20	32	3.7596	2.9390	0.5647
50	79	3.7024	2.9242	0.5670
80	126	3.6878	2.9204	0.5676
100	158	3.7256	2.9302	0.5661

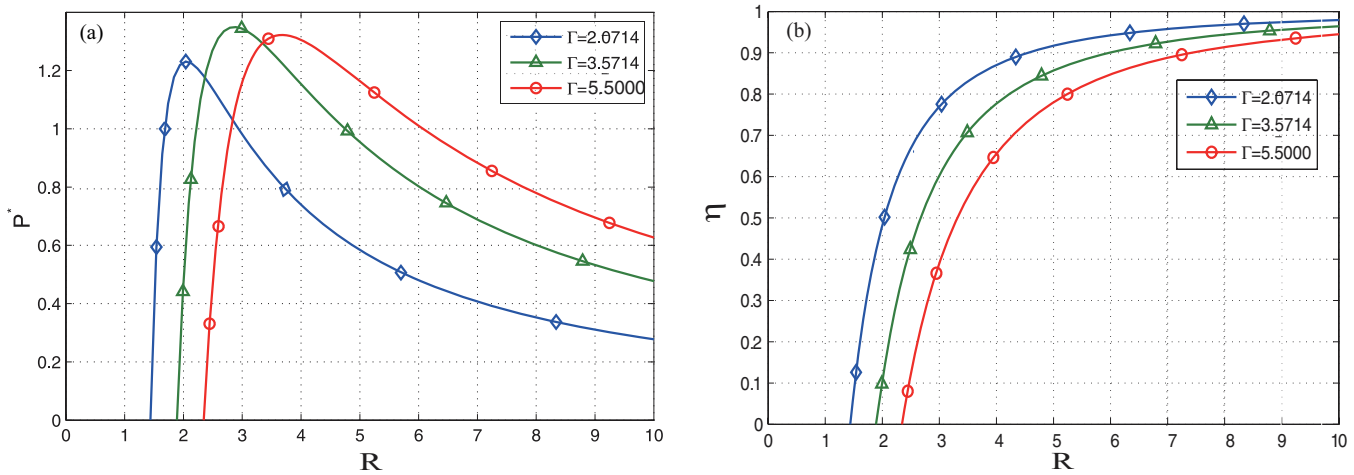

FIG. 4. (Color online) For different values of Γ at given particle number $N = 3$, dimensionless power output P^* (a) and efficiency η (b) as functions of R, respectively. The values of Γ are adopted 2.0714 with $M = 4$, 3.5714 with $M = 5$, and 5.5000 with $M = 6$, respectively.

an example, for an engine model with $N = 3$ particles, the optimal energy-level number is given by $M = 5$, since the ratio $\Gamma_m \equiv 3.5714$ is closest to $\Gamma_{\rm opt} = 3.7532$. It can be seen from Table I that, when N as well as M increases, the physical values of R_m, Γ_m, and η_m approach the corresponding the optimal values at the possible maximum power output, $R_{\rm opt}$, $\Gamma_{\rm opt}$, and $\eta_{\rm opt}$, respectively, implying that the EPMP given in Eq. (22) would be reached when N becomes large enough (see more details in the Appendix).

Figure 4 displays the dimensionless power output (a) and efficiency (b) as functions of the ratio R for different values of Γ. It is shown that the efficiency η increases as Γ (R) decreases (increases), while there exists a maximum value of dimensionless power output P^* as a function of R (Γ) for given Γ (R). Combing Eqs. (25) and (20), we plot the performance

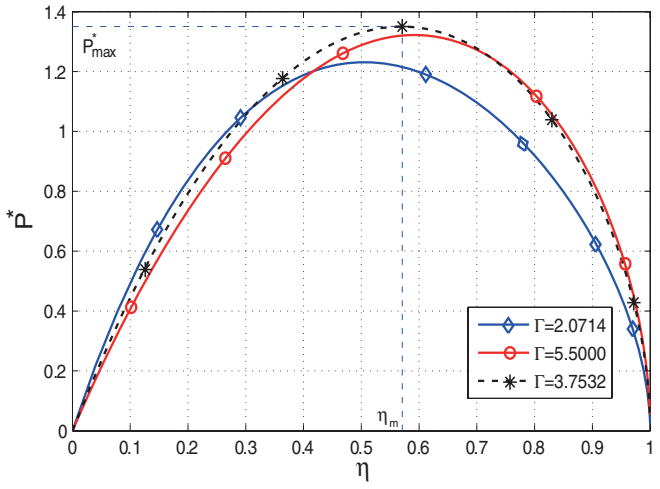

FIG. 5. (Color online) Dimensionless power output P^* versus efficiency η in the case of $N = 3$. The values of dimensionless power output, with $\Gamma = 2.0714$ ($M = 4$), and with $\Gamma = 5.5000$($M = 6$), are represented by a red and a blue solid line with circles and squares, respectively. The dimensionless power output for $\Gamma_{\rm opt} = 3.7532$ (which is determined under the possible maximum power) is denoted by a black dashed line with asterisks.

characteristic curve of $P^* - \eta$, as shown in Fig. 5. Figure 5 displays that the curve of the dimensionless power output versus the efficiency is parabola-like. It can be seen from Fig. 5 that, when $P^* < P^*_{\rm max}$, there are two different efficiencies for given power output, where one is smaller than η_m and the other is lager than η_m. When $\eta < \eta_m$, the dimensionless power output P^* increases as the efficiency decreases, such that the efficiencies, larger than η_m, are the optimal values for the heat engine. That is, the optimal region of the efficiency is given by

$$\eta_m \leqslant \eta < 1. \tag{23}$$

The value of η_m is the allowable value of the lower bound of the optimal efficiency. The ratio R determining the structure of the engine model should satisfy the relation $R \geqslant R_m$, where R_m is determined from Eq. (21) and thus depends on the value of Γ. In constructing the structure of the engine cycle, the condition that $L_C = R_m L_A$ must be fulfilled in order for the engine to operate in the optimal region given in Eq. (23).

V. GENERAL CASE

In order to consider the general case, we will consider 1D power-law potentials whose energy spectrum are parameterized by the form of [33–35]

$$\varepsilon_n = \hbar\omega n^\sigma, \tag{24}$$

where ω is the trap frequency, n is a positive integer quantum number, and σ is the index of the single-particle energy spectrum. The frequency ω can be expressed in terms of the trap width L as $\omega = \lambda L^{-\theta}$, where λ is a constant for a given potential and θ is a trap exponent [36] in dependence of the trapping potential. There are several special cases as an prominent example: (i) $\sigma = \theta = 2$ for a box and a harmonic potential, $\lambda = \frac{\pi^2\hbar^2}{2m}$ for a box potential, while $\lambda = \frac{\hbar^2}{m}$ for a harmonic potential [21], with m being the particle mass. (ii) $\sigma = \theta = 1$ and $\lambda = \pi\hbar c$ for extremely relative particles in a box potential [37]. (iii) $\sigma = \frac{4}{3}$, $\theta = 2$, and $\lambda = \frac{\hbar^2}{m}$ for a quartic potential [33].

The power output $P = W/\tau_0$, with τ_0 defined in Eq. (17), is given by

$$P = \frac{\lambda\hbar\bar{v}G_1}{2}\frac{1}{L_A^{\theta+1}}\frac{R^\theta - \Gamma}{R^{\theta+1} - R}\ln\Gamma, \tag{25}$$

where $G_1 \equiv \sum_{i=1}^{N} i^\sigma$, $G_2 = \sum_{i=M-N+1}^{M} i^\sigma$, and $\Gamma = G_2/G_1$, yielding the positive work condition

$$R > \Gamma^{1/\theta}. \tag{26}$$

The efficiency of the generalized engine model is then given by

$$\eta = 1 - \frac{\Gamma}{R^\theta}. \tag{27}$$

Defining the dimensionless power output $P^* = P/(\frac{G_1\lambda\hbar}{2L_A^{\sigma+1}})$ and using the extremal condition of $(\frac{\partial P^*}{\partial R})_{R=R_m} = 0$, we have

$$R_m^{\theta+1} + \Gamma[-R_m(\theta+1) + \theta] = 0, \tag{28}$$

in a similar manner adopted in Sec. III. For a box trap, Eq. (28) becomes identical to Eq. (21). Under the positive work condition that was given by Eq. (26), we can obtain the optimal value $R = R_m$ and thus determine corresponding efficiency η_m of the model working with a given potential. For simplicity, through our discussions the working system has been assumed to be in a state that the fermions occupy the consecutive energy levels. On the other hand, the EPMP, in which the optimal values of both Γ and R are only determined by Eq. (28), is rarely dependent of the choice of the energy levels. Thus, the EMP, as the one closest to the EPMP, will be rarely influenced by our choice of the energy levels for given particle number N.

VI. RELATIONSHIP BETWEEN EFFICIENCY OF THE QUANTUM ENGINE CYCLE AND THAT OF THE CORRESPONDING QUANTUM CARNOT CYCLE

In this section we discuss the relationship between the efficiency of the QHE cycle mentioned above and that of the Carnot cycle consisting of two quantum adiabatic and two isothermal processes. In a quantum isothermal process, the N-particle Fermi system as the working substance is coupled with heat baths at constant temperatures, T_h and T_c, respectively, whereas occupation probabilities are kept constant during any adiabatic process, as shown in Fig. 2. (The N-particle system is coupled to heat baths at constant temperatures $T = T_h$ and $T = T_c$, instead of $E = E_h$ and $E = E_c$, in the processes $A \to B$ and $C \to D$, respectively.)

We first consider an M-level system of a single particle in a 1D power-law trap. For a three-level system with a single particle, $\xi_n^{(1)} = |a_n|^2$ $(n = 1,2,\ldots,M)$. These occupation probabilities should satisfy the Boltzmann distribution $\xi_{n+1}^{(1)} = \xi_n^{(1)} e^{-\Delta_n(L)/k_BT}$ with $\Delta_n(L) \equiv [(n+1)^\sigma - 1]\lambda/L^\theta$ and $n = 1,2,\ldots,M-1$, where λ was defined below Eq. (24). According to the condition that $\sum_{n=1}^{M}\xi_n^{(1)} = 1$, the occupation probability $\xi_1^{(1)}$ of the ground state is written in terms of trap size L and temperature T:

$$\xi_1^{(1)} = \frac{1}{\sum_{n=1}^{M} e^{-(n^\sigma-1)\lambda/(k_BTL^\theta)}}, \tag{29}$$

which shows that the occupation probability depends on the actual number of energy levels involved. The canonical partition function Z_N for an ideal N-particle Fermi system can be determined by the exact recurrence relation [38–40] $Z_N = \frac{1}{N}\sum_{k=1}^{N}(-1)^{k+1} z_k Z_{N-k}$, where $z_k = \sum_{n=1}^{\infty}\exp[-k\lambda n^\sigma/(k_BTL^\theta)]$ and $Z_0 = 1$. Using the above recurrence relation, one can readily obtain the following recursive scheme for the occupation probabilities [39]:

$$\xi_n^{(N)} = \exp[-\lambda n^\sigma/(k_BTL^\theta)]\frac{Z_{N-1}}{Z_N}\left[-\xi_n^{(N-1)} + 1\right], \tag{30}$$

where $\xi_n^{(N)}$ and $\xi_n^{(N-1)}$ represent the occupation probabilities of the nth state for system with N and $N-1$ particles, respectively. The expressions of occupation probabilities for an ideal Fermi system with an arbitrary number of particles [41] can be found using the recursive scheme.

Let $S(J)$ be the entropies of the working substance at different instants $J = A, B, C$, and D, the heat exchanges Q_h and Q_c are $Q_h = T_h[S(B) - S(A)]$ and $Q_c = T_c|S(D) - S(C)|$. According to the first law of thermodynamics, the work W per cycle can be calculated as

$$W = Q_h - Q_c = (T_h - T_c)[S(B) - S(A)]. \tag{31}$$

In obtaining Eq. (31) the relations $S(C) = S(B)$ and $S(D) = S(A)$ for two adiabats have been used. The efficiency of the quantum Carnot engine, $\eta_c = \frac{W}{Q_h}$, is thus given by

$$\eta_C = 1 - \frac{T_c}{T_h}. \tag{32}$$

Because the occupations probabilities $\xi_n^{(N)}$ with $n = 1,2,\ldots,M$ in Eq. (30) are fixed in the adiabatic process, there exists a relation $\frac{T_c}{T_h} = (\frac{L_B}{L_C})^\theta$.

In the branch $A \to B$ with fixed initial potential width L_A, the total energy of the M-level system of N noninteracting fermions can be expressed as $E(L) = \lambda\sum_{n=1}^{M} L^{-\theta}\xi_n^{(N)}$. The maximum value $L_B = \sqrt{\frac{G_2}{G_1}}L_A$ of potential width, where $G_1 = \sum_{n=1}^{N} n^\sigma$ and $G_2 = \sum_{n=M-N+1}^{M} n^\sigma$ at instant B is achieved, only under the conditions that $\xi_n^{(N)} = 1$ with $n = 1,2,\ldots,N$ at instant A and $n = M-N+1, M-N+2,\ldots,M-1, M$ at instant B. It is clear that this condition can be completely satisfied in the isoenergetic process described above. However, it is impossible to achieve a state in which N fermions occupy either the highest or lowest N energy levels in the presence of a heat bath. We find from Eq. (30) that, for the quantum isothermal process $A \to B$ in which $E(L_B) \leqslant E(L_A)$, this condition cannot be fulfilled and thus $L_B < \sqrt{\frac{G_2}{G_1}}L_A$. As a consequence, we have the inequality

$$\eta = 1 - \frac{E_c}{E_h} < 1 - \frac{T_c}{T_h}. \tag{33}$$

That is, under the same conditions, the efficiency of our engine model is bounded from above the Carnot value. Note that the inequality Eq. (33) is independent of the form of the potential, the particle number, as well as the energy-level number.

VII. SUMMARY AND DISCUSSIONS

As a summary, we have extended the previous models in Refs. [2,21,23] by considering a QHE that consists of an arbitrary number of fermions in a trap. Considering the finite-time processes, we explicitly derive the expressions of power output and efficiency and demonstrate that the power output can be enhanced, but without decrease in efficiency, either by increasing particle number as well as energy levels involved in the engine model or by decreasing minimum value of the potential width. We find that the EPMP is universal and independent of any parameters, including the maximum and minimum values of potential width, the particle number, and energy-level number. The optimal region as well as positive work condition of the engine cycle has been found, which can provide a new theoretical basis for the performance evaluation and improvement of the quantum heat engine. We discuss the generalized engine cycle model, which proceeds with an ideal Fermi gas with an arbitrary number of particles trapped in an arbitrary power-law potential. This generalized model is a microscopic analogue of the classical Carnot engine and its efficiency is proved to be bounded from above the Carnot value.

A natural extension of our work would be to take into account other Hamiltonians [36,42]. For example, if the working substance consists of N bosons in a trap, the case becomes more complex [43]. We also expect that the performance characteristics of other models exhibiting interaction between particles [24,36,42] can be studied in the future.

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APPENDIX: THE PROBABILITY OF FINDING A PHYSICAL VALUE OF Γ CLOSE TO THE BEST VALUE $\Gamma_{\rm opt} = 3.7532$

Now we turn to discussion on the probability of finding a physical value much closer to the optimal value of $\Gamma_{\rm opt}$ (as as well as the EPMP $\eta_{\rm opt}$). Since $\Gamma = \frac{G_2}{G_1}$, where G_2 and G_1 were defined in the text, the ratio $\Gamma = \frac{2N^2+6M(M-N+1)-3N+1}{(2N+1)(N+1)}$, leading to

$$M_{\rm opt} = \tfrac{1}{2}(N-1)+\tfrac{1}{6}\sqrt{3(N+1)[(4\Gamma_{\rm opt}-1)N+2\Gamma_{\rm opt}+1]}. \tag{A1}$$

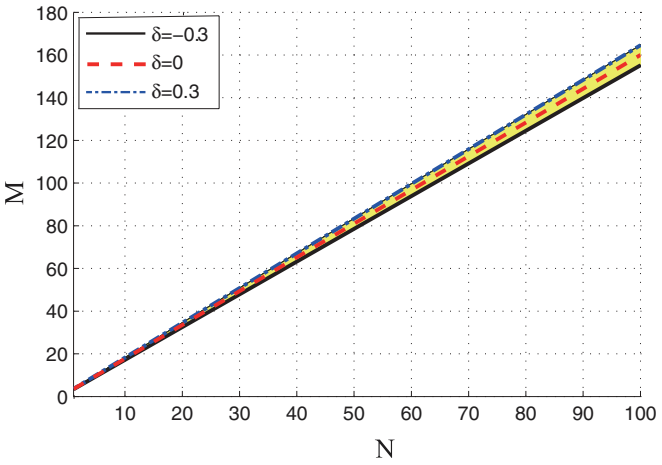

FIG. 6. (Color online) M as a function of N. The upper $[M = M(\Gamma_{\rm opt}+\delta,N)]$ and lower $[M = M(\Gamma_{\rm opt}-\delta,N)]$ bounds of M are denoted by a black solid line and a blue dash-dotted line, respectively. The optimal M $[M = M(\Gamma_{\rm opt},N)]$ at possible maximum power output is the red dashed line. The probability of finding a physical value of Γ_m approaching the optimal $\Gamma_{\rm opt}$ is proportional to the blue region between the lower and upper bounds.

Assuming $\Gamma_m = \Gamma_{\rm opt} \pm \delta$, where $\delta << 1$, both Γ_m and $\Gamma_{\rm opt}$ were defined in Sec. IV, M can be written as

$$M_m = \tfrac{1}{2}(N-1) + \tfrac{1}{6}\sqrt{3(N+1)[(4\Gamma_{\rm opt}\pm 4\delta-1)N+2\Gamma_{\rm opt}\pm 2\delta+1]}. \tag{A2}$$

Let $\varrho \equiv |M_m - M_{\rm opt}|$, it is not difficult to verify that

$$\varrho = \frac{2\delta+\delta/N}{4\Gamma-1+(2\Gamma+1)/N}\sqrt{(N+1)[N(4\Gamma-1)+2\Gamma+1]}. \tag{A3}$$

In obtaining Eq. (A3), we have used the approximate expansion $\sqrt{1\pm\frac{4\delta+2\delta/N}{4\Gamma-1+(2\Gamma+1)/N}} = 1\pm\frac{1}{2}\frac{4\delta+2\delta/N}{4\Gamma-1+(2\Gamma+1)/N}+\ldots$, provided that $\delta << 1$. We find the leading order of the expansion of Eq. (A3) is satisfied with a form as $\varrho \propto N$, when N tends to be large enough. Since the probability of finding a physical value approaching the optimal $\Gamma_{\rm opt}$ is proportional to the value of ϱ, it increases as N increases. As an example, in Fig. 6 we plot the region of $\varrho' = M(N,\Gamma_{\rm opt}+\delta) - M(N,\Gamma_{\rm opt}-\delta)$ in the (N,M) plane, adopting the values of $\delta = 0.3000$ and $\Gamma_{\rm opt} = 3.7532$. It can seen from Fig. 6 that, when the particle number N increases, the probability of finding a physical value that close to the optimal $\Gamma_{\rm opt}$ increases, implying that the optimal value of $\Gamma_{\rm opt}$ as well as $\eta_{\rm opt}$ would be reached when N tends to be large enough.

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