Scaling behavior for a class of quantum phase transitions

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We show that for quantum phase transitions with a single bosonic zero mode at the critical point, like the Dicke model and the Lipkin-Meshkov-Glick model, metric quantities such as fidelity, that is, the overlap between two ground states corresponding to two values λ_1 and λ_2 of the controlling parameter λ, depend only on the ratio $\eta = (\lambda_1 - \lambda_c)/(\lambda_2 - \lambda_c)$, where $\lambda = \lambda_c$ at the critical point. This scaling property is valid also for time-dependent quantities such as the Loschmidt echo, provided time is measured in units of the inverse frequency of the critical mode.

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I. INTRODUCTION

Due to its quantum nature, a quantum phase transition (QPT) [1], defined as a drastic change of fundamental properties of ground states, may have properties quite different from those of thermal phase transitions. This point has received much attention in recent years. In particular, some concepts and quantities in the field of quantum information, e.g., entanglement and fidelity, have been found quite useful in characterizing the occurrence of a QPT. For example, the overlap between two ground states corresponding to two nearby values λ_1 and λ_2 of the controlling parameter λ,

$$L_p(\lambda_1,\lambda_2) = \left|\langle 0_{\lambda_1}|0_{\lambda_2}\rangle\right|, \tag{1}$$

has been proposed as a probe of quantum criticality [2]: the dramatic change of the wave function at a QPT implies a decrease of the overlap L_p in the neighborhood of the critical point. Hence, the fidelity L_p can be used to detect the occurrence of a QPT (see Ref. [2–10] and references therein).

Similarly to thermal phase transitions, an important aspect of QPTs is the dependence of relevant quantities on the controlling parameter λ. For example, the characteristic energy scale usually takes the form $|\lambda - \lambda_c|^{\varphi}$, with λ_c indicating the critical point and $\varphi > 0$ a critical exponent. This is for the case in which only one value of the parameter λ is of relevance. For quantities like fidelity, which instead depend on two values λ_1 and λ_2 of the controlling parameter, one should understand whether their behavior in the critical region encodes universal properties about a QPT. With regard to the fidelity, the question is whether its drop near a QPT can be used not only to detect the QPT itself but also to determine the critical exponents [10]. In this context, understanding scaling properties is relevant.

When studying fidelity, either as the overlap (1) of ground states or as the survival probability of an initial state prepared in the ground state $|0_{\lambda_2}\rangle$ of Hamiltonian $\hat{H}(\lambda_2)$ and evolved under a different Hamiltonian $\hat{H}(\lambda_1)$, two values λ_1 and λ_2 of the controlling parameter are involved. For the sake of clarity, in what follows, we use the name fidelity for the former quantity, $L_p(\lambda_1,\lambda_2)$, and use Loschmidt echo (LE) for the latter [11]. An interesting question is about the dependence of the fidelity and LE on λ_1 and λ_2. In the case that the fidelity $L_p(\lambda_1,\lambda_2)$ goes to zero when λ_1 approaches λ_c for a fixed λ_2, it is clear that for each λ_2' there exists a value λ_1' such that $L_p(\lambda_1,\lambda_2) = L_p(\lambda_1',\lambda_2')$. This implies that the relative positions of λ_1 and λ_2 with respect to λ_c, rather than their exact positions, play the crucial role. Hence, the question arises of whether the fidelity may be invariant under rescaling of the controlling parameter.

In this paper, we show that for QPTs possessing only one bosonic zero mode at the critical point, metric quantities like fidelity depend only on the ratio $\eta = (\lambda_1 - \lambda_c)/(\lambda_2 - \lambda_c)$. That is, these physical quantities are invariant under linear rescaling of the controlling parameter with respect to the critical point. We also show that this scaling property is valid for time-dependent quantities such as the LE, provided time is measured in units of the inverse frequency of the critical mode. The class of QPTs possessing such features includes important physical models, like the Dicke [12] and the Lipkin-Meshkov-Glick (LMG) models [13].

The article is organized as follows. In Sec. II we discuss our scaling argument for static metric quantities like the fidelity, extending in Sec. III such scaling to time-dependent quantities like the LE. The scaling for time-dependent quantities is then illustrated by means of the semiclassical theory. We then illustrate the fidelity and LE scaling in two relevant physical models, the Dicke model (Sec. IV) and the LMG model (Sec. V). We finish with concluding remarks in Sec. VI.

II. SCALING ARGUMENT

As only the lowest energy levels are concerned close to the critical point, we assume that the Hamiltonian describing a QPT can be approximately written in terms of n harmonic oscillators:

$$\hat{H}(\lambda) = \sum_{i=1}^{n} e_i(\lambda)\hat{c}_i^{\dagger}(\lambda)\hat{c}_i(\lambda), \tag{2}$$

where $\hat{c}_i^{\dagger}(\lambda)$ and $\hat{c}_i(\lambda)$ are *bosonic* creation and annihilation operators for the ith mode. The ground state $|0_{\lambda}\rangle$ for the parameter λ is defined by $\hat{c}_i(\lambda)|0_{\lambda}\rangle = 0$. For two parameter

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values λ_1 and λ_2, one may write

$$\hat{c}_i^\dagger(\lambda_1) = \sum_{j=1}^{n}[P_{ij}\hat{c}_j^\dagger(\lambda_2) + Q_{ij}\hat{c}_j(\lambda_2)], \tag{3}$$

where P_{ij} and Q_{ij} are functions of λ_1 and λ_2, with $P_{ij} = \delta_{ij}$ and $Q_{ij} = 0$ for $\lambda_1 = \lambda_2$. We discuss the case in which there is only one zero mode at the critical point: $e_1(\lambda) \sim |\lambda - \lambda_c|^\varphi$ for λ close to λ_c, while $e_i(\lambda_c) \neq 0$ for $i \neq 1$. In this case, Eq. (3) reduces to

$$\hat{c}_1^\dagger(\lambda_1) = P_{11}\hat{c}_1^\dagger(\lambda_2) + Q_{11}\hat{c}_1(\lambda_2), \tag{4}$$

with a corresponding expression for $\hat{c}_1(\lambda_1)$. From the bosonic commutation relations it follows that $|P_{11}|^2 - |Q_{11}|^2 = 1$. Let us write explicitly the phases of P_{11} and Q_{11}, as $P_{11} = |P_{11}|e^{i(\theta_c+\theta_r)}$ and $Q_{11} = |Q_{11}|e^{i(\theta_c-\theta_r)}$. In the representation of $\hat{H}(\lambda_2)$, the change of the pair $(\hat{c}_1(\lambda_2),\hat{c}_1^\dagger(\lambda_2))$ to $(e^{-i\theta_r}\hat{c}_1(\lambda_2),e^{i\theta_r}\hat{c}_1^\dagger(\lambda_2))$ does not bring any change to the physics; hence, the phase θ_r can be absorbed by $\hat{c}_1(\lambda_2)$ and $\hat{c}_1^\dagger(\lambda_2)$. The phase θ_c is the relative phase between the pair of operators $(\hat{c}_1(\lambda_2),\hat{c}_1^\dagger(\lambda_2))$ at λ_2 and the pair $(\hat{c}_1(\lambda_1),\hat{c}_1^\dagger(\lambda_1))$ at λ_1, which generates relative phases between the set of basis states $|n_{\lambda_1}\rangle = [\hat{c}_1^\dagger(\lambda_1)]^n|0_{\lambda_1}\rangle|/\sqrt{n!}$ and $|m_{\lambda_2}\rangle = [\hat{c}_1^\dagger(\lambda_2)]^m|0_{\lambda_2}\rangle|/\sqrt{m!}$.

Let us consider a physical quantity A depending on two values λ_1 and λ_2 of the controlling parameter (for instance, A might be the fidelity), written in the vicinity of the critical point as a function of the annihilation operators for the zero mode:

$$A = \langle 0_{\lambda_i}|\hat{A}(\hat{c}_1(\lambda_1),\hat{c}_1(\lambda_2))|0_{\lambda_j}\rangle \quad (i,j = 1,2). \tag{5}$$

In what follows, we focus on quantities A that do not depend on the phase θ_c [14]. In particular, *metric quantities*, like $C_{nm} \equiv |\langle n_{\lambda_1}|m_{\lambda_2}\rangle|$, belong to this class. Such quantities include, for instance, the fidelity $L_p = C_{00}$ and the participation ratio χ of an eigenstate of, e.g., $\hat{H}(\lambda_2)$, $|m_{\lambda_2}\rangle = \sum_n \langle n_{\lambda_1}|m_{\lambda_2}\rangle|n_{\lambda_1}\rangle$, with respect to the basis of the eigenstates of $\hat{H}(\lambda_1)$; by definition, $\chi = 1/\sum_n |\langle n_{\lambda_1}|m_{\lambda_2}\rangle|^4 = 1/\sum_n C_{mn}^4$. In the study of these quantities, we can take $\theta_c = 0$. Then, since the phase θ_r can be absorbed by $\hat{c}_1(\lambda_2)$ and $\hat{c}_1^\dagger(\lambda_2)$, P_{11} and Q_{11} are just their absolute values, with $P_{11} \geqslant 1$. Using the ground state definition $\hat{c}_1(\lambda_1)|0_{\lambda_1}\rangle = 0$, Eq. (4), and the expansion $|0_{\lambda_1}\rangle = \sum_m \langle m_{\lambda_2}|0_{\lambda_1}\rangle|m_{\lambda_2}\rangle$, we can express $|0_{\lambda_1}\rangle$ as a function of P_{11}, Q_{11}, $\hat{c}_1^\dagger(\lambda_2)$, and $|0_{\lambda_2}\rangle$. After inserting the obtained expression for $|0_{\lambda_1}\rangle$ into Eq. (5), we find that A is a function of P_{11} and Q_{11} only.

The dependence of P_{11} on λ_1 and λ_2 can be written as $P_{11} = F(\Delta\lambda_1,\Delta\lambda_2)$, where $\Delta\lambda_i = \lambda_i - \lambda_c$ $(i = 1,2)$. We study λ_1 and λ_2 belonging to the same phase, so that $\eta = \Delta\lambda_1/\Delta\lambda_2 > 0$. We assume, as is natural for a QPT with an infinitely degenerate zero mode at the critical point, that $F(\Delta\lambda_1,\Delta\lambda_2)$ goes to infinity in the limit $\Delta\lambda_1 \to 0$ with $\Delta\lambda_2 \neq 0$, as well as in the limit $\Delta\lambda_2 \to 0$ with $\Delta\lambda_1 \neq 0$. For the sake of simplicity, we assume that F is a monotonic function of $\Delta\lambda_2$, when $\Delta\lambda_2$ changes from a given $\Delta\lambda_1$ to 0 [15]. Then, given $F(\Delta\lambda_1,\Delta\lambda_2) = d$, for any λ_1' there must exist a λ_2' such that $F(\Delta\lambda_1',\Delta\lambda_2') = d$. This implies that there exists a function $\Delta\lambda_2 = g(\Delta\lambda_1,d)$, such that $F(\Delta\lambda_1,g(\Delta\lambda_1,d)) = d$ for any $\Delta\lambda_1$; hence,

$$\partial F/\partial\Delta\lambda_1 + (\partial F/\partial\Delta\lambda_2)(\partial g/\partial\Delta\lambda_1) = 0. \tag{6}$$

For a given d and a sufficiently small $\Delta\lambda_1$, the Taylor expansion reads $g = g_0 + g'\Delta\lambda_1$, where $g' = \partial g/\partial\Delta\lambda_1$ with d fixed. We recall that in the limit $\Delta\lambda_1 \to 0$, $F(\Delta\lambda_1,\Delta\lambda_2)$ goes to infinity if $\Delta\lambda_2$ is nonzero. Hence, for any given d, $\Delta\lambda_2$ must go to zero in order that $\lim_{\Delta\lambda_1\to 0} F(\Delta\lambda_1,\Delta\lambda_2) = d$. That is, $\lim_{\Delta\lambda_1\to 0} g(\Delta\lambda_1,d) = 0$; as a result, $g_0 = 0$. Therefore, when the above Taylor expansion works for all fixed values of d, we have $\partial g/\partial\Delta\lambda_1 = g/\Delta\lambda_1 = \Delta\lambda_2/\Delta\lambda_1$. Substituting this result into Eq. (6), we find

$$\Delta\lambda_1\,\partial F/\partial\Delta\lambda_1 + \Delta\lambda_2\,\partial F/\partial\Delta\lambda_2 = 0. \tag{7}$$

This equation has the solution $F = F(\ln\Delta\lambda_1 - \ln\Delta\lambda_2) = F(\ln\eta)$. Hence, P_{11} is a function of η. Then, due to the relation $P_{11}^2 - Q_{11}^2 = 1$, Q_{11} is also a function of η. Since the quantity A is a function of P_{11} and Q_{11}, we can conclude that A depends only on the ratio $\eta = (\lambda_1 - \lambda_c)/(\lambda_2 - \lambda_c)$

III. TIME-DEPENDENT QUANTITIES

We now consider time-dependent metric quantities $A(t)$, with the dynamics described by the Hamiltonian (2). Since the frequencies $\omega_i(\lambda) = e_i(\lambda)/\hbar$ depend on λ, $A(t)$ usually cannot be a function of η only. However, for systems with a single zero mode at the critical point, the η scaling still applies, provided time is rescaled: $t \to \tau \equiv \omega_1(\lambda)t$.

As an illustration we show in the following that in the vicinity of a QPT with a single zero mode at the critical point the decay of the quantum Loschmidt echo depends only on the scaling parameter η and the rescaled time τ. The LE gives a measure for the stability of the quantum motion under slight variation of the Hamiltonian [16–18]. It is defined by $M_L(t) = |m(t)|^2$, where

$$m(t) = \langle\Psi_0|\exp[i\hat{H}(\lambda_2)t/\hbar]\exp[-i\hat{H}(\lambda_1)t/\hbar]|\Psi_0\rangle. \tag{8}$$

Here, $\hat{H}(\lambda_1) = \hat{H}(\lambda_2) + \epsilon\hat{V}$, with $\epsilon = \lambda_1 - \lambda_2$. Extensive investigations have been performed in recent years to understand the decaying behavior of the LE in different regimes, depending on the chaotic or integrable nature of the dynamics, on the system's dimensionality, and on the perturbation strength (see Refs. [19–34] and references therein). Furthermore, recent investigations have shown that the LE may be employed to characterize QPTs, since it exhibits extrafast decay in the vicinity of critical points [34–39].

Here we consider a system initially prepared in the ground state $|0_{\lambda_2}\rangle$ of $\hat{H}(\lambda_2)$. Then the LE is in fact the survival probability,

$$M_L(t) = \left|\langle 0_{\lambda_2}|e^{-i\hat{H}(\lambda_1)t/\hbar}|0_{\lambda_2}\rangle\right|^2. \tag{9}$$

In the critical region, $\hat{H}(\lambda_2)$ represents a harmonic oscillator and, therefore, its ground state can be written as a Gaussian wave packet. As shown in Ref. [31], when the classical motion is periodic with a period T_p, semiclassical theory predicts that for $t > T_p$ the LE has an initial Gaussian decay followed by a power law decay. Indeed, to a second-order term of the perturbation expansion,

$$M_L(t) \simeq b_0(1 + \xi^2t^2)^{-1/2}e^{-\Gamma t^2/(1+\xi^2t^2)}, \tag{10}$$

where $b_0 \sim 1$, $\Gamma = (\frac{\epsilon W}{\hbar}\frac{\partial U}{\partial p_0})^2/2$, and $\xi = |\frac{\epsilon W^2}{2\hbar}\frac{\partial^2 U}{\partial p_0^2}|$, with the derivatives evaluated at the center p_0 of the initial Gaussian wave packet. Here, $U = \frac{1}{T_p}\int_0^{T_p} V\,dt$ and W is a measure of the width of the initial Gaussian packet in the momentum space. It is seen that M_L has a Gaussian decay $e^{-\Gamma t^2}$ for short times and a $1/\xi t$ decay for long times.

Let us consider the case in which $\hat{H}$ has, close to the critical point, two lowest-energy relevant modes, of which only the first one has zero frequency at the critical point. The first mode moves very slowly, so that effectively the system moves periodically with the frequency of the second mode. That is, for times much shorter than the period T_1 of the first mode (which diverges at the critical point), the classical motion is approximately periodic with the period of the second mode. Hence, the period $T_p = 2\pi/\omega_2(\lambda_1)$. Note that for long times the first mode dominates and the LE oscillates with a period related to $T_1 = 2\pi/\omega_1(\lambda_1)$ (detailed later). Therefore, the above semiclassical prediction works within times longer than T_p and shorter than T_1.

Starting from the classical expression of the Hamiltonian,

$$H(\lambda) = \omega_1(\lambda)I_1(\lambda) + \omega_2(\lambda)I_2(\lambda), \tag{11}$$

we obtain

$$\xi t = \frac{\xi}{\omega_1(\lambda_1)}\tau \sim \frac{\partial^2}{\partial p_0^2}\left\langle\frac{H(\lambda_1) - H(\lambda_2)}{e_1(\lambda_1)}\right\rangle\tau \tag{12}$$

$$\simeq \frac{\partial^2}{\partial p_0^2}\left\langle\frac{I_1(\lambda_1)}{\hbar} - \frac{e_1(\lambda_2)}{e_1(\lambda_1)}\frac{I_1(\lambda_2)}{\hbar} + \frac{\Delta H_2}{e_1(\lambda_1)}\right\rangle\tau, \tag{13}$$

where $\Delta H_2 = I_2(\lambda_1)\omega_2(\lambda_1) - I_2(\lambda_2)\omega_2(\lambda_2)$. The second mode has no singularity at λ_c; hence from Taylor expansion of ΔH_2 we obtain $\Delta H_2 \sim (\lambda_1 - \lambda_2)$. Thus, $\frac{\Delta H_2}{e_1(\lambda_1)} \sim |1-\eta^{-1}|^{\varphi}|\lambda_1 - \lambda_2|^{1-\varphi}$. Therefore, when λ_1 is sufficiently close to λ_2, $\frac{\Delta H_2}{e_1(\lambda_1)}$ can be neglected for $\varphi < 1$ and $\eta \neq 0$. Then, since $I_1(\lambda_1)$ and $I_1(\lambda_2)$ have no singularity at λ_c and $\frac{e_1(\lambda_2)}{e_1(\lambda_1)} = \eta^{-\varphi}$, we find that in the very neighborhood of λ_c, $\xi t \simeq F(\eta)\tau$. Similarly, Γt^2 can be written as $G(\eta)\tau^2$. We can therefore conclude that

$$M_L(t) \simeq b_0\{1 + [F(\eta)\tau]^2\}^{-1/2} e^{-G(\eta)\tau^2/\{1+[F(\eta)\tau]^2\}} \tag{14}$$

is a function of η and of the rescaled time τ.

IV. SCALING FOR THE DICKE MODEL

This model [12] provides a physically significant example of our scaling behavior. It describes the interaction between a single bosonic mode and a collection of N two-level atoms and finds applications in quantum optics, condensed matter physics, and quantum information. In terms of the collective operator $\hat{\mathbf{J}}$ for the N atoms, the Dicke Hamiltonian is written as (hereafter we take $\hbar = 1$)

$$\hat{H}(\lambda) = \omega_0\hat{J}_z + \omega\hat{a}^\dagger\hat{a} + (\lambda/\sqrt{N})(\hat{a}^\dagger + \hat{a})(\hat{J}_+ + \hat{J}_-). \tag{15}$$

In the thermodynamic limit $N \to \infty$, the system undergoes a QPT at $\lambda_c = \frac{1}{2}\sqrt{\omega\omega_0}$, with a normal phase for $\lambda < \lambda_c$ and a super-radiant phase for $\lambda > \lambda_c$. The Hamiltonian can be diagonalized in this limit [12], taking, up to a constant energy term, the form Eq. (2), with $n = 2$ modes. In the normal phase,

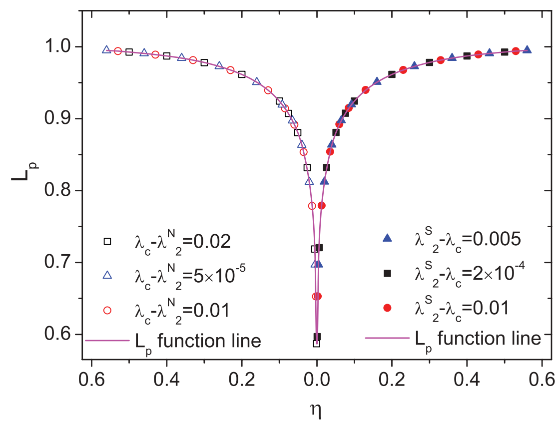

FIG. 1. (Color online) Dependence of the fidelity L_p on the scaling parameter η for different values of λ_1 and λ_2. Note that the values of η on both sides of the zero point are positive. Symbols on the left-hand side of $\eta = 0$ represent fidelity in the normal phase, with the superscript N of λ_2 standing for normal phase (open symbols). Symbols on the right-hand side of $\eta = 0$ are for the super-radiant phase (superscript S of λ_2, full symbols). Data are in agreement with the analytical result of Eq. (18) (solid curve).

the energies of the two harmonic oscillators read

$$e_{1,2}(\lambda) = \left\{\tfrac{1}{2}\left[\left(\omega^2 + \omega_0^2\right) \pm \sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 16\lambda^2\omega\omega_0}\right]\right\}^{1/2}, \tag{16}$$

ordered so that $e_1(\lambda) < e_2(\lambda)$. It is seen that $e_1(\lambda) = 0$ for $\lambda = \lambda_c$; hence, the ground level of $\hat{H}(\lambda_c)$ is infinitely degenerate and the system undergoes a QPT at λ_c. On the other hand, $e_2(\lambda) \neq 0$ at the critical point. In the super-radiant phase, the Dicke Hamiltonian can still be diagonalized in the thermodynamic limit, resulting in a two-mode form, with the energies

$$e_{1,2}(\lambda) = \left\{\frac{1}{2}\left[\omega^2 + \frac{\omega_0^2}{\mu^2} \pm \sqrt{\left(\frac{\omega_0^2}{\mu^2} - \omega^2\right)^2 + 4\omega^2\omega_0^2}\right]\right\}^{1/2}, \tag{17}$$

where $\mu \equiv \omega\omega_0/4\lambda^2$ and $e_1(\lambda) < e_2(\lambda)$. It is easy to see that $e_1(\lambda) = 0$ and $e_2(\lambda) \neq 0$ for $\lambda = \lambda_c$. Thus, the ground level of $\hat{H}(\lambda_c)$ is also infinitely degenerate (and with a single zero mode) at the critical point from the super-radiant-phase side.

As can be shown both analytically and numerically the fidelity in this model uniquely depends on the scaling parameter η. Indeed, in both phases, it has been found (see the Appendix) that

$$L_p = \frac{\sqrt{2}\sqrt[4]{\eta^{\varphi}}}{\sqrt{\eta^{\varphi} + 1}} = \frac{\sqrt{2}\sqrt[8]{\eta}}{\sqrt{\sqrt{\eta} + 1}}, \tag{18}$$

with the critical exponent $\varphi = 1/2$, $[\omega_1(\lambda) \sim |\lambda - \lambda_c|^{1/2}]$ [40]. For $\eta \ll 1$, $L_p \propto \eta^{\varphi/4}$. The analytical result (18) is in agreement with numerical simulations shown in Fig. 1: data for different values of λ_1 and λ_2 collapse on a single universal curve.

In the derivation of Eq. (18), we have used Eq. (16), which is obtained by diagonalizing an effective form of the exact Dicke Hamiltonian (15), given by Eq. (2), with $n = 2$ modes. It is therefore important to assess the validity of the effective Hamiltonian in computing the fidelity L_p. The effective

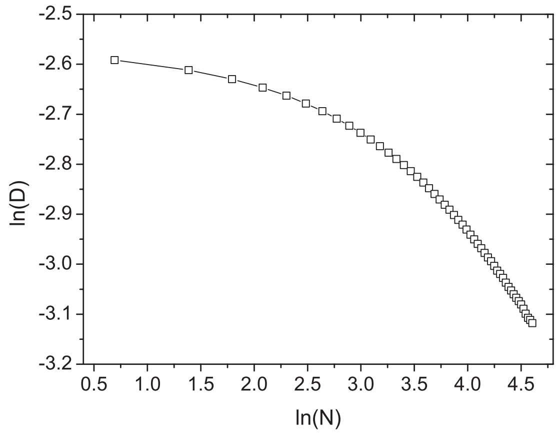

FIG. 2. The quantity $\ln D$ versus $\ln N$ in the Dicke model, with parameters $\omega_0 = \omega = 1$, $\lambda_1 = 0.495$, and $\lambda_2 = 0.45$ ($\lambda_c = 0.5$).

Hamiltonian leads to an error inversely proportional to N, where N is the number of atoms. For those quantities, for which there exist some contributions proportional to N, the effective Hamiltonian gives poor predictions [41–43]. However, for quantities like the fidelity L_p of ground states, there is no such contribution. Hence, the effective Hamiltonian is expected to correctly describe the behavior of L_p in the critical region. To substantiate this expectation, we have compared the prediction $L_p(\lambda_1,\lambda_2)$ of Eq. (18) with $L_p^N(\lambda_1,\lambda_2)$, which is the corresponding fidelity numerically computed by direct diagonalization of the exact Hamiltonian (15) in a truncated Hilbert space. The truncated Hilbert space is obtained for a finite number N of the atoms and by taking the lowest N levels of the bosonic mode. We have studied the variation of D with N, where

$$D = \left|L_p^N(\lambda_1,\lambda_2) - L_p(\lambda_1,\lambda_2)\right|. \tag{19}$$

With increasing N, the quantity D exhibits a decay faster than power law (see Fig. 2) and slower than exponential. Therefore, it is reasonable to expect that the effective Dicke Hamiltonian provides the correct physical picture when computing the fidelity of ground states in the large-N limit.

Next, we discuss the LE. For the Dicke model, it can be analytically proved that the LE is an oscillating function of time with period $T = \pi/\omega_1(\lambda_1) = T_1/2$. This period diverges when λ_1 approaches the critical point λ_c, and for times shorter than $T/2$ the LE decays according to the above semiclassical prediction. Indeed, numerical simulations in Fig. 3 show that the LE is a function of η and of the rescaled time $\tau = \omega_1(\lambda_1)t$. Moreover, in the super-radiant phase the LE decays in the same manner as in the normal phase. Finally, we have studied the minimum value of the LE, denoted by M_p, as a function of λ_1 and λ_2. Since this quantity is time independent, according to previous scaling arguments it should be a function of the ratio η only. This expectation is confirmed by our numerical simulations (see the inset of Fig. 3).

V. SCALING FOR THE LMG MODEL

In the two-orbital Lipkin-Meshkov-Glick model for N interacting particles, in terms of the total spin operator for its collective motion, S_α ($\alpha = x,y,z$), the Hamiltonian can be

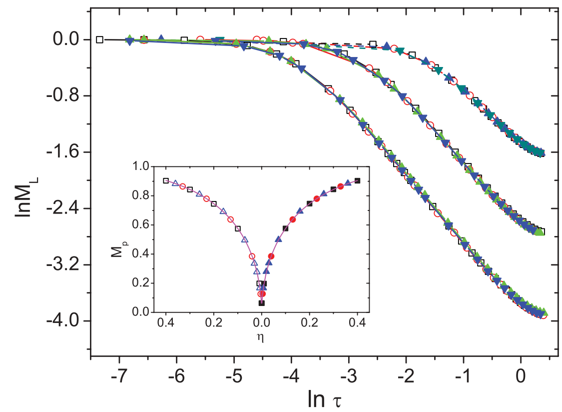

FIG. 3. (Color online) Dependence of the Loschmidt echo M_L on the rescaled time $\tau = \omega_1(\lambda_1)t$, for various values of λ_1 and λ_2, with different symbols representing different pairs (λ_1,λ_2). The three curves correspond, from top to bottom, to $\eta = 10^{-2}$, 10^{-3}, and 10^{-4}. An initial Gaussian decay is followed by $1/t$ decay, as predicted in Eq. (14). Inset: Dependence of the minimum value M_p of M_L on η for various values of λ_1 and λ_2 (open symbols stand for the normal phase and solid ones for the super-radiant phase). The fitting curves are given by $M_p = 2\sqrt{\eta}/(1+\eta)$ in both phases.

written as

$$H(\gamma,h) = -\frac{2}{N}\left(S_x^2 + \gamma S_y^2\right) - 2hS_z + (1+\gamma)/2. \tag{20}$$

As shown in Ref. [44], in the thermodynamic limit, making use of the Holstein-Primakoff transformation and of a standard Bogoliubov transformation, the Hamiltonian can be diagonalized,

$$H(\gamma,h) = \Delta a_\Theta^\dagger a_\Theta, \tag{21}$$

where

$$\Delta = 2[(h-1)(h-\gamma)]^{1/2}, \quad \tanh\Theta = \frac{1-\gamma}{2h-1-\gamma} \tag{22}$$

for $h > 1$,

$$\Delta = 2[(1-h^2)(1-\gamma)]^{1/2}, \quad \tanh\Theta = \frac{h^2-\gamma}{2-h^2-\gamma} \tag{23}$$

for $h < 1$, and $a_\Theta^\dagger$ and a_Θ are bosonic creation and annihilation operators.

Equations (22) and (23) show that when h approaches 1 from either side, $\Delta \to 0$. This implies that the system undergoes a quantum phase transition at the critical point $h_c = 1$. The phase with $h > 1$ is usually called the symmetric phase and the phase with $h < 1$ the broken phase [44].

In the thermodynamic limit, the same scaling law as in Eq. (18) can be derived analytically in the vicinity of the critical point. In fact, for a fixed γ the ground state $|0\rangle_{\Theta_2}$ for $h = h_2$ has the following expansion on the basis $|n\rangle_{\Theta_1}$ of the eigenstates of Hamiltonian (21) at $h = h_1$ [45]:

$$|0\rangle_{\Theta_2} = \frac{1}{\sqrt{C}}\sum_{n=0}^{\infty}\sqrt{\frac{(2n-1)!!}{(2n)!!}}\tanh^n\left(\frac{\Theta_2-\Theta_1}{2}\right)|2n\rangle_{\Theta_1}, \tag{24}$$

where C is a normalization constant,

$$C=\left[1-\tanh^2\left(\frac{\Theta_2-\Theta_1}{2}\right)\right]^{-1/2}. \tag{25}$$

Then, it is easy to find that

$$L_p(h_1,h_2)=\left[1-\tanh^2\left(\frac{\Theta_2-\Theta_1}{2}\right)\right]^{1/4}. \tag{26}$$

In the vicinity of the critical point $h_c=1$, the right-hand side of Eq. (26) can be simplified further. In fact, in the symmetric phase, from Eq. (22), one obtains, up to terms of higher order in $h-h_c$,

$$\tanh\frac{\Theta}{2}=1-2\left(\frac{h-1}{1-\gamma}\right)^{1/2}. \tag{27}$$

This gives

$$\tanh\left(\frac{\Theta_2-\Theta_1}{2}\right)=\frac{\eta^{1/2}-1}{\eta^{1/2}+1}, \tag{28}$$

where $\eta=(h_1-1)/(h_2-1)$. After inserting (28) into Eq. (26), we obtain the same expression (18) for L_p as for the Dicke model.

For the LE, making use of analytical results in the symmetric phase [45] and its generalization to the broken phase, similar scaling behaviors as shown in Fig. 3 have also been found.

VI. CONCLUSIONS

To summarize, we have proved a scaling property for time-independent metric quantities such as the fidelity and the participation ratio. The scaling is valid for models like the Dicke model and the LMG model, whose QPT can be described in terms of a single bosonic zero mode. Moreover, time-dependent quantities such as the Loschmidt echo also exhibit the same scaling provided time is measured in units of the inverse frequency of the critical mode.

Our scaling arguments showing the η dependence of static quantities can be generalized to the cases of more than one zero mode, provided appropriate new restrictions on the coefficients P_{ij} and Q_{ij} in Eq. (3) are introduced. On the other hand, the scaling for time-dependent quantities cannot be extended in a straightforward way to the case of more than one zero mode, when the corresponding frequencies $\omega_i(\lambda)$ have different scaling behaviors and, in contrast to the case of a single zero mode, there is no natural rescaling of time by means of a single frequency.

Finally, we remark that in our theory we compute the fidelity in the thermodynamic limit. The results obtained imply that many-body systems whose QPT can be described in terms of a single bosonic zero mode do not exhibit the Anderson orthogonality catastrophe [46]. That is to say, the ground states corresponding to two nearby values of the controlling parameter are not orthogonal at the thermodynamic limit, provided these two values belong to the same phase.

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APPENDIX: DERIVATION OF Eq. (18)

In this Appendix, we give a derivation of Eq. (18) in the main text for the fidelity of two ground states at λ_1 and λ_2. For this purpose, we use the following expression of the fidelity given in Ref. [2]:

$$L_p=\frac{2\left\{\left[\det A_{\lambda_2}\right]/\left[\det A_{\lambda_1}\right]\right\}^{1/4}}{\left[\det\left(1+A_{\lambda_1}^{-1}A_{\lambda_2}\right)\right]^{1/2}}, \tag{A1}$$

where $A_\lambda=U^{-1}M_\lambda U$, $M_\lambda=\mathrm{diag}[e_1^\lambda,e_2^\lambda]$, and U is an orthogonal matrix,

$$U=\begin{bmatrix} c & -s \\ s & c\end{bmatrix}.$$

Here, $c=\cos\gamma$ and $s=\sin\gamma$, with

$$\gamma=\tfrac{1}{2}\arctan\left[4\lambda\sqrt{\omega\omega_0}/(\omega^2+\omega_0^2)\right].$$

It is straightforward to verify the following relations:

$$\det A_\lambda=e_1^\lambda e_2^\lambda, \tag{A2}$$

$$\det\left(1+A_{\lambda_1}^{-1}A_{\lambda_2}\right)=1+\mathrm{Tr}\left(A_{\lambda_1}^{-1}A_{\lambda_2}\right)+\left[\det A_{\lambda_1}\right]^{-1}\det A_{\lambda_2}, \tag{A3}$$

and

$$\mathrm{Tr}\left(A_{\lambda_1}^{-1}A_{\lambda_2}\right)=\frac{e_1^{\lambda_2}}{e_1^{\lambda_1}}+\frac{e_2^{\lambda_2}}{e_2^{\lambda_1}}. \tag{A4}$$

In the normal phase of the Dicke model, the energies $e_{1,2}^\lambda$ are given by Eq. (16). In the neighborhood of the critical point λ_c, from Eq. (16) we get, up to terms of higher order in $\lambda_c-\lambda$,

$$e_1^\lambda=\left[\frac{8\lambda_c(\lambda_c-\lambda)\omega\omega_0}{\omega_0^2+\omega^2}\right]^{1/2}. \tag{A5}$$

Then, we have $e_1^{\lambda_2}/e_1^{\lambda_1}=(1/\eta)^{1/2}$. Using again Eq. (A5), we obtain

$$\frac{\det A_{\lambda_2}}{\det A_{\lambda_1}}=\sqrt{1/\eta}, \tag{A6}$$

$$\mathrm{Tr}\left(A_{\lambda_1}^{-1}A_{\lambda_2}\right)=\sqrt{1/\eta}+1, \tag{A7}$$

where $e_2^{\lambda_2}/e_2^{\lambda_1}=1$ has been used in the vicinity of the critical point. Substituting the above results into Eq. (A1), one finds Eq. (18). By the same method, the same expression of the fidelity can be obtained in the super-radiant phase.

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