

Thermodynamic phase transitions for Pomeau-Manneville maps

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We study phase transitions in the thermodynamic description of Pomeau-Manneville intermittent maps from the point of view of infinite ergodic theory, which deals with diverging measure dynamical systems. For such systems, we use a distributional limit theorem to provide both a powerful tool for calculating thermodynamic potentials as also an understanding of the dynamic characteristics at each instability phase. In particular, topological pressure and Rényi entropy are calculated exactly for such systems. Finally, we show the connection of the distributional limit theorem with non-Gaussian fluctuations of the algorithmic complexity proposed by Gaspard and Wang [Proc. Natl. Acad. Sci. USA **85**, 4591 (1988)].

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I. INTRODUCTION

The major goal of statistical mechanics is to explain the macroscopic properties of complex systems in terms of a very small number of parameters by using probabilistic approaches. As is well known, its primary motivation was the study of thermodynamical properties of matter based on random behavior of their very large number of constituents (atom and molecules). Almost a century after Boltzmann's seminal work [1], such approaches were extended to dynamical systems theory [2–4], a branch that is currently known as thermodynamic formalism [4]. In this scenario, the chaotic dynamics of an ensemble of trajectories plays the role of randomness in the many-body dynamics, even for one degree-of-freedom dynamical systems. The thermodynamic approach has proven to be a powerful tool in the ergodic theory of hyperbolic and expanding dynamical systems [4]. Later, there has been growing interest, mostly by theoretical physicists, in extending this approach to more general dynamical systems (see Ref. [5] and references therein), particularly those that exhibit fractal sets [6–10] or some kind of intermittent behavior [11–16].

Here we deal with phase transitions for Pomeau-Manneville (PM) maps $x_{t+1} = f(x_t)$ where f takes the form

$$f(x) = x(1 + ax^{1/\alpha}) \bmod 1, \quad (1)$$

with $a > 0$ and $\alpha > 0$ [17]. The remarkable characteristic of such systems is the intermittent behavior due to the presence of the indifferent fixed point $x = 0$, i.e., $f(0) = 0$ and $f'(0) = 1$. It is important to stress that the global form of f far from $x = 0$ is less relevant here. For example, systems behaving like (1) on $[0, x_*)$, where $f(x_*) = 1$, exhibit the same statistical behavior of Eq. (1) since the map on $[x_*, 1]$ is given by some well-behaved function f_1 such that $f_1(x_*) = 0$ and $f_1(1) = 1$. Systems of the type (1) have diverging invariant measure $\mu(x)$ near their indifferent fixed points for $0 < \alpha < 1$. More specifically, the invariant density $\omega(x)$ of map (1), where $d\mu(x) = \omega(x)dx$, behaves as

$$\omega(x) \sim bx^{-1/\alpha}, \quad (2)$$

near $x = 0$ [18]. Therefore diverging measure regime of Eq. (2) leads to a very slow laminar phase near $x = 0$

alternating with fast turbulent one elsewhere. Due to this peculiarity, the dynamics of the system (1) exhibit subexponential instability of the type $|\delta x_t| \sim |\delta x_0| \exp(\lambda_\alpha t^\alpha)$ for $0 < \alpha < 1$ [19]. On the other hand, $\alpha > 1$ leads to the finiteness of invariant measure, which is naturally related to the usual chaos and ordinary Lyapunov exponents.

It is important to point out here the connection between subexponential instability and the so-called “sporadic randomness,” a phenomenon initially studied by Gaspard and Wang [20]. These authors conjectured that the Kolmogorov-Chaitin algorithmic complexity C_t for map (1) is proportional to the number of entrances N_t into a given phase space cell after a large number of iterations. In this assumption, recently confirmed in Ref. [19] by means of a Pesin-type identity, the statistics of N_t is ruled by non-Gaussian fluctuations involving Feller's renewal results [21]. Subsequently, thermodynamic phase transitions of PM map (1) for $0 < \alpha < 2$ was studied by Wang [12] employing the same approach of [20]. It is also interesting to note that the sporadic randomness has not only been verified in PM intermittent maps (e.g., Ref. [19]), but also suggested as a distinguishing feature in weather systems [22], noncoding DNA sequences [23], and some linguistic texts [24].

The purpose of this work is twofold. First, we will revisit the pioneering results of Ref. [12], but now from the point of view of infinite ergodic theory [25] (for some applications, see also [19,26]). In this first part some results involving phase transitions of the so-called topological pressure are considerably improved. The topological pressure can be interpreted as a free energy density associated with the ensemble of trajectories. We also discuss the phase transition related to the Rényi entropy, extending the results observed in Ref. [11] to the diverging measure (nonergodic) regime of PM map (1). Finally, we show the connection between Feller's sporadic statistics and the infinite ergodic theory. Second, it aims at understanding the phase transition problem from a dynamical point of view since singular behavior of thermodynamical quantities does not tell everything about dynamic characteristics of a system. The approach employed here also show us precisely what happens at each phase, particularly in the subexponential regime of map (1).

II. TOPOLOGICAL PRESSURE

In the thermodynamic formalism, systems of the type (1) exhibit continuous phase transition, a situation where

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thermodynamic quantities vary continuously but not analytically when some external parameter of the system is changed. The paradigmatic example in the usual statistical mechanics is the ferromagnetic material at zero external magnetic field: Ferromagnets lose their spontaneous magnetization when heated above a specific critical temperature T_c and the derivative of the magnetization with respect to magnetic field (susceptibility) diverges at T_c and zero field. As shown in Ref. [12], the many-body model that most closely resembles (1) is the Fisher-Felderhof droplet model of condensation [27].

Let us first consider the topological pressure $P(\beta)$, a kind of negative Helmholtz free energy of thermodynamic formalism, defined as [5]

$$P(\beta) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln Z_t(\beta), \quad (3)$$

with the corresponding partition function $Z_t(\beta)$ given by

$$Z_t(\beta) = \sum_{\{x_i\}} \exp \left[-\beta \sum_{k=0}^{t-1} \ln |f'[f^k(x_i)]| \right]. \quad (4)$$

The set of points $\{x_i\}$ in Eq. (4) is chosen as follows. First, consider a partition of phase space into disjoint boxes Δ_i so that transitions between nearest-neighbor configurations (i, i') are possible, i.e., $f(\Delta_i) \cap \Delta_{i'} \neq \emptyset$. For each allowed sequence i_0, \dots, i_{t-1} , there is a subset $\Delta_x(i_0, \dots, i_{t-1})$ of phase space defined by

$$\Delta_x(i_0, \dots, i_{t-1}) = \{x : f^k(x) \in \Delta_{i_k}, k = 0, \dots, t-1\}. \quad (5)$$

The size of subsets Δ_{i_k} goes to zero as $t \rightarrow \infty$. Then, for very large but still finite t , we pick a representative point x_i , one from each subset, and collect them as the set $\{x_i\}$. It is important to stress, however, that the analytical determination of this set is not usually a practical task. We can circumvent this problem by replacing the summation over x_i by the integration over the conditional measure as follows (h is an arbitrary function):

$$\sum_{\{x_i\}} h(x_i) \sim \int_{x \in [0,1]} d\sigma(x) |\Delta_x(i_0, \dots, i_{t-1})| h(x), \quad (6)$$

where $\sigma(x)$ represents the measure of initial condition x and $|\Delta|$ denotes the Lebesgue measure of Δ . As $t \rightarrow \infty$ we have the following property [28]:

$$|\Delta_x(i_0, \dots, i_{t-1})| = |\Delta_x(i_1, \dots, i_{t-1})| |f'(x)|, \quad (7)$$

where, on the right side of Eq. (7), we can replace i_1 by i_2 and $|f'(x)|$ by $|f'[f(x)]| |f'(x)|$, and so forth, leading to

$$|\Delta_x(i_0, \dots, i_{t-1})| = \prod_{k=0}^{t-1} |f'[f^k(x)]|. \quad (8)$$

Finally, we can rewrite Eq. (4) by means of Eqs. (6) and (8) yielding

$$Z_t(\beta) \sim \int d\sigma(x) \exp \left[(1 - \beta) \sum_{k=0}^{t-1} \ln |f'[f^k(x)]| \right]. \quad (9)$$

Before attempting to estimate Eq. (9) we will make use of the Aaronson-Darling-Kac (ADK) theorem [25], which is precisely applicable to PM systems of type (1). For such systems, this theorem ensures that, for a positive function

ϑ integrable over μ and an arbitrary measure σ of initial conditions absolutely continuous with respect to the Lebesgue measure, we have

$$\frac{1}{t^\gamma} \sum_{k=0}^{t-1} \vartheta[f^k(x)] \xrightarrow{d} \xi_\gamma c_\gamma(t) \int \vartheta d\mu, \quad (10)$$

as $t \rightarrow \infty$, where ξ_γ is a non-negative Mittag-Leffler random variable of index $\gamma \in (0, 1]$ and expected value $E(\xi_\gamma) = 1$. The corresponding Mittag-Leffler probability density function $\rho_\gamma(\xi)$ is given by [21,29]

$$\rho_\gamma(\xi) = \frac{\Gamma^{1/\gamma}(1 + \gamma)}{\gamma \xi^{1+1/\gamma}} g_\gamma \left[\frac{\Gamma^{1/\gamma}(1 + \gamma)}{\xi^{1/\gamma}} \right], \quad (11)$$

where g_γ stands for the one-sided Lévy stable density, whose Laplace transform is $\tilde{g}(u) = \exp(-u^\gamma)$ (see Refs. [29,30] for a detailed discussion). For PM maps of the type (1), the index γ is

$$\gamma = \begin{cases} \alpha, & 0 < \alpha < 1, \\ 1, & \alpha \geq 1, \end{cases} \quad (12)$$

whereas the coefficient $c_\gamma(t)$ in Eq. (10) takes the asymptotic form [31]

$$c_\gamma(t) \sim \begin{cases} \frac{1}{ba} \left(\frac{a}{\alpha}\right)^\alpha \frac{\sin(\pi\alpha)}{\pi\alpha}, & 0 < \alpha < 1, \\ (b \ln t)^{-1}, & \alpha = 1, \\ 1, & \alpha > 1, \end{cases} \quad (13)$$

as $t \rightarrow \infty$, recalling that $b = \lim_{x \rightarrow 0} x^{1/\alpha} \omega(x)$. For $\alpha > 1$ we have introduced the Birkhoff ergodic case $\gamma = 1$, for which the corresponding Mittag-Leffler density reduces to $\rho_1(\xi) = \delta(1 - \xi)$, as in the $\alpha = 1$ case. Evidently, we can choose $\vartheta = \ln |f'|$ in the ADK formula (10).

Consider now the algorithmic complexity C_t of PM map (1), valid for all $\alpha > 0$ [19]:

$$C_t(x) \sim \sum_{k=0}^{t-1} \ln |f'[f^k(x)]|, \quad (14)$$

as $t \rightarrow \infty$. Equations (10) and (13) lead to (see also Ref. [19])

$$\frac{C_t}{\langle C_t \rangle} \xrightarrow{d} \xi_\gamma, \quad (15)$$

as $t \rightarrow \infty$, where the ADK average $\langle C_t \rangle = h_\gamma t^\gamma$ is given in terms of the average of generalized Kolmogorov-Sinai entropy [19]

$$h_\gamma = c_\gamma \int d\mu \ln |f'|. \quad (16)$$

Going back to Eq. (9), we can overcome the integration problem over arbitrary σ considering it absolutely continuous with respect to the Lebesgue measure. Such condition is sufficiently broad to assure that our results involving phase transitions typically do not depend on the initial condition distributions. This is somewhat surprising in the case of nonergodic regimes, i.e., $0 < \alpha < 1$ in the present case. After applying Eqs. (15) and (16) in the ADK formula (10), we have

$$Z_t(\beta) \sim \int_0^\infty d\xi \rho_\gamma(\xi) \exp[-(\beta - 1)h_\gamma t^\gamma \xi]. \quad (17)$$

Note that Eq. (17) is just the Laplace transform of ρ_γ , which is given by the Mittag-Leffler special function $E_\gamma(u)$, namely [29]

$$\tilde{\rho}_\gamma(u) = E_\gamma(u) = \sum_{n=0}^{\infty} \frac{[\Gamma(1+\gamma)u]^n}{\Gamma(1+n\gamma)}, \quad (18)$$

with $u = (1-\beta)h_\gamma t^\gamma$. Now, considering the asymptotes $E_\gamma(u) \sim \gamma^{-1} \exp(u^{1/\gamma})$ as $u \rightarrow \infty$ [32] and $E_\gamma(u) \sim 0$ as $u \rightarrow -\infty$ [33], we have finally for all $\alpha > 0$ and β near 1

$$P(\beta) \sim \begin{cases} [h_\gamma(1-\beta)]^{1/\gamma}, & \beta < 1, \\ 0, & \beta \geq 1, \end{cases} \quad (19)$$

observing that $h_\gamma = 0$ ($c_\gamma \rightarrow 0$) for $\alpha = 1$. Note that Eq. (19) is in accordance with the results first obtained in Ref. [12] for $0 < \alpha < 2$ and later extended for all $\alpha > 0$ in Ref. [16]. It is noteworthy here that, unlike these approaches, the prefactor h_γ in Eq. (19) is obtained exactly, given by Eq. (16) for all $\alpha > 0$.

III. RÉNYI ENTROPY

Let us consider now the phase transition related to the Rényi entropy [5]:

$$K(\beta) = \frac{1}{1-\beta} \lim_{t \rightarrow \infty} \frac{1}{t} \ln \sum_{j=0}^{t-1} (p_j^{(t)})^\beta, \quad (20)$$

where $p_j^{(t)} = p_j(i_0, \dots, i_{t-1})$ usually denotes the probability that a randomly chosen initial condition (σ distributed) on the phase space falls into Δ_j at time $t-1$. In view of the fact that we are also dealing with nonergodic regimes ($0 < \alpha < 1$), we set $p_j^{(t)}$ such that

$$p_j^{(t)}(q) = \frac{\tau_j^q}{\sum_{k=0}^{t-1} \tau_k^q}, \quad (21)$$

where $p_j^{(t)}(q=1) = p_j^{(t)}$. In Eq. (21) we consider the amount of time τ_j spent in state Δ_j instead of its length $|\Delta_j|$, which is usually considered for ergodic systems (for which $\tau_j \propto |\Delta_j|$). Then the partition function Z_t takes the asymptotic form

$$Z_t(q) \sim \exp[tP(q)] \sim \sum_{k=0}^{t-1} \tau_k^q. \quad (22)$$

Recalling that $\sum_j [p_j^{(t)}(q)]^\beta \sim \exp[(1-\beta)K(\beta, q)t]$, we have for $q=1$

$$K(\beta) = \frac{P(\beta) - \beta P(1)}{1-\beta}, \quad (23)$$

also valid for ergodic systems [5]. From the topological pressure (19) we then have

$$K(\beta) \sim \begin{cases} h_\gamma^{1/\gamma} (1-\beta)^{-1+1/\gamma}, & \beta < 1, \\ 0, & \beta \geq 1. \end{cases} \quad (24)$$

Note that for $\gamma=1$, i.e., ergodic regimes, $K(\beta) = h_{KS}$ for $\beta < 1$, where $h_1 = h_{KS}$ is the Kolmogorov-Sinai entropy, whereas $K(\beta) = 0$ for $\beta \geq 1$. Therefore Eq. (24) extends for nonergodic regimes ($0 < \alpha < 1$) the nonanalytic behavior of Rényi entropy at $\beta=1$ observed in Ref. [11].

IV. ALGORITHMIC COMPLEXITY SATISFIES THE ADK THEOREM

In Ref. [12], as well as in Ref. [20], the algorithmic complexity C_t of a piecewise version of the PM map was considered as the random number of entrances N_t into a given phase space cell (A_0) during t iterations of the map, i.e., $C_t \sim N_t$. The statistics p_α of N_t employed is well known from Feller's renewal theorems [21], and it was applied in the estimation of $P(\beta)$. The accordance with Eq. (19) can be understood by observing that p_α is, in fact, a Mittag-Leffler probability density function. The statistics of N_t for the case $0 < \alpha < 1$ is given by Ref. [21]

$$P_\alpha \left(N_t \geq c_1 \frac{t^\alpha}{q^\alpha} \right) \sim G_\alpha(q), \quad (25)$$

as $t \rightarrow \infty$, where P_α and G_α stand for the cumulative distribution functions of p_α and g_α , respectively. Applying the change of variable $q = r\xi^{-1/\alpha}$, with $r^\alpha = \alpha\Gamma(\alpha)$ [29], and after introducing the normalized random variable $\xi = N_t/(N_t)$, we have

$$p_\alpha(N_t) dN_t \sim \rho_\alpha(\xi) d\xi, \quad (26)$$

as $t \rightarrow \infty$, where $\langle N_t \rangle = c_1 t^\alpha / \alpha \Gamma(\alpha)$ and ρ_α is the Mittag-Leffler density (11). For the case $\alpha > 1$, but different from 2, we have [21]

$$P_\alpha(N_t \geq c_2 t - c_3 t^{1/\kappa} q) \sim G_\kappa(q), \quad (27)$$

where $\kappa = \alpha$ for $1 < \alpha < 2$ and $\kappa = 2$ for $\alpha > 2$. Now G_κ stands for the cumulative distribution function of the two-sided stable density g_κ [12]. We can consider the same normalized variable ξ of Eq. (26), but now with $\langle N_t \rangle = c_2 t$. Then Eq. (27) becomes

$$p_\alpha(N_t) dN_t \sim \frac{1}{\epsilon} g_\kappa \left(\frac{1-\xi}{\epsilon} \right) d\xi, \quad (28)$$

where $\epsilon = (c_3/c_2)t^{-1+1/\kappa}$ goes to 0 as $t \rightarrow \infty$. This leads to $\delta(1-\xi)d\xi$ on the right hand side of Eq. (28). Already in the Gaussian case $\kappa=2$, ϵ^2 is proportional to the variance, also leading Eq. (28) to the same Dirac δ function. The same occurs for $\alpha=2$, where we also have Eq. (28) for $\kappa=2$, but replacing c_3 by $c_3\sqrt{\ln t}$ and also leading to $\epsilon \rightarrow 0$ as $t \rightarrow \infty$.

V. FINAL REMARKS

We revisit here the problem of thermodynamic phase transitions for PM maps (1) by using the infinite ergodic theory, in particular the ADK theorem. The topological pressure $P(\beta)$ and Rényi entropy $K(\beta)$ are calculated exactly for such systems, exhibiting both phase transitions at the same critical value $\beta_c = 1$. Our results also shed some light on the role of the measure of initial conditions σ in the calculation of these thermodynamic functions. Such quantities are invariant under σ since it is absolutely continuous with respect to the Lebesgue measure. This result is somewhat surprising in the case of the nonergodic regime of the PM map (1), showing once more the strength of the ADK theorem.

From a dynamical point of view, the thermodynamic formalism allows us to obtain important quantities that characterize nonlinear systems. We can mention, for instance,

the Pesin formula relating Lyapunov exponent Λ to the Kolmogorov-Sinai entropy $h_1 = h_{KS}$, namely $h_{KS} = [P(\beta) - P'(\beta)]_{\beta \rightarrow 1_-} = \Lambda$ [5]. For nonergodic regimes $0 < \alpha < 1$, however, the topological pressure (19) gives us the trivial relation $h_{KS} = \Lambda = 0$. In fact, the dynamic instability is stretched exponential for such cases, of the form $|\delta x_t| \sim |\delta x_0| \exp(\lambda_\alpha t^\alpha)$, rather than exponential $\alpha = 1$. For such cases we can consider, once more, the infinite ergodic theory approach. It has recently been shown that the Pesin relation can be extended in a nontrivial way provided one introduces a convenient subexponential generalization of the Lyapunov

exponent and Kolmogorov-Sinai entropy [19]. Moreover, the generalizations of such quantities behave like Mittag-Leffler random variables with h_α as the first moment [19,26]. A quest for new constitutive relations involving $P(\beta)$ that lead directly to these results in the thermodynamic formalism probably deserve further investigations.

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