Radiative or neutron transport modeling using a lattice Boltzmann equation framework

H. Bindra[1,*] and D. V. Patil[2,†]

[1]*Levich Institute, City College of New York, New York City, New York 10031, USA*

[2]*The CUNY Energy Institute, City College of New York, New York City, New York 10031, USA*

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In this paper, the lattice Boltzmann equation (LBE)-based framework is used to obtain the solution for the linear radiative or neutron transport equation. The LBE framework is devised for the integrodifferential forms of these equations which arise due to the inclusion of the scattering terms. The interparticle collisions are neglected, hence omitting the nonlinear collision term. Furthermore, typical representative examples for one-dimensional or two-dimensional geometries and inclusion or exclusion of the scattering term (isotropic and anisotropic) in the Boltzmann transport equation are illustrated to prove the validity of the method. It has been shown that the solution from the LBE methodology is equivalent to the well-known P_n and S_n methods. This suggests that the LBE can potentially provide a more convenient and easy approach to solve the physical problems of neutron and radiation transport.

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I. INTRODUCTION

The radiation transport equation (RTE) has been used for astrophysical studies, and it plays an important role in the design of furnaces and burners. In recent times, there are several everyday examples in which matter-radiation interactions are becoming important, e.g., in radiation therapy to cure cancer. The RTE and neutron transport equation (NTE) address practical applications such as nuclear reactor physics, solar collectors, and medical imaging. In order to advance and improve the design of these systems, it is essential to understand the interacting physics of radiation transport and material systems. However, due to the complexity and large difference in their mathematical formulation, they are usually treated independent of one another. Recently, researchers have made efforts to study these coupled interactions [1]. In nuclear engineering, the coupling of neutronics and thermal hydraulics is gaining importance for better safety analysis of nuclear reactors [2,3]. It is difficult to obtain insightful results from the existing nonunified methods of solution using computational fluid dynamics (CFD) and neutron transport codes. A unified algorithm to solve CFD and neutron transport is necessary. Recently, Asinari and co-workers [4,5] described the advantage of having common data structures for radiation intensity and fluid flow in radiative heat transfer and fluid mechanics problems. In this context, the lattice Boltzmann equation (LBE) method-based approach for fluid dynamics has been developed extensively, and hence, it may be useful to solve the RTE and NTE with the LBE as well. The RTE and NTE are similar in their mathematical formulations; henceforth, only the RTE will be used for description purposes. In this paper, a framework of a kinetic-based LBE method is used for the solution of the RTE.

The RTE can be considered as a linear variant of the Boltzmann equation [6,7]. The general Boltzmann equation is used to describe an evolution of the particle density distribution for a near-equilibrium rarefied gas whereas the linear Boltzmann transport equation captures the variation in the density distribution of neutrons and radiative particles due to transport. The major difference in the general Boltzmann equation and the linear Boltzmann equation is the formulation of a scattering kernel. In the RTE, the scattering of radiation particles or neutrons is from the media of propagation. Furthermore, for the RTE, interparticle collisions are negligible. This implies neither collisions nor the nonlinear term in the Boltzmann equation, which represents relaxation to equilibrium due to interparticle collisions. Here, "no interparticle collisions" is equivalent to the Boltzmann equation in the Knudsen gas limit. Various analytical, approximate, and numerical procedures of solving problems described by these transport equations have been devised in the past [6,8]. In the RTE analysis, the following numerical approaches are widely used, namely, the discrete ordinate method (DOM) developed by Chandrashekhar [9] (the transport theory coding for nuclear engineering applications historically has been based on the DOM) (for recent literature on the DOM, Refs. [10,11] and references therein may be referenced), the discrete transfer method (DTM) [12], the collapsed dimension method (CDM) [13], and the finite volume method (FVM) [14]. The DOM is computationally efficient [15]; however, other methods have advantages in terms of angular discretization in various types of geometries.

In recent years, the LBE method has been a popular choice for numerically solving the Boltzmann equation with a simplified interparticle collision term (most often the Bhatnagar, Gross, Krook (BGK) model [16]) on a discrete phase-space lattice [17–19]. Over the years, the LBE has been applied to solve the conductive-radiative and convective-radiative heat transfer problems using a decoupled approach [20,21] wherein the radiation transport was solved using the FVM or DOM. Recently, the LBE itself has been adopted for solving radiation transport problems [5,22] in which one-dimensional (1D) and two-dimensional (2D) examples of radiative transfer are discussed. Furthermore, the scattering term from the media of propagation is not considered in Ref. [22]. However, this term is important to model as it is present in most of the radiation transport cases [23,24] and is the only angular variation term which contributes to the transport processes. In the present paper, a systematic inclusion of the scattering kernel within

*hbindra@che.ccny.cuny.edu

†dpatil@che.ccny.cuny.edu

the lattice Boltzmann method framework for solving problems related to radiative or neutron transport is formulated.

The numerical examples with relevant physical significance are examined to show the implementation of the proposed method. The first example is a Heaviside source problem which represents a nuclear system of discrete radiation or neutron sources in a moderator-fuel matrix. This numerical exercise also examines present data against the results by Ma *et al.* [22]. In the next case, a classic problem of the isotropic scattering of radiation in the atmosphere [24] is chosen. The final work demonstrates an extension of the approach to the 2D geometry using a standard D2Q4 lattice to obtain the solution. The classical benchmark example of radiative heat transfer in a closed square cavity is studied. In this example, first, a case with no scattering and a uniform source term is solved. This is followed by the solution of an isotropic scattering case with one wall radiated and the system at equilibrium. Finally, we show the anisotropic scattering case for both backward and forward scattering with one wall radiated and the system at equilibrium.

The rest of the paper is organized as follows. In Sec. II, theoretical formulations for the radiative transport equation and corresponding LBE are described. In Sec. III, fundamental radiative transport cases, i.e., numerical examples, are investigated. Final remarks and conclusions are made in Sec. IV.

II. MATHEMATICAL FORMULATION

A. The radiative or neutron transport equation

The general form of the one-speed (monoenergetic) transport equation for the radiation or neutrons is a linear Boltzmann integrodifferential equation as

$$\frac{1}{\nu}\frac{\partial\Psi(r,\Omega,t)}{\partial t}+\Omega\cdot\nabla\Psi(r,\Omega,t)+\kappa_a\Psi(r,\Omega,t) = \int_{4\pi}\Psi(r,\Omega',t)f(r,\Omega\rightarrow\Omega',t)\partial\Omega'. \quad (1)$$

Here, Ω is the angular direction, and $\Psi(r,\Omega,t)$ is the radiation particle distribution function in the corresponding angular direction at a given location r and at time t. Furthermore, ν is the particle speed, and κ_a is an absorption coefficient. Ω' represents the after-scattering angular direction. It is noted that Ω is used to denote the collision term in the LBE literature and should not be confused with the meaning adopted in this paper. The term $f(r,\Omega\rightarrow\Omega',t)$ represents the radiative scattering term. Practically relevant problems such as atmospheric scattering, radiative heat transfer in a furnace, and optimization of a solar collector are described as three-dimensional (3D) forms of this transport equation. Analytical solutions for these 3D problems are tedious or for some cases not possible, and hence, numerical solutions are preferred. Few standard numerical methods such as the finite difference method exist for the treatment of the space and time dependence of the integrodifferential equation [Eq. (1)]. Furthermore, the angular dependence of Eq. (1) is treated usually by employing the DOM or S_n method [9]. In this approach, a finite number of discrete angular directions are chosen, and the integral term is approximated with a summation over the chosen discrete directions as

$$\frac{1}{\nu}\frac{\partial\Psi(r,\Omega_i,t)}{\partial t}+\Omega_i\cdot\nabla\Psi(r,\Omega_i,t)+\kappa_a\Psi(r,\Omega_i,t) = \sum_j\Psi(r,\Omega_j,t)f(r,\Omega_i\rightarrow\Omega_j,t)\partial\Omega_j. \quad (2)$$

Here, the subscripts i and j represent discrete directions. Hence, the integrodifferential equation [Eq. (1)] is reduced to a system of linear differential equations. In the literature, these systems of linear differential equations are solved by conventional finite-element or finite-volume techniques [25,26].

Furthermore, an accurate approach to provide benchmark solutions to these integrodifferential equations in a less complicated but impractical scenario is the so-called P_n method or the method of spherical harmonics [6,27]. In this method, an angular distribution in $\Psi(r,\Omega,t)$ is expanded into spherical harmonics as

$$\Psi(r,\Omega,t)=\frac{1}{4\pi}\sum_{n=0}^{\infty}P_n(\Omega)\psi(r,t). \quad (3)$$

However, for the irregular geometries and complex problems, other methods may be useful. For a further understanding of these convectional numerical methods, readers are referred to the original papers [20,21]. In this work, a recently developed LBE technique for the solutions of the RTE and NTE is compared with the benchmarks from the P_n and S_n methods.

B. The lattice Boltzmann equation method for the RTE and NTE

The LBE method is shown to be derived from the Boltzmann equation with the use of the DOM [28] and by employing the low-Mach-number approximation [29,30]. A new approach for solving the radiation transport with the LBE is recently formulated based on the Chapman-Enskog expansion [22]. The LBE for a 3D system without the scattering term given in Ref. [22] is written as

$$\Psi_\Omega(r+\nu\Omega\Delta t,t+\Delta t)-\Psi_\Omega(r,t) = -\frac{\Delta t}{\tau}\left[\Psi_\Omega(r,t)-\Psi_\Omega^{\rm eq}(r,t)\right] + S_\Omega(r,t)\Delta t-\nu\kappa_a\Psi_\Omega(r,t)\Delta t. \quad (4)$$

The above equation represents the LBE evolution with the combination of the collision and streaming steps. Here, τ is the relaxation time, and the equilibrium intensity $\Psi_\Omega^{\rm eq}(r,t)$ obeys the conservation equations for energy, flux, and momentum, i.e., the zeroth, first, and second moments of intensity, respectively. $S_\Omega(r,t)$ represents the rate of energy emission. For further details on this approach, readers are referred to Ref. [22]. However, in the case of radiative and neutron transport, interparticle collisions are negligible. Hence, the collision kernel is neglected in the linear Boltzmann equation representing the RTE and NTE. This implies the term $-(1/\tau)[\Psi_\Omega(r,t)-\Psi_\Omega^{\rm eq}(r,t)]$ is redundant in Eq. (4) for radiation transport. Therefore, Eq. (4) may be simply modified for the nonscattering linear LBE as

$$\Psi_\Omega(r+\nu\Omega\Delta t,t+\Delta t)-\Psi_\Omega(r,t) = S_\Omega(r,t)\Delta t-\nu\kappa_a\Psi_\Omega(r,t)\Delta t. \quad (5)$$

Next, it is proposed here that a similar technique as described in Ref. [22] may be utilized to solve the general form of the linear Boltzmann transport equation inclusive of the scattering term. The scattering integral term may be substituted with the summation over the chosen lattice directions. The approach is similar to the method of discrete ordinates in which the integral is summed over discrete directions [31,32]. Abe [28] derived the LBE method using the discrete ordinate form of the integrodifferential Boltzmann equation. Here, a similar methodology is followed for the general form of the RTE with the scattering function $p(\Omega,\Omega')$. The LBE form can be be written as

$$\begin{aligned}\Psi_\Omega(r+\nu\Omega\Delta t,t+\Delta t)-\Psi_\Omega(r,t)\\ =S_\Omega(r,t)\Delta t-\nu\kappa_a\Psi_\Omega(r,t)\Delta t\\ +\sum_{\Omega'}W(\Omega')p(\Omega,\Omega')\Psi_{\Omega'}(r,t)\Delta t.\end{aligned} \tag{6}$$

Here, $W(\Omega')$ is the weighting function. The scattering term of Asinari *et al.* [4] is included as the "collision" term although the treatment of the scattering integral is similar. It is noted that the formulation in Ref. [4] is subjected to the case with isotropic scattering. In this paper, we demonstrate the application of the above formulation to the anisotropic scattering RTE. In the next section, classical (1D and 2D) radiative or neutron transport cases are described. Furthermore, the analytical or conventional numerical method is used for each case to obtain the benchmark data. The details of these methods and the corresponding LBE formulation used is given.

III. NUMERICAL EXAMPLES

A. The Heaviside source problem using the D1Q2 lattice

This example problem was solved by Ma *et al.* [22] using the LBE developed in their work and was shown to be equivalent to the analytical solution. The Heaviside source problem represents nuclear reactor systems in which the neutron or radiation generation source is present only in certain discrete locations such as the nuclear fuel and moderator matrix. The simplest case is shown with the source present in a half-domain of the slab geometry. Here, it is shown that using a 1D form (without the relaxation term, which is redundant as stated earlier), the solution compares well with the analytical solution (see Fig. 1). The following steady-state problem with a

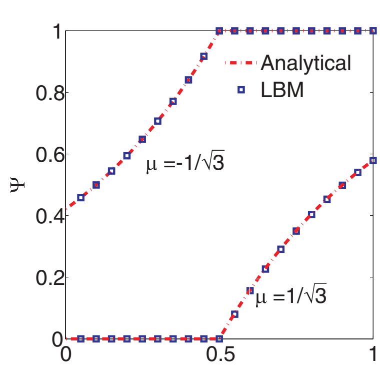

FIG. 1. (Color online) Radiative intensity comparison with $\kappa_a = 1$. The maximum relative error between the LBE and analytical solutions is less than 10^{-5}.

Heaviside source term has been solved with the new approach:

$$\mu\frac{\partial\Psi(x,\mu)}{\partial x}+\kappa_a\Psi(x,\mu)=S(x)=H(x-0.5),\quad x\in(0,1),\quad \mu=\pm\frac{1}{\sqrt{3}}. \tag{7}$$

Here, the following boundary conditions are imposed:

$$\Psi(x_w,\mu)=\frac{S(x_w)}{\kappa_a},\quad x_w=\begin{cases}0, & \mu>0,\\ 1, & \mu<0.\end{cases} \tag{8}$$

For the LBE method, a pseudotime marching is performed with a two-velocity lattice model in 1D (D1Q2). The evolution equation is given as

$$\begin{aligned}\Psi_\mu(x+\mu\Delta t,t+\Delta t)-\Psi_\mu(x,t)\\ =S_\mu(x,t)\Delta t-\kappa_a\Psi_\mu(x,t)\Delta t.\end{aligned} \tag{9}$$

The exact steady-state analytical solution to Eq. (7) is obtained as

$$\begin{aligned}\Psi(x,\mu)=\Psi(x_w,\mu)\exp\left[-\frac{\kappa_a}{\mu}(x-x_w)\right]\\ +\frac{1}{\kappa_a}\left\{1-\exp\left[-\frac{\kappa_a}{\mu}(x-0.5)\right]\right\}H(x-0.5)\\ +\frac{1}{\kappa_a}\left\{\exp\left[-\frac{\kappa_a}{\mu}(x-0.5)\right]\right.\\ \left.-\exp\left[-\frac{\kappa_a}{\mu}(x-x_w)\right]\right\}H(x-x_w).\end{aligned} \tag{10}$$

The two particle velocity directions $\pm\mu$ in this example are considered with a magnitude of $\frac{1}{\sqrt{3}}$ as specified in Ref. [22]. The mesh size N for the LBE calculations is 500, and $\Delta t = 1/N$. The initial condition for the distribution functions is $\Psi_{\pm\mu} = 0$, and 500 time iterations are required to obtain the steady-state solution.

B. Isotropic scattering using the D1Q2 lattice

Isotropic scattering means the incident ray is scattered to all directions uniformly. The scattering of solar radiation in a thick uniform atmosphere is a perfect example of isotropic scattering. Although in this problem, an isotropic source is also added to present a more general case. The general one-dimensional form of the one-speed transport equation is the integrodifferential equation with the scattering function $f(x,\mu\to\mu')$:

$$\begin{aligned}\frac{1}{\nu}\frac{\partial\Psi(x,\mu,t)}{\partial t}+\mu\frac{\partial\Psi(x,\mu,t)}{\partial x}+\kappa_a\Psi(x,\mu,t)\\ =\frac{c}{2}\int_{-1}^{1}\Psi(x,\mu',t)f(x,\mu\to\mu',t)\partial\mu'+\frac{S(x,\mu,t)}{2}.\end{aligned} \tag{11}$$

The isotropic scattering is the most common and simplest case in which the scattering function has no directional dependence $[f(x,\mu\to\mu')=g(x)]$. We will now show an extension of the LBE-based approach to solve this isotropic scattering problem. The steady-state example problem for demonstration and comparison purposes is Eq. (12) wherein $[f(x,\mu\to\mu')=1]$. The corresponding discrete time-dependent LBE form for the transport equation can be written as Eq. (13):

$$\mu \frac{\partial \Psi(x,\mu)}{\partial x} + \kappa_a \Psi(x,\mu) = \frac{c}{2}\int_{-1}^{1} \Psi(x,\mu')\partial\mu' + \frac{S(x,\mu)}{2}, \quad (12)$$

$$\Psi_\mu(x+\mu\Delta t, t+\Delta t) - \Psi_\mu(x,t) = \frac{S_\mu(x,t)}{2}\Delta t - \kappa_a \Psi_\mu(x,t)\Delta t + \frac{c}{2}\sum_\mu \Psi_\mu(x,t)\Delta t. \quad (13)$$

It should be noted that the velocity magnitude for two directions of motion for the 1D (D1Q2) problem will remain fixed as $\mu = \pm(1/\sqrt{3})$ as the directions can only be the zeros of the second-order Legendre polynomial [23]. The previous works on solving similar integrodifferential problems semianalytically in 1D without a source term include Refs. [23,24]. The source term $S(x,\pm\mu)$ is a constant S_0, and the boundary conditions given by

$$\Psi(x_w,\mu) = 0, \quad \mu \in [-1,1]. \quad (14)$$

Here, we will compare our LBE solution with the P_1 approximation of Eq. (12). The P_1 approximation leads to the formation of two first-order coupled ODEs [Eqs. (15) and (16)] instead of the integrodifferential equation for $x \in (0,1)$:

$$\frac{d\phi_0(x)}{dx} + 3\phi_1(x) = 0, \quad (15)$$

$$\frac{d\phi_1(x)}{dx} + (1-c)\phi_0(x) = S_0. \quad (16)$$

The boundary condition given by Eq. (14) has a P_1 approximation form as

$$\frac{1}{2}\phi_0(x_w) + \frac{3}{2}\mu\phi_1(x_w) = 0, \quad x_w = \begin{cases} 0, & \mu > 0, \\ 1, & \mu < 0. \end{cases} \quad (17)$$

Furthermore, the analytical solution for this P_1 approximate form is given as

$$\phi_0(x) = A\left[\cosh(\lambda x) + \frac{1}{\mu_0\lambda}\sinh(\lambda x)\right] + \frac{3S_0}{\mu_0\lambda^3}\sinh(\lambda x) + \frac{3S_0}{\lambda^2}, \quad (18)$$

where

$$A = \frac{-3S_0}{\lambda^2}\frac{1+\cosh\lambda + \frac{\sinh\lambda}{\mu_0\lambda}}{2\cosh\lambda + \left(\mu_0\lambda + \frac{1}{\mu_0\lambda}\right)\sinh\lambda}, \quad \lambda = \sqrt{3(1-c)}, \quad \mu_0 = |\mu|. \quad (19)$$

The quantity ϕ_0 implies the average flux, and ϕ_1 implies the average current in a physical system (see Fig. 2). In the LBE method, the quantities ϕ_0 and ϕ_1 are

$$\phi_0(x) = \int_{-1}^{1} \Psi(x,\mu)d\mu = \sum_\mu \Psi(x,\mu), \quad (20)$$

$$\phi_1(x) = \int_{-1}^{1} \mu\Psi(x,\mu)d\mu = \sum_\mu \mu\Psi(x,\mu). \quad (21)$$

The LBE numerical simulation setup has a mesh size of $N = 1000$ and $\Delta t = 1/N$. The Ψ values are initialized to zero. The steady state is achieved in 5000 time steps with a maximum absolute error over analytical values being on the order of 10^{-3}.

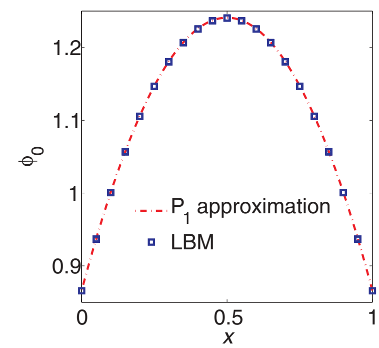

FIG. 2. (Color online) Steady-state average flux ϕ_0 computed by the LBE method and the P_1 approximation with $c = 1$, $\kappa_a = 1$, and $S_0 = 1$.

C. Radiative transfer in a square enclosure using the D2Q4 lattice

The radiative heat transfer in a square black body is the benchmark problem existing in the literature [33–36]. Hence, a black-body radiative heat transfer problem in a closed square (2D) enclosure is solved using the lattice Boltzmann approach described in this paper. Similarly, 2D radiative transfer has been solved using the LBM approach [4] and compared against FVM results. Beforehand, it may be necessary to compare against the discrete angular direction method (DOM). It is understood that for the DOM as well as the LBM, the level of accuracy increases with an increasing number of discrete angular directions and discrete velocities, respectively. An exact comparison between two schemes may be performed only when the number of discrete directions are equivalent, such as in the cases of S_2 and D2Q4. Therefore, the following study has been conducted to provide this comparison. In the case of D2Q4, there are four particle velocities ($\pm\mu,\pm\eta$) with equal magnitudes of $\frac{1}{\sqrt{3}}$ (Fig. 3). It is shown that the solution conforms well with the previously developed [35] discrete ordinate (S_2) method for this problem. Although more accurate DOM schemes have been discussed in previous works [34–36], in this work we intend to compare the LBE (D2Q4) with the S_2 method as both these methods have only four discrete angular directions. The details of this radiative heat transfer problem along with the appropriate boundary conditions are described

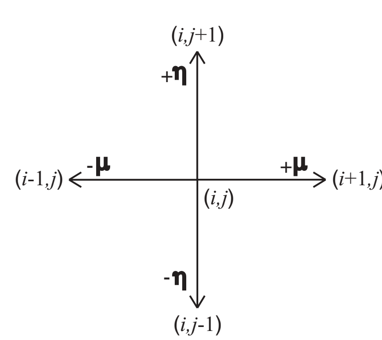

FIG. 3. A 2D lattice geometry.

as follows:

$$\Omega \cdot \nabla\Psi(r,\Omega) = -\kappa_a \Psi(r,\Omega) + \kappa_a \int_{4\pi} \Psi(r,\Omega)d\Omega + S(r,\Omega), \quad (22)$$

$$\Psi(r_w,\Omega) = \Psi_b(r_w,\Omega), \quad r_w = \begin{cases} 0, & \Omega > 0, \\ 1, & \Omega < 0. \end{cases} \quad (23)$$

The two terms $\kappa_a \int_{4\pi} \Psi(r,\Omega)d\Omega$ and the nonradiative volumetric heat source term $S(r,\Omega)$ are usually [34–36] combined to represent the black-body radiation intensity term $\kappa_a \Psi_b(r,\Omega)$:

$$\Omega \cdot \nabla\Psi(r,\Omega) = -\kappa_a \Psi(r,\Omega) + \kappa_a \Psi_b(r,\Omega). \quad (24)$$

The quantity Ψ_b is the effective black-body radiation intensity at the temperature of the medium or wall, T_b. We consider two cases for this study: an isothermal medium and a medium under radiative equilibrium. In the case of an isothermal medium Ψ_b, the effective black-body radiation and the temperature of the medium or wall T_b both are constant. Black walls surrounding the gas and/or medium are kept at a fixed temperature and radiative intensity, and the flux is estimated inside the enclosure and boundaries. However, the medium in a radiative equilibrium implies $S(r,\Omega) = 0$ and $\kappa_a \Psi_b(r,\Omega) = \kappa_a \int_{4\pi} \Psi(r,\Omega)d\Omega$. The integral $\int_{4\pi} \Psi(r,\Omega)d\Omega$ can be numerically expressed as $\frac{1}{4}\sum_i \Psi_i$, and $1/4$ is the weight factor due to the presence of four directions in the D2Q4 lattice [4]. The LBE (note the time-dependent form) of this problem can be written as

$$\Psi_i(x + e_{i,x}\Delta t, y + e_{i,y}\Delta t, t + \Delta t) - \Psi_i(x,y,t) = -\kappa_a \Psi_i(x,y,t)\Delta t + \frac{\kappa_a}{4}\sum_i \Psi_i(x,y,t)\Delta t. \quad (25)$$

The S_n form of the same problem is given as

$$\mu_i \frac{\Psi_i(x+\Delta x,y) - \Psi_i(x,y)}{\Delta x} + \eta_i \frac{\Psi_i(x,y+\Delta y) - \Psi_i(x,y)}{\Delta y} = -\kappa_a \Psi_i(x,y) + \kappa_a \Psi_b(x,y). \quad (26)$$

The numerical solution to this problem is obtained using both these methods with the same mesh size $N_x = N_y = 200$ and $\Delta t = 1/N_x$. The initial condition is $\Psi = 0$ for all grid points. A steady-state solution is achieved in 1200 time iterations for both methods. The results from both these methods are compared in Fig. 4. Although the numerical accuracy and computational effort is similar for both these methods, the formulation for the LBE is much more convenient as compared to the equivalent DOM form. Moreover, due to the widespread popularity of the LBE method, more accurate forms and variations for complex geometries are available in the case of the LBE.

D. Anisotropic radiative scattering using the D2Q4 lattice

The scattering is not necessarily isotropic in real-life problems due to anisotropic properties of the participating media. Hence, it is important to study the anisotropic form of radiation scattering. This aspect has been studied in the past by analytical and numerical methods [37,38]. The D2Q4 lattice applied in the previous isotropic scattering example is extended to the anisotropic scattering case. Here, forward and backward scattering are employed, replacing the isotropic scattering. The general form of radiative transfer with the nonisotropic case can be expressed with the following modification of the scattering term in Eq. (22):

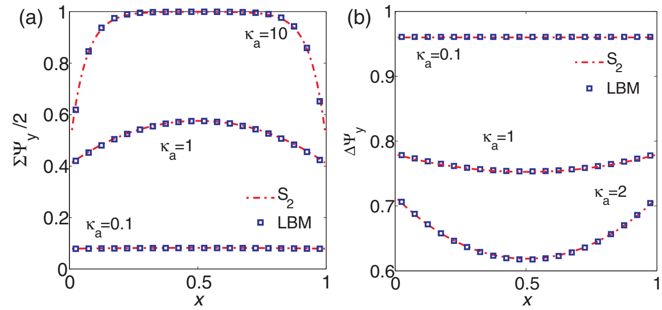

FIG. 4. (Color online) Angular radiative heat flux solution with the LBE and S_2 methods. (a) Isothermal cavity case $\Psi_b = 0$ for all four walls and $\Psi_b = 1$ for the medium. The plot shows $\sum\Psi/2$ at the wall surface. (b) The radiative equilibrium case $\Psi_b = 1$ for one wall and the other three walls at $\Psi_b = 0$ and $S = 0$ in the medium. The plot shows $\Delta\Psi$ at the hot wall surface.

$$\Omega \cdot \nabla\Psi(r,\Omega) = -\kappa_a \Psi(r,\Omega) + \kappa_a \int_{4\pi} \Psi(r,\Omega')p(\Omega,\Omega')d\Omega' + S(r,\Omega), \quad (27)$$

wherein $p(\Omega,\Omega')$ represents the anisotropic scattering function. We consider two types of anisotropic scattering as forward (+) and backward (−) scattering such that

$$p(\zeta,\zeta') = 1 \pm \zeta\zeta'. \quad (28)$$

Here, ζ represents any of the four directions in the D2Q4 lattice. In the example problem, we choose the case of radiative equilibrium in which one wall of the square enclosure is exposed to $\Psi_b = 1$ and the other three walls are kept at $\Psi_b = 0$. In the implementation of D2Q4, the scattering integral is resolved into four lattice directions, and the respective integrals are computed as

$$\oint \Psi(r,\zeta')p(\zeta,\zeta')d\zeta' = \int_{-1}^{1} \Psi(r,\mu')p(\zeta,\mu')d\mu' + \int_{-1}^{1} \Psi(r,\eta')p(\zeta,\eta')d\eta'. \quad (29)$$

For the D2Q4 LBE, each of these scattering integrals is simplified and converted to its numerical form in the following manner:

$$\int_{-1}^{1} \Psi(r,\mu')p(\zeta,\mu')d\mu' = \int_{-1}^{0} \Psi(r,\mu')(1 \pm \zeta\mu')d\mu' + \int_{0}^{1} \Psi(r,\mu')(1 \pm \zeta\mu')d\mu', \quad (30)$$

$$\int_{-1}^{1} \Psi(r,\mu')(1 + \zeta\mu')d\mu' = \Psi(r,-\mu)\left(1 \mp \frac{\zeta}{2}\right) + \Psi(r,\mu)\left(1 \pm \frac{\zeta}{2}\right). \quad (31)$$

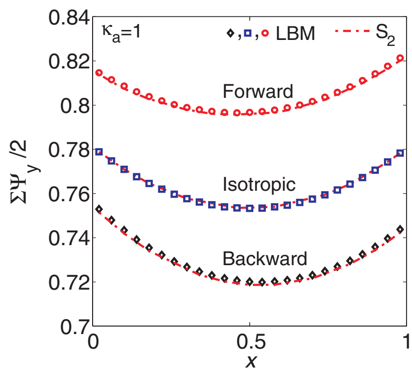

FIG. 5. (Color online) Angular radiative heat flux solution with the LBE for anisotropic scattering cases. Radiative equilibrium case $\Psi_b = 1$ for one wall and the other three walls at $\Psi_b = 0$ and $S = 0$ in the medium. The plot shows $\Delta\Psi$ at the hot wall surface.

The LBE for this RTE with forward (+) and backward (−) scattering can be written as

$$\Psi_i(x + e_{i,x}\Delta t, y + e_{i,y}\Delta t, t + \Delta t) - \Psi_i(x,y,t) = -\kappa_a \Psi_i(x,y,t)\Delta t + \frac{\kappa_a}{4}\sum_i \left[\Psi_i(x,y,t)\left(1 \pm \frac{e_i}{2}\right) + \Psi_{-i}(x,y,t)\left(1 \mp \frac{e_i}{2}\right)\right]\Delta t. \tag{32}$$

The solution to the forward and backward scattering case is compared with the isotropic case ($\kappa_a = 1$) in Fig. 5. The solution is obtained with a similar mesh size and number of iterations as in Sec. III C. The results of the LBE method for these scattering cases are compared with the results from the S_2 method and are shown to be equivalent (error less than 1%) in Fig. 5.

IV. CONCLUSIONS

In this work, the recently proposed LBE for solving radiative transfer problems is reinvestigated and extended. The nonlinear interparticle collision term or relaxation term used in the previous work to solve the radiative transfer problem with no scattering is shown to be redundant. The scattering integral term in the integrodifferential transport equation is replaced by summation over the Gaussian quadrature. Furthermore, with the help of numerical examples it is shown to yield accurate results as obtained by conventionally used spherical harmonics P_n or discrete ordinate S_n methods. In 2D systems, the D2Q4 lattice setup is equivalent to the S_2 method. Higher-order accurate D2Q9 and S_4 in the LBE method and the DOM, respectively, have different numbers of discrete directions. Hence, direct comparison is not possible in this case. Therefore, in the near future, attention to the development of a higher-order accurate LBE scheme for radiative or neutron transport is required.

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