Breaking of upper hybrid oscillations in the presence of an inhomogeneous magnetic field

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We present space-time evolution of large-amplitude upper hybrid modes in a cold homogeneous plasma in the presence of an inhomogeneous magnetic field. Using the method of Lagrange variables, an exact space-time-dependent solution is obtained in parametric form. It is found that the magnetic field inhomogeneity causes various nonlinearly excited modes to couple, resulting in phase mixing and eventual breaking of the initially excited mode. The occurrence of wave breaking is seen by the appearance of spikes in the density profile. These results will be of relevance to laboratory and space plasma situations in which the external magnetic field is inhomogeneous.

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I. INTRODUCTION

The search for analytical solutions expressing space-time evolution of normal modes in plasmas seems to have been attracting attention for quite some time. This is because analytic solutions facilitate the understanding of basic experimental situations in which large amplitude modes are excited, thus leading to deeper insight into the underlying phenomenon. Starting from some physically realizable initial state, the excited modes generally evolve in nonlinear fashion and reach toward a final state. Phase mixing and breaking of such excited modes play a central role in nonlinear plasma theory. It is well known that nonlinear oscillations in a cold homogeneous unmagnetized plasma exhibit coherent oscillatory behavior up to a critical value of initial amplitude in perturbations. Beyond the critical amplitude, waves and oscillations break [1]. Inclusion of various physical effects, such as relativistic effects, inhomogeneity in the ion density, etc., in wave dynamics leads to wave breaking via a phenomenon called phase mixing [2–6]. In such situations, the characteristic frequency of the mode under consideration becomes space-dependent. In response to wave dynamics, plasma particles situated at different locations oscillate at different local frequencies. As a result, mixing of phases corresponding to various parts of the mode occurs and the mode breaks. The wave breaking and phase mixing phenomena in an unmagnetized plasma have been studied extensively. In the present paper, we have studied the wave breaking phenomenon in a magnetized plasma corresponding to the high-frequency upper hybrid mode. We show here that upper hybrid modes in a cold plasma break via phase mixing in the presence of an externally applied inhomogeneous magnetic field. Inhomogeneity in a magnetic field is prevalent in almost all realistic situations involving a magnetized plasma. Examples include laboratory-based plasma experiments such as the magnetized plasma linear device, tokamak, etc., and space plasma situations such as magnetospheric plasma.

A number of problems regarding nonlinear waves and oscillations in a homogeneous plasma have been solved using Lagrangian coordinates [7–9]. All of the ones mentioned above have been solved exactly and illustrate the nature of nonlinear evolution of the modes in space and time. Variants of nonlinear solutions achieved by introducing Lagrangian variables have also been obtained [10–13]. In addition, instead of treating plasma as homogeneous, periodicity in ion density background has also been assumed by several authors. The effect of such a spatially periodic ion density fluctuation on electron plasma waves has been studied analytically as well as in numerical simulation and reported in Ref. [14]. Further, taking periodic or hyperbolic-secant-square pulse and cavity type ion density, exact nonlinear solutions have been obtained in parametric form [3,4]. These studies show that ion density fluctuations enhance the amplitude of plasma oscillations, resulting in peaks in the density profile. Physically, inhomogeneity in ion density makes the local plasma frequency space-dependent. As a result, plasma waves phase-mix. In this paper, we investigate the effects of inhomogeneity in the *magnetic field* on upper hybrid modes in a cold plasma where ion density is taken as independent of both space and time. Inclusion of inhomogeneity in the magnetic field makes the electron cyclotron frequency space-dependent. This causes phase mixing of upper hybrid oscillations. Performing a linear analysis, we show that mode coupling happens between finite-amplitude magnetic field fluctuation and initial perturbation corresponding to the mode under study. We further obtain an exact solution to this problem by using Lagrangian coordinates in one space variable. Nonlinear analysis shows the appearance of density spikes, indicating the breaking of such modes.

The paper is organized as follows. In Sec. II, the basic equations which describe space-time evolution of upper hybrid modes in a cold magnetized plasma are given along with linear analysis. In Sec. III, the Lagrangian variables are introduced for nonlinear analysis and an exact analytical solution is presented in parametric form. In the next subsection, an approximate analytic solution together with the frequency-amplitude relationship of oscillations using the homotopy perturbation method are presented [15,16]. This yields an approximate expression for phase-mixing time. Finally, in Sec. IV we summarize our results.

II. BASIC EQUATIONS AND LINEAR ANALYSIS

In one dimension, in the presence of an external inhomogeneous magnetic field, the space-time evolution of an upper hybrid mode under electrostatic approximation is governed by the following equations:

$$\left(\frac{\partial}{\partial t}+v_{\rm ex}\frac{\partial}{\partial x}\right)n_e=-n_e\frac{\partial v_{\rm ex}}{\partial x}, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+v_{\rm ex}\frac{\partial}{\partial x}\right)v_{\rm ex}=-\frac{e}{m_e}E_x-\frac{eB(x)}{m_ec}v_{\rm ey}, \tag{2}$$

$$\left(\frac{\partial}{\partial t}+v_{\rm ex}\frac{\partial}{\partial x}\right)v_{\rm ey}=\frac{eB(x)}{m_ec}v_{\rm ex}, \tag{3}$$

$$\left(\frac{\partial}{\partial t}+v_{\rm ex}\frac{\partial}{\partial x}\right)E_x=4\pi en_0v_{\rm ex}, \tag{4}$$

where the external magnetic field $B(x)$ is directed along the z direction and the other symbols have their usual meaning. The plasma is assumed to be cold and collisionless, and ions form an immobile homogeneous neutralizing background. To extract the physical content from the above set of equations, we first perform a linear analysis. Linearizing the above equations (1)–(4), we obtain the following evolution equation for perturbed electron density:

$$\frac{\partial^2\tilde{n}_e}{\partial t^2}+\left[\omega_p^2+\Omega_e^2(x)\right]\tilde{n}_e=0. \tag{5}$$

To find an analytical solution corresponding to the linearized Eq. (5), we need to specify the functional form of the externally applied inhomogeneous magnetic field. As a typical example, we take the magnetic field having sinusoidal variation in space, i.e., $B(x)=B_0\cos(\alpha x)$, where α is the inverse of the magnetic field variation length L_B. In the limit $L_B\to\infty$, the external magnetic field becomes constant with magnitude B_0. With the initial conditions for the perturbations, $\tilde{n}_e(x,0)=\delta\cos(kx)$ and $\dot{\tilde{n}}_e(x,0)=0$, where δ is the amplitude of perturbation and k is the inverse of perturbation scale, we get

$$\tilde{n}_e(x,t)=\delta\cos(kx)\cos\left[\sqrt{1+\bar{\Delta}\cos(2\alpha x)}\bar{\omega}_{uh0}t\right], \tag{6}$$

where $\omega_{uh0}=\sqrt{\omega_p^2+\Omega_{e0}^2}$ with $\Omega_{e0}=eB_0/m_ec$, $\bar{\omega}_{uh0}=\sqrt{1-\Delta}\omega_{uh0}$ with $\Delta=(\Omega_{e0}^2/2)/\omega_{uh0}^2<1/2$, and $\bar{\Delta}=\Delta/(1-\Delta)<1$. Notice that the frequency of oscillation has a spatial dependence which leads to phase mixing of upper hybrid oscillations. This can be seen explicitly by expressing the solution in terms of Bessel functions of the first kind, J_l, as

$$\tilde{n}_e(x,t)=\frac{\delta}{2}\sum_{l=-\infty}^{\infty}J_l\left(\frac{\bar{\Delta}\bar{\omega}_{uh0}t}{2}\right)\left[\cos\left(\bar{\omega}_{uh0}t+\frac{l\pi}{2}\right)\{\cos(k+2l\alpha)x+\cos(k-2l\alpha)x\}\right.$$
$$\left.+\sin\left(\bar{\omega}_{uh0}t+\frac{l\pi}{2}\right)\{\sin(k+2l\alpha)x-\sin(k-2l\alpha)x\}\right]. \tag{7}$$

The result given in Eq. (7) shows mode coupling. This can be interpreted as follows: At $t=0$, the electrostatic energy is injected into a single mode with wave number k. As time goes on, energy flows toward higher and higher modes with wave numbers $(k\pm l\alpha)$. At the same time, primary disturbance dies out in a time scale $\omega_{uh0}t_{\rm mix}\simeq\frac{2}{\Delta}\sqrt{1-\Delta}$ because the amplitude of the primary mode varies as $\sim\delta J_0(\bar{\Delta}\bar{\omega}_{uh0}t/2)$. The physical reason for this occurrence of mode coupling is that the characteristic frequency of the oscillation is space-dependent [see Eq. (6)], i.e., $\omega/\bar{\omega}_{uh0}=\sqrt{1+\bar{\Delta}\cos(2\alpha x)}$. Therefore, finite-amplitude magnetic field fluctuations can interact with the initial perturbation, which causes phase mixing, and the higher the value of the amplitude of magnetic field inhomogeneity, the less time it takes the upper hybrid oscillations to phase-mix. From Fig. 1, we see that at phase-mixing time ($\omega_{uh0}t\sim 9$), the initial coherence pattern is almost destroyed. Beyond phase mixing time, wave packets are observed, which is depicted in Fig. 1 corresponding to time $\omega_{uh0}t\sim 400$. The high "k" modes shown in Fig. 1 arise due to coupling of the initially excited mode "k" with the mode number of the magnetic field inhomogeneity "α" and its harmonics. Hence their spatial scale is entirely determined by the scale length of the magnetic field inhomogeneity and is not associated with any intrinsic plasma spatial scale. It should be noted that one should not expect an appearance of density spikes within a linear analysis. This is because linear analysis is valid only when the amplitude of oscillation is small, i.e., $\delta<<1$.

III. NONLINEAR ANALYSIS

Now we wish to analyze Eqs. (1)–(4) in a nonlinear fashion by keeping the "convective nonlinearity" term $v_{\rm ex}\partial/\partial x$. For

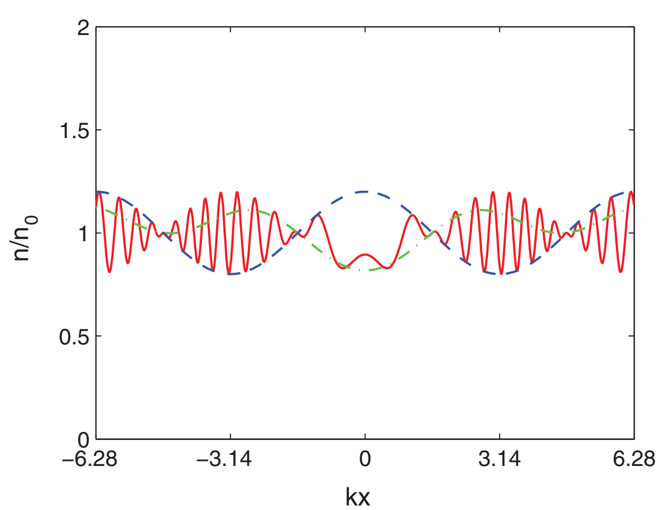

FIG. 1. (Color online) Normalized plasma density n/n_0 as a function of kx for various values of $\omega_{uh0}t$, with $\delta=0.2$, $\Delta=0.2$, and $\alpha/k=0.2$ (linear case). The blue (dashed) line shows the profile at $\omega_{uh0}t=0$, the green (dash-dotted) line at $\omega_{uh0}t=9$, and the red (solid) line at $\omega_{uh0}t=400$.

this purpose, we introduce Lagrangian transformation. The transformation relation from Eulerian variables (x,t) to Lagrangian variables (ξ,τ) can be expressed through an auxiliary variable ψ as

$$\xi = x - \psi,\ \tau = t,\ \psi = \int_0^\tau v_{\rm ex}(\xi,\tau)d\tau. \tag{8}$$

From Eq. (8), it can be easily seen that the spatial and temporal derivatives transform as

$$\frac{\partial}{\partial x} = \left[1 + \int_0^\tau \frac{\partial v_{\rm ex}(\xi,\tau)}{\partial \xi}d\tau\right]^{-1}\frac{\partial}{\partial \xi},\quad \frac{\partial}{\partial t} + v_{\rm ex}\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau}. \tag{9}$$

In terms of Lagrangian variables and with $B(x) = B_0\cos(\alpha x)$, Eqs. (1)–(4) give, respectively,

$$n_e(\xi,\tau) = n_e(\xi,0)\left(1 + \frac{\partial \psi}{\partial \xi}\right)^{-1}, \tag{10}$$

$$\frac{\partial^2 \psi}{\partial \tau^2} = -\frac{eE_x}{m_e} - \Omega_{e0}\cos\{\alpha(\xi+\psi)\}v_{\rm ey}, \tag{11}$$

$$\frac{\partial v_{\rm ey}}{\partial \tau} = \Omega_{e0}\cos\{\alpha(\xi+\psi)\}\frac{\partial \psi}{\partial \tau}, \tag{12}$$

$$\frac{\partial E_x}{\partial \tau} = 4\pi e n_0\frac{\partial \psi}{\partial \tau}. \tag{13}$$

Combining Eqs. (11)–(13), we obtain

$$\frac{\partial^3 \psi}{\partial \tau^3} + \big[\omega_p^2 + \Omega_{e0}^2\cos\{2\alpha(\xi+\psi)\} + \Omega_{e0}^2\sin(\alpha\xi)\sin\{\alpha(\xi+\psi)\}\big]\frac{\partial \psi}{\partial \tau} = 0, \tag{14}$$

where we have used an initial condition $v_{\rm ey}(x,0) = 0$. Moreover, using the initial conditions $n_e(x,0) = n_0(1+\delta\cos kx)$, $v_{\rm ex}(x,0) = 0$, or equivalently using $\psi(\xi,0) = (\partial\psi/\partial\tau)\,|_{(\xi,0)} = 0$ and $(\partial^2\psi/\partial\tau^2)\,|_{(\xi,0)} = (\omega_p^2\delta/k)\sin(k\xi)$, two consecutive integrations of Eq. (14) yield

$$\frac{\partial^2 \phi}{\partial \bar\tau^2} + \bar\omega_p^2\phi + \frac{2\Delta}{\bar\alpha}\left[\sin\{\bar\alpha(\bar x+\phi)\} - \sin\left\{\bar\alpha\left(\bar x + \frac{\phi}{2}\right)\right\} + \sin\left(\frac{\bar\alpha\phi}{2}\right)\right] - \delta\bar\omega_p^2\sin\bar x = 0 \tag{15}$$

and

$$\left(\frac{\partial \phi}{\partial \bar\tau}\right)^2 = 2\delta\bar\omega_p^2\phi\sin\bar x + \frac{4\Delta}{\bar\alpha^2}\left[\cos\{\bar\alpha(\bar x+\phi)\} + \cos(\bar\alpha\bar x) - 2\cos\left\{\bar\alpha\left(\bar x+\frac{\phi}{2}\right)\right\} + 2\cos\left(\frac{\bar\alpha\phi}{2}\right)\right] - \frac{8\Delta}{\bar\alpha^2} - \bar\omega_p^2\phi^2, \tag{16}$$

respectively. Here we have defined $\phi = k\psi$, $\bar\tau = \omega_{uh0}\tau$, $\bar x = k\xi$, $\bar\alpha = 2\alpha/k$, and $\bar\omega_p^2 = \omega_p^2/\omega_{uh0}^2 \equiv (1-2\Delta)$. Moreover, from Eq. (16) we have

$$\bar\tau = \int_0^\phi d\phi\left[2\delta\bar\omega_p^2\phi\sin\bar x + \frac{4\Delta}{\bar\alpha^2}\left[\cos\{\bar\alpha(\bar x+\phi)\} + \cos(\bar\alpha\bar x) - 2\cos\left\{\bar\alpha\left(\bar x+\frac{\phi}{2}\right)\right\} + 2\cos\left(\frac{\bar\alpha\phi}{2}\right)\right] - \frac{8\Delta}{\bar\alpha^2} - \bar\omega_p^2\phi^2\right]^{-1/2}. \tag{17}$$

From Eq. (10), we get

$$n_e(\xi,\tau) = n_0(1+\delta\cos\bar x)\left(1+\frac{\partial\phi}{\partial\bar x}\right)^{-1}, \tag{18}$$

where

$$\begin{aligned}\frac{\partial\phi}{\partial\bar x} = &\left\{2\delta\bar\omega_p^2\phi\sin\bar x + \frac{4\Delta}{\bar\alpha^2}\left[\cos\{\bar\alpha(\bar x+\phi)\} + \cos(\bar\alpha\bar x) - 2\cos\left\{\bar\alpha\left(\bar x+\frac{\phi}{2}\right)\right\} + 2\cos\left(\frac{\bar\alpha\phi}{2}\right)\right] - \frac{8\Delta}{\bar\alpha^2} - \bar\omega_p^2\phi^2\right\}^{1/2}\\ &\times\frac{1}{2}\int_0^\phi d\phi\left\{2\delta\bar\omega_p^2\phi\cos\bar x - \frac{4\Delta}{\bar\alpha}\left[\sin\{\bar\alpha(\bar x+\phi)\} + \sin(\bar\alpha\bar x) - 2\sin\left\{\bar\alpha\left(\bar x+\frac{\phi}{2}\right)\right\}\right]\right\}\\ &\times\left\{2\delta\bar\omega_p^2\phi\sin\bar x + \frac{4\Delta}{\bar\alpha^2}\left[\cos\{\bar\alpha(\bar x+\phi)\} + \cos(\bar\alpha\bar x) - 2\cos\left\{\bar\alpha\left(\bar x+\frac{\phi}{2}\right)\right\} + 2\cos\left(\frac{\bar\alpha\phi}{2}\right)\right] - \frac{8\Delta}{\bar\alpha^2} - \bar\omega_p^2\phi^2\right\}^{-3/2}.\end{aligned} \tag{19}$$

Finally from Eq. (8), the coordinate transformation relation becomes

$$kx = \phi + \bar x. \tag{20}$$

Equations (17), (18), and (20) represent the exact solution in parametric form:

$$t = t(\phi,\bar x),\quad n_e = n_e(\phi,\bar x),\quad x = x(\phi,\bar x). \tag{21}$$

To interpret the exact solutions expressed in parametric form, a conventional way is to look at the corresponding phase-space diagram. The phase-space diagram $(\phi_{\bar\tau},\phi)$ for various values of $\bar x$ is shown in Fig. 2. This figure illustrates the dependence of velocity field on the initial position of different fluid elements because each individual fluid element is characterized by one specific value of ξ. Thus, although the motion of each fluid element is periodic, the local time period of oscillations of various fluid elements situated at different initial positions is different. Therefore, ϕ is a periodic function of τ, where the time period is a function of $\bar x$, hence it can be written as

$$\phi = \phi\,[\bar x,\omega(\bar x)\tau]\,.$$

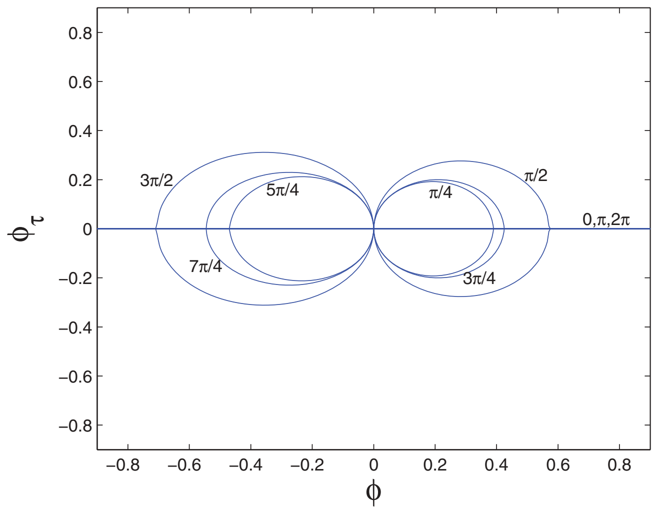

FIG. 2. (Color online) Phase-space diagrams $(\phi_{\bar{\tau}},\phi)$ for various values of $k\xi$ as indicated in the figure with $\delta = 0.45$, $\Delta = 0.2$, and $\alpha/k = 0.2$ (exact case).

Therefore,

$$\frac{\partial\phi}{\partial\bar{x}} = \frac{\partial\phi}{\partial\bar{x}} + \dot{\phi}\left[\tau\frac{d\omega(\bar{x})}{d\bar{x}}\right],$$

where the dot represents differentiation with respect to $\omega(\bar{x})\tau$. Because ϕ is periodic, its derivative is also periodic and goes through both signs. Therefore, at some finite τ, the denominator of Eq. (18) in the paper will become zero and n_e becomes singular. Thus space-dependent frequency leads to fine scale mixing of various parts of the oscillation, which results in a breakdown of the coherent oscillatory motion of fluid elements.

A. Approximate analysis

In the above section, we obtained an exact solution of the problem in parametric form. To make the problem physically more transparent, we now obtain an approximate solution for ϕ and then compare with the exact solution. This approximate analysis also yields an expression for phase-mixing time. Keeping up to the ϕ^2 term in Eq. (15), we obtain the following approximate equation:

$$\frac{\partial^2\phi}{\partial\bar{\tau}^2} + (1-\Delta)\phi + \Delta\cos(\bar{\alpha}\bar{x})\,\phi - \frac{3}{4}\Delta\bar{\alpha}\sin(\bar{\alpha}\bar{x})\,\phi^2 - \delta\bar{\omega}_p^2\sin\bar{x} = 0. \quad (22)$$

Now we solve Eq. (22) using the homotopy perturbation method [15,16] subjected to the following initial conditions: $\phi(\bar{x},0) = (\partial\phi/\partial\bar{\tau})\,|_{(\bar{x},0)} = 0$. The homotopy to this equation can be constructed as follows:

$$\frac{\partial^2\phi}{\partial\bar{\tau}^2} + \beta^2(1-\Delta)\phi + p\left[(1-\beta^2)(1-\Delta)\phi + \Delta\cos(\bar{\alpha}\bar{x})\phi - \frac{3}{4}\Delta\bar{\alpha}\sin(\bar{\alpha}\bar{x})\,\phi^2 - \delta\bar{\omega}_p^2\sin\bar{x}\right] = 0, \quad (23)$$

where p has been introduced as an embedding small parameter, $p \in [0,1]$. Notice that, when $p = 0$, Eq. (23) takes the form of a well known nonrelativistic linear harmonic oscillator equation, and if $p = 1$, it turns out to be the original Eq. (22). Looking for the periodic solution, we expand ϕ in powers of small parameter p as

$$\phi = \sum_{i=0}^{\infty} p^i\phi_i.$$

Substituting ϕ into Eq. (23), we collect various powers of p:

$$p^0: \ \frac{\partial^2\phi_0}{\partial\bar{\tau}^2} + \beta^2(1-\Delta)\phi_0 = 0, \quad \phi_0(\bar{x},0) = 0, \quad (\partial\phi_0/\partial\bar{\tau})\,|_{\bar{\tau}=0} = 0 \quad (24)$$

yields the zeroth-order solution $\phi_0(\bar{x},\bar{\tau}) = 0$;

$$p^1: \frac{\partial^2\phi_1}{\partial\bar{\tau}^2} + \beta^2(1-\Delta)\phi_1 + (1-\beta^2)(1-\Delta)\phi_0 + \Delta\cos(\bar{\alpha}\bar{x})\phi_0 - \frac{3}{4}\Delta\bar{\alpha}\sin(\bar{\alpha}\bar{x})\phi_0^2 - \delta\bar{\omega}_p^2\sin\bar{x} = 0,$$
$$\phi_1(\bar{x},0) = 0, \quad (\partial\phi_1/\partial\bar{\tau})\,|_{\bar{\tau}=0} = 0. \quad (25)$$

Substituting ϕ_0 in the above equation, the solution of Eq. (25) is

$$\phi_1 = \frac{f(\bar{x})}{\bar{\beta}^2}\{1-\cos(\bar{\beta}\bar{\tau})\}, \quad (26)$$

where $f(\bar{x}) = \delta(\bar{\omega}_p^2)\sin\bar{x}$ and $\bar{\beta} = \sqrt{1-\Delta}\beta$. The next-order equation in p is

$$p^2: \frac{\partial^2\phi_2}{\partial\bar{\tau}^2} + \beta^2(1-\Delta)\phi_2 + (1-\beta^2)(1-\Delta)\phi_1 + \Delta\cos(\bar{\alpha}\bar{x})\phi_1 - \frac{3}{4}\Delta\bar{\alpha}\sin(\bar{\alpha}\bar{x})\,(2\phi_0\phi_1) = 0,$$
$$\phi_2(\bar{x},0) = 0, \quad (\partial\phi_2/\partial\bar{\tau})\,|_{\bar{\tau}=0} = 0. \quad (27)$$

To remove secular terms in ϕ_2, we require a vanishing coefficient of $\cos\bar{\beta}\bar{\tau}$, which yields the characteristic frequency of oscillation as

$$\beta = \{1 + \bar{\Delta}\cos(\bar{\alpha}\bar{x})\}^{1/2}, \quad (28)$$

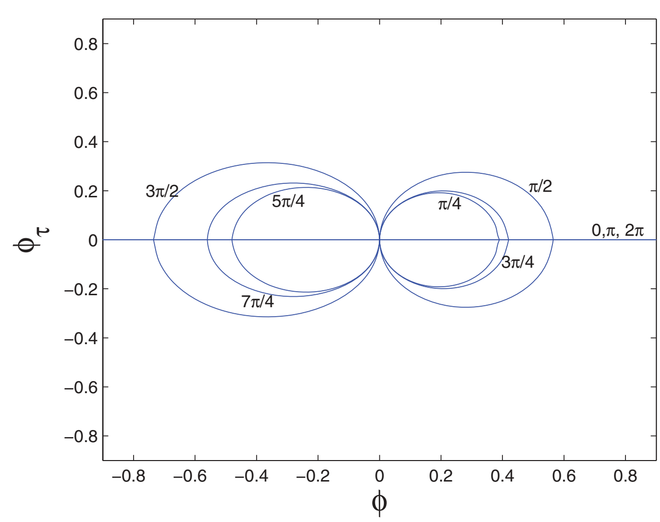

FIG. 3. (Color online) Phase-space diagrams $(\phi_{\bar{\tau}},\phi)$ for various values of $k\xi$ as indicated in the figure with $\delta = 0.45$, $\Delta = 0.2$, and $\alpha/k = 0.2$ (approximate case).

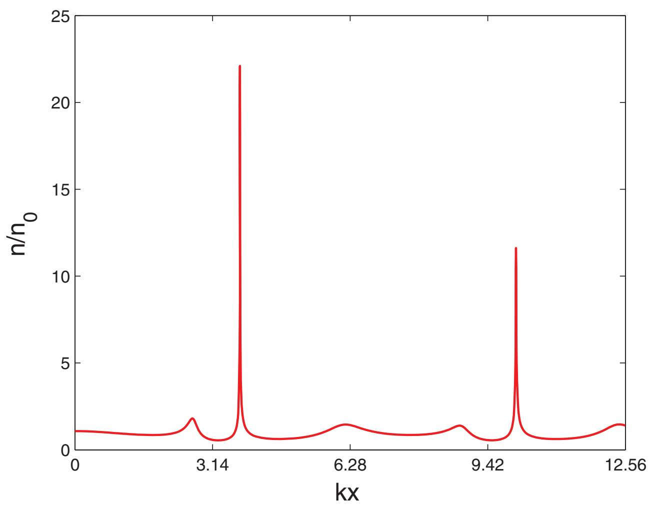

FIG. 4. (Color online) Normalized plasma density n/n_0 as a function of kx in the wave breaking situation, with $\delta = 0.45$, $\Delta = 0.2$, and $\alpha/k = 0.2$ (nonlinear approximate case), corresponding to the wave breaking time $\omega_{uh0}t_{\rm wb} = 54.7$.

where $\bar{\Delta} = \Delta/(1-\Delta)$. Now the solution of Eq. (23) up to first order can be written as

$$\phi = \lim_{p\to 1}[\phi_0 + p\phi_1] = \frac{f(\bar{x})}{\bar{\beta}^2}\{1-\cos(\bar{\beta}\bar{\tau})\}, \qquad (29)$$

where β is given by Eq. (28), which clearly shows spatial dependence of frequency, indicating phase mixing of upper hybrid oscillations. Now using the above expression for ϕ, the equation for phase-space curves is given by

$$\left(\frac{\partial\phi}{\partial\bar{\tau}}\right)^2 + (1-\Delta)\left\{1+\bar{\Delta}\cos(\bar{\alpha}\bar{x})\right\}\phi^2 - 2\delta\bar{\omega}_p^2\phi\sin\bar{x} = 0, \qquad (30)$$

which is an approximate form of Eq. (16). Figure 3 shows the phase-space plot as governed by Eq. (30) for the same values of parameters as used in Fig. 2. This shows that Eq. (30) is indeed a good approximation to Eq. (16), and hence can be used with confidence for small values of (α/k). Using the above approximate expression for ϕ, the electron density and coordinate transformation relation can be expressed as

$$n_e(\bar{x},\bar{\tau}) = n_0(1+\delta\cos\bar{x})\left[1+\frac{1}{\bar{\beta}^2}\frac{\partial f}{\partial\bar{x}}\{1-\cos(\bar{\beta}\bar{\tau})\} + \bar{\tau}\frac{f}{\bar{\beta}^2}\frac{\partial\bar{\beta}}{\partial\bar{x}}\sin(\bar{\beta}\bar{\tau}) - 2\frac{f}{\bar{\beta}^3}\frac{\partial\bar{\beta}}{\partial\bar{x}}\{1-\cos(\bar{\beta}\bar{\tau})\}\right]^{-1}, \qquad (31)$$

$$kx = \frac{f(\bar{x})}{\bar{\beta}^2}\{1-\cos(\bar{\beta}\bar{\tau})\} + \bar{x}. \qquad (32)$$

The phase-mixing time (i.e., the time at which the wave breaks, $\tau_{\rm wb}$) is given by the zero of the denominator of the density equation and is approximately given by

$$\omega_{uh0}\tau_{\rm wb} \simeq \frac{2\sqrt{1-\Delta}}{\Delta}\frac{(1-\Delta)(1+\bar{\Delta}\cos 2\alpha\xi)^{3/2}}{\delta(1-2\Delta)(2\sin k\xi\sin 2\alpha\xi)}\left(\frac{\alpha}{k}\right)^{-1}. \qquad (33)$$

The normalized electron density is plotted against kx in Fig. 4 corresponding to the approximate solution. We see that density spikes appear indicating breaking of oscillations in a time $\omega_{uh0}\tau_{\rm wb}$. Notice that the wave breaking time is inversely proportional to the ratio (α/k). For $\alpha = 0$ (or, equivalently, $L_B \to \infty$), the wave breaking time becomes infinity, which corresponds to coherent oscillations in the presence of a constant magnetic field. Therefore, we conclude that upper hybrid modes always phase-mix and eventually break in the presence of a finite-amplitude inhomogeneous magnetic field.

IV. SUMMARY

In summary, we have investigated the space-time evolution of upper hybrid modes in a cold plasma with an immobile homogeneous neutralizing ion background in the presence of an externally applied inhomogeneous magnetic field. Our analysis shows that upper hybrid oscillations in the presence of an inhomogeneous magnetic field phase-mix and eventually break. Physically, the magnetic field inhomogeneity causes the upper hybrid frequency to acquire a spatial dependence. As a result, different parts of the upper hybrid oscillation oscillate at different local frequency, causing a mixing of phases of the neighboring oscillators. A manifestation of this process is seen in mode coupling, where, as time progresses, energy flows from the initially excited mode "k" to higher modes $k \pm l\alpha$, where α is the inverse of the magnetic field inhomogeneity scale length. Finally, a time comes when neighboring fluid elements cross, resulting in a spike in the density profile. Nonlinear analysis of the problem gives an estimate of the phase-mixing time, which scales with the perturbation amplitude δ and the inhomogeneity scale length α as $\omega_{uh0}\tau_{\rm wb} \simeq \frac{2\sqrt{1-\Delta}}{\Delta}\frac{(1-\Delta)(1+\bar{\Delta}\cos 2\alpha\xi)^{3/2}}{\delta(1-2\Delta)(2\sin k\xi\sin 2\alpha\xi)}(\frac{\alpha}{k})^{-1}$. We note that inclusion of thermal effects, collisional drag, or viscous damping might remove the singularity in the density profile [17]. We would further like to add that, to our knowledge, there is no specific experiment that has examined the breaking of upper hybrid modes in a plasma in an inhomogeneous magnetic field. It would be interesting to carry out experiments in this direction and look for signatures of wave breaking of upper hybrid modes, such as a burst of energetic electrons.

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