PHYSICAL REVIEW E **86**, 016104 (2012)

Relationship between clustering coefficient and the success of cooperation in networks

M. N. Kuperman and S. Risau-Gusman
Consejo Nacional de Investigaciones Científicas y Técnicas, Buenos Aires, Argentina and Instituto Balseiro, Centro Atómico Bariloche,
8400 San Carlos de Bariloche, Rio Negro, Argentina

(Received 22 December 2011; published 10 July 2012)

In recent years the prisoner's dilemma has become a paradigm for the study of the emergence of cooperation in spatially structured populations. Such a structure is usually assumed to be given by a graph. In general, the success of cooperative strategies is associated with the possibility of forming globular clusters, which in turn depends on a feature of the network that is measured by its clustering coefficient. In this work we study the dependence of the success of cooperation on this coefficient for regular networks. Additionally, for both stochastic and deterministic dynamics we show that there is a strong dependence on the initial composition of the population. This hints at the existence of several different mechanisms that could promote or hinder cluster expansion. We have studied in detail some of these mechanisms by concentrating on completely ordered networks (large clustering coefficient) or completely random networks (vanishing clustering coefficient).

DOI: 10.1103/PhysRevE.86.016104 PACS number(s): 89.65.−s, 89.75.Fb

I. INTRODUCTION

The emergence of cooperation in different real systems has been puzzling researchers in several areas devoted to the study of systems involving social, economic, or biological organization. Even though each of these systems is conformed by single units with natural competitive tendencies, the emergence of collective behaviors is undeniable. While natural selection operates through competition, cooperation is essential to the evolution and emergence of higher degrees of complexity. The struggle between competition and cooperation is then one of the keys in understanding the self-organization of complex systems conformed by interacting units. Still, many questions arise regarding how such opposites forces can coexist. The first models, based on utilitarian precepts ruled by the natural selection or the rational choice, showed that the cooperative behavior can only be a very unstable behavioral strategy, easily invaded by a more exploitative attitude. Despite the conclusive arguments against the possibility of steady cooperation expressed by these works, the ubiquity of cooperation in nature confronted those theoretical results [1].

The survival of the cooperative behavior turned then into a classical problem of game theory. In this context, the paradigmatic prisoner's dilemma game [2] has been widely studied in different versions. It is usually formulated as a standard model for the confrontation between cooperative and selfish behaviors. For many years it was implemented in zero-dimensional systems, where every player can interact with any other, until the crucial effects of spatial distribution were finally noticed [3,4]. Since then, several mechanisms for the evolution of cooperation have been proposed. Some of them are summarized in Ref. [5]: kin selection, direct reciprocity, indirect reciprocity, group selection, and network reciprocity. Here we have chosen to focus on this last mechanism, which is associated with the fact that a cooperative individual can take advantage of the topology of the network to form clusters of cooperators that are often resilient to the invasion of defectors.

Studies about the effect of network reciprocity have dominated the literature on spatial distributed games in recent years [4,6–11] (see, e.g., Ref. [8], where many references can be found). These works stressed the need of studying the evolution of the strategies of players of a game beyond the simplifying assumption of a well-mixed population, where everybody interacts equally likely with everybody else. The observation that real populations are not well mixed and the fact that spatial structures could affect the evolution of a game and the strategies of the players demanded a new approach. A natural step was to consider complex networks as models for the underlying topology characterizing the spatial or social structures. In the case of a game played on top of a network or graph, the individuals of a population are located on the vertices of the graph. The edges of the graph determine the links through which individuals can interact. In a spatial model for the prisoner's dilemma, the players are classified either as cooperators or defectors and it is assumed that every agent can play with only its neighbors.

It has been shown that extremely simple rules determine whether network reciprocity can favor cooperation [10]. Here we concentrate on the relationship between the possibility of forming clusters of cooperators and the global success of cooperating strategies. This has been discussed and analyzed in many works [4,5,11–20]. In one of the pioneering works on spatially extended games [4] the authors have analyzed several shapes for a cluster of cooperators and tested the stability of each one against the invasion by defectors. They found that cooperators can survive and grow only if they form clusters. Other works that studied the clustering effect [5,12] have shown that, in certain conditions, cooperators can survive by forming clusters within which they benefit from mutual cooperation that in turn protects them from the exploitation by defectors near the borders of the cluster. However, an inverse relationship between the formation of clusters and the success of cooperation has also been reported [13]. Recently, the details of the effects of the topological aspect of the spatial distribution of players on the final outcome of an evolutionary game have been studied with increasing interest. In Refs. [14–16] the authors studied different scale-free networks with controlled clustering coefficient and found that while the increase of clustering favors the chance of survival of the cooperative behavior, the conclusions about the net effect of a higher clustering are not straightforward inasmuch as there are other aspect to be taken into account. In those works the authors suggested the idea of competing effects and mentioned the existence of an optimal clustering. Other authors [17–20] have studied different families of networks. In Ref. [17] the authors studied the evolution of a game in lattices and networks with several topologies, concluding that clustering is the factor that facilitates cooperation, but at the same time, when the connectivity of the network exceeds a certain threshold, cooperation cannot survive. The development of cooperation on small-world networks was studied in Ref. [18], where the authors also included noise effects. In Ref. [19] the authors worked with homogeneous small worlds, which exhibit a homogeneous connectivity distribution. These networks are analogous to those presented here and their construction will be explained in detail later. The rationale for the use of such networks is to separate the roles of heterogeneity and small-world effects on the dynamics of a game, concluding that they exhibit opposite behaviors in the evolution of cooperation. In Ref. [20] the authors studied the evolution of the prisoner's dilemma on two different types of homogeneous networks: the homogeneous small-world network and the regular ring graph. They verified the prevalence of a sort of resonancelike effect on the steady proportion of survivor cooperators as a function of the clustering coefficient, previously reported in Ref. [21].

In contrast, a few authors unveiled the fact that the outcome of many evolutionary games on networks is strongly correlated with the initial conditions [8]. In Ref. [22], for example, the authors study the evolutionary prisoner's dilemma game on scale-free networks, centering their analysis on the influence of the initial distribution of strategies. The authors study not only different initial proportions of cooperators but also several types of initial distributions for defectors, located according the degree of the nodes. They show how the initial configurations can determine the steady cooperation level.

All these works teach us that even though we are not in a position to describe with full quantitative detail the path to cooperation in a real complex system, it is still possible to show how cooperation can be macroscopically stable. However, in general, the systems are characterized by using the average of the fraction of cooperation present in the steady state. It is not clear that this average is representative of what happens in individual runs.

In this article we will address some of the problems mentioned above, but from a microscopic point of view. We show in detail the process behind the fact that although the possibility of forming a cluster in some networks is directly related to their clustering coefficients C, the cooperation success does not increase monotonically with it. We recall that the link between the probability of cluster formation and clustering coefficient is not a universal property of networks in general. Lattices, for example, are a well-known exception.

We have studied only networks that have the same degree distribution but differ on their clustering coefficient in order to isolate as much as possible the contribution of this last quantity. We have found not only that the equilibrium fraction of cooperators is a nonmonotonic function of C, but also that the strong dependence on the composition of the initial population acquires a new significance. We have found that for an initial distribution with few cooperators, the average values are not enough to characterize the typical behavior of the system. Furthermore, we explain in which way the initial distribution of cooperators is crucial in defining the future behavior of the system and show which are the underlying dynamic mechanisms. We show that the nonmonotonicity in the dependence of the clustering coefficient can be explained as the combination of two competing mechanisms that do depend monotonically on it but in opposite ways. A theoretical approximation allows us also to study the dependence of these mechanisms on the initial condition.

TABLE I. Payoff table for the prisoner's dilemma: The strategy in each row gets the payoff given by the table when playing again the strategies in the columns.

	C	**D**
C	r	s
D	t	p

Recent experiments have shown that humans do not always update their strategies by imitating the strategy of their most successful neighbors. In the last section we briefly discuss the success of cooperation as a function of C for players that have a given probability p_r of randomly changing their strategy. We show that even though when p_r is close to 1 the success of cooperation evidently does not depend on the network (as all systems end up with roughly half of the population cooperating at any given time), for rather large values of p_r there is still a strong dependence on C of the number of cooperators in the steady state.

II. MODEL

The prisoner's dilemma is a caricature of a real situation in which selfish and altruist tendencies compete. It has been the subject of study of game theory for the past 60 years [2,5,23]. Its name and formal elaboration is attributed to Tucker, who mentioned it in a classroom in 1950, but it was not until 1952 that the first results about it were published [24].

The formulation of the prisoner's dilemma as a game is rather simple. It is played by two players who must choose their moves between two strategies: to cooperate C or to defect D. The reward, or *payoff*, obtained by each player after one round of the game is given by Table I. Each element in the payoff matrix represents the payoff of a player using the strategies in the rows and that when confronting a player choosing the strategies in the columns. A defector **D** receives t, the temptation to defect, when its opponent is a cooperator **C**, who in turn gets s, the sucker's payoff. In the case of mutual cooperation, each player obtains a reward r, while mutual defection punishes both players with the payoff p. The payoffs of the prisoner's dilemma must satisfy the additional constraints $t > r > p > s$ and $2r > t + s$. Other relationships between the parameters define the snowdrift and stag hunt games [8].

In some versions of the game a different set of parameters is used ($r = c - b$, $s = -b$, $t = c$, and $p = 0$ [10]) to account for a slightly different interpretation of the game: A cooperator C is someone who pays a cost c for any other individual to receive a benefit b. In turn, a defector does not distribute any

TABLE II. Reduced payoff table for the prisoner's dilemma: The strategy in each row gets the payoff given by the table when playing again the strategies in the columns.

	C	**D**
C	1	$1 - t$
D	t	0

benefits and gets those delivered by the cooperators at no cost. To simplify the analysis, in the following we use a reduced version of the payoff table (Table II), which has only one free parameter. It has been shown that this parameter reduction preserves the most relevant features of the original prisoner's dilemma [4].

In order to study the possibility that the players can change their strategies as a result of their previous interactions, thus generating an evolutionary dynamics of strategies, many authors started to work with the iterated prisoner dilemma, in which players interact by iteratively playing the game several times. The history of successes or failures of each player is recorded in what is called the cumulative payoff. How the players use the information accumulated in their own and others' cumulative payoffs is what defines the rules of evolution. Operationally, the evolutionary dynamics acts at a certain instance of the game, for example, after everybody has played against everybody else, when players decide whether or not to change strategies, following certain update rules. Before all the players start again playing the game, all the cumulative payoffs are set to 0. The spectrum of rules of evolution is wide and ranges from purely deterministic to purely stochastic dynamics [10,25–28].

Complementary to the evolutionary aspects mentioned above, many authors started to analyze spatial games in order to cope with the limitations associated with the assumption that players were always part of a well-mixed population. [3,4,8]. The evolutionary behavior of the populations of surviving strategies of spatial games on networks can be affected by several features of the underlying topology such as the degree distribution of the graph, the average distance between nodes, or the clustering coefficient [6,7,11,27,29].

The concept that cooperators can survive by grouping in clusters has been discussed and analyzed in many works [4,10–20]. Intuitively, the reasoning goes as follows. The effect of the cluster would be to screen the nodes interior from the presence of defectors. As defectors can only get an advantage from their interaction with cooperators, only those located next to the border of a cluster of cooperators should collect any benefits. In turn, although the cooperators at the border of the cluster should have lower payoffs because of their interaction with defectors, their cooperator neighbors at the interior of the cluster should perform better than the defectors at the border. Thus, imitating the internal cooperators should always be more convenient than imitating the bordering defectors, which should lead to the survival, and even expansion, of the cluster of cooperators. The problem is that all these arguments, as well as the very definition of "cluster," depend crucially on the structure of the network. The most relevant feature in this regard is the clustering coefficient C, which measures how connected the neighborhood of each node is, on average. The existence of local transitive relationships, closely related to the clustering [31], is what defines the possibility of survival of small clusters of cooperators. Paradoxically, it will also be responsible for the negative effect that an isolated cooperator may have on incipient cooperative clusters.

Here we use the definition of global clustering coefficient of Watts and Strogatz [30]. For each node i, its local clustering coefficient is defined as the quotient between the number of links joining nodes of the neighborhood of i divided by the total number of possible links $(k_i - 1)k_i/2$. Then C is defined as the average over i of all local clustering coefficients. We study the influence of C on the evolutionary dynamics of the iterated prisoner's dilemma, but keeping the degree distribution constant, to disentangle both contributions. Furthermore, we analyze regular networks (i.e., with the same number of neighbors for every node) with different values of C, generated with the following algorithm. Starting from an ordered network (defined below), we select at random two pairs of connected nodes. Then we "cut" both connections and connect each individual to one of the individuals it had not been connected to before. In other words, the connections are swapped. If this change gives a network with smaller C, it is accepted and the network is updated. If it does not decrease C, the change is only accepted with a fixed (and typically small) probability. This process goes on until the clustering coefficient has reached the desired value. Note that this procedure leaves the degree distribution of the original network unchanged. When the desired clustering coefficient is very low, it is to be expected that the resulting network is very close to a regular random network, independently of the starting one. In contrast, if C is not small it is to be expected that the effect of the starting network is much larger. For this reason we use two different starting networks: ring networks, where each node is connected symmetrically to the closest k nodes, and two-dimensional lattice networks. The networks generated from these two classes are called, respectively, random ring networks or random lattice networks. Three different starting lattice networks are used: regular square lattices ($k = 4$), triangular lattices ($k = 6$), and square lattices where each node is connected to its Moore neighborhood ($k = 8$). For all values of k ring networks can be considered as one dimensional because for a given cluster of nodes the size of the surface is independent of the volume, whereas for lattice networks the relationship is $V \approx S^2$.

Throughout our simulations, we have considered two types of evolutionary dynamics, one deterministic [8] and the other stochastic [10]. In both cases each player either copies the strategy of one of its neighbors or sticks to the same strategy used in the previous round. In the deterministic dynamics each player copies the strategy of its most successful neighbor if the payoff of that neighbor is larger than its own. In the probabilistic dynamics, previously used in Ref. [10], it copies the strategy of a neighbor chosen at random, with a probability proportional to its relative payoff. Its own strategy is also included in the pool of eligible strategies. As the results we have obtained are qualitatively the same for both types of dynamics, in the following we focus on the deterministic dynamics and comment briefly on the small differences obtained when using the stochastic dynamics.

III. NUMERICAL RESULTS

In all the cases we consider regular networks with 1000–5000 nodes with even degrees between 4 and 8. Simulations with larger systems are computationally much more costly. Nevertheless, we have performed some simulations with more than 100 000 agents and we have checked that the curves obtained are qualitatively very similar. In contrast, for systems with approximately 100 agents the variation between systems becomes so great that averages do not give any meaningful information.

The state of the nodes is synchronically updated and the payoff of each player is not cumulative in time. As we are interested in the dependence on the initial conditions we use two different initial concentrations of cooperators $\rho_c(0) = 0.1$ and 0.5 for every network analyzed in this paper. A thorough analysis of the full range of initial condition is presented in next section.

If the equilibrium value of ρ_c is plotted as a function of the payoff parameter t, leaving all the other parameters constant, a piecewise constant function is obtained as shown in Fig. 1. This has also been previously noticed [29], but with a different payoff table (in the case considered in Ref. [29] a cooperator gets 0 payoff when playing against a defector). To understand the origin, and quantify the limits, of these steps, we must consider the necessary conditions for the propagation of the cooperating behavior. For a cooperator to have a chance to turn a defecting neighbor into a cooperating one, its payoff should be at least larger than that of the defecting neighbor. This leads to the condition $n_{CC} + (k - n_{CC})(1 - t) > n_{DC}t$, where n_{CC} is the number of cooperator neighbors of the cooperator and n_{DC} is the number of cooperator neighbors of the defector. The condition on t can be written as $t > k/(k - n)$, where $n = n_{CC} - n_{DC}$. Note that, as $n_{CC} \leqslant k - 1$ and $n_{DC} \geqslant 1$, n is a natural number that must satisfy $1 \leqslant n \leqslant k - 2$. This gives a maximum of $k - 1$ possible steps. Note, however, that in some networks the range of possible values for n is smaller and therefore the number of steps of ρ_c is at most $k - 2$. In general, for networks with the same number of k the number of possible steps will be smaller for the networks with smaller clustering coefficients. As an example consider the two extreme cases of a tree and a lattice network with $k = 8$: Whereas the tree has the maximum possible of steps, the lattice network can have at most four steps. In all cases the last step corresponds to $\rho_c = 0$ because for those values of t a cooperator, regardless of the composition of its neighborhood, is not able to turn a defecting neighbor into a cooperating one. Furthermore, it is also possible that, because of geometrical constraints, ρ_c also vanishes for other steps. For the networks analyzed in this paper, we have confirmed that only the height of the steps depends on C. Furthermore, simulations show that only for the first two steps is the final number of cooperators nonvanishing (see Fig. 1). For these reasons we have only analyzed the dependence of ρ_c in these first two steps, i.e., we have used only two values of t, t_1, and t_2, which satisfy $1 < t_1 < k/(k-1)$ and $k/(k-1) < t_2 < k/(k-2)$.

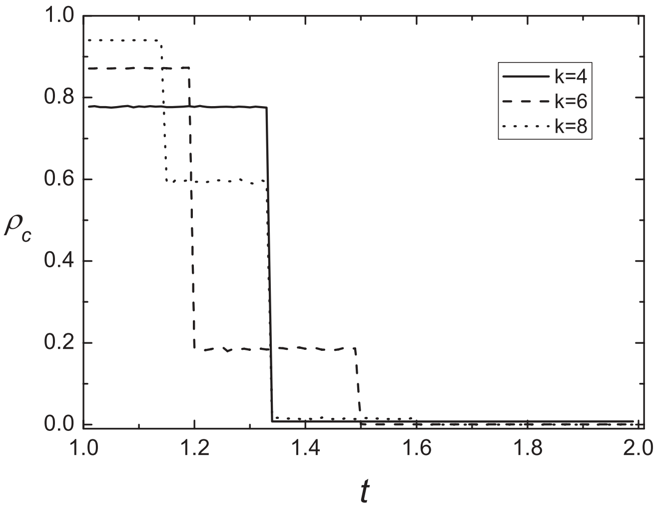

FIG. 1. Steady cooperator density ρ_c as a function of the parameter t for three different networks: $k = 4$ (lattice), a random lattice with $C = 0.2$ starting from a lattice with $k = 6$, and a random lattice with $C = 0.2$ starting from a lattice with $k = 8$.

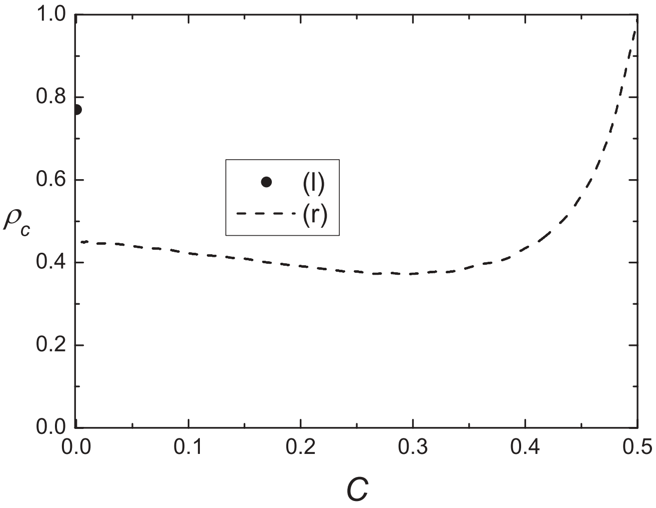

FIG. 2. Steady cooperator density ρ_c as a function of the clustering coefficient C for $k = 4$ and $\rho_c(0) = 0.5$. In the legend (l) and (r) refer to lattice and ring networks, respectively.

In Figs. 2–5 we plot the numerical results obtained from computational simulations with 1000–5000 agents. Each curve corresponds to the average fraction of cooperators in the steady state as a function of the clustering of the networks. The highest clustering value corresponds to the ordered network (lattice or ring) and networks get increasingly disordered as C is decreased.

We begin by analyzing what happens for evolutions whose initial state consists of the same number of cooperators and defectors [i.e., $\rho_c(0) = 0.5$]. In this case, there is always a nonvanishing fraction of agents that are cooperators in the steady state. As can be seen in Figs. 2 and 3, there are some features that are common to all the families of networks analyzed. The first is that, for each class of network, the behaviors of the curves are qualitatively the same for the two values of t used. The only difference is that, as is to be expected, curves for t_1 are below curves for t_2. Another important feature is that the final fraction of cooperators for ordered networks (largest clustering coefficient) is always larger than what is obtained in completely random networks (smallest clustering coefficient). Even though this seems to confirm the

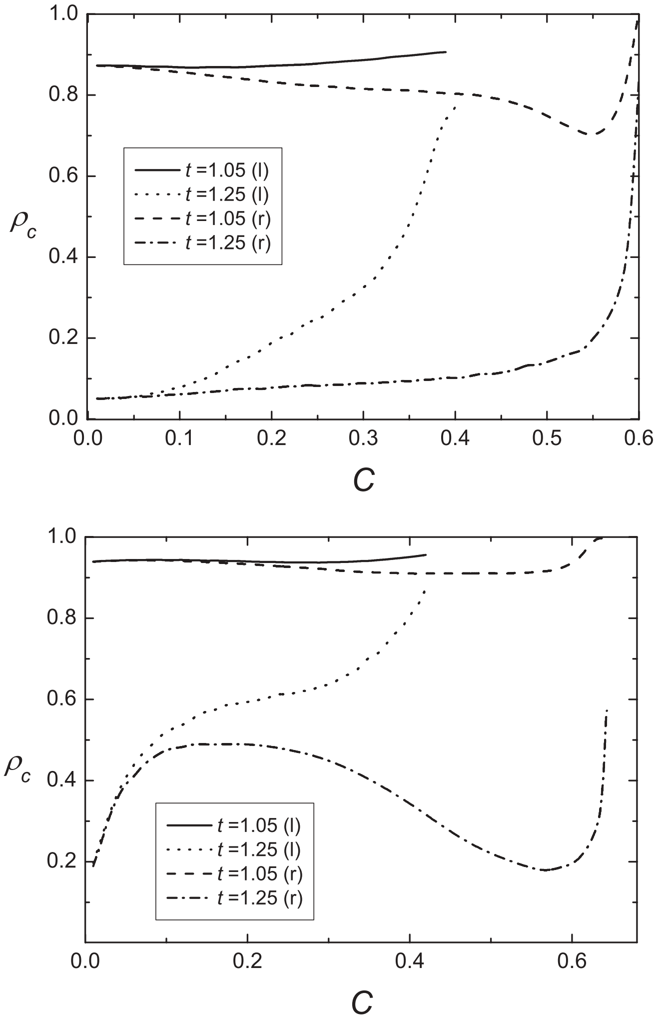

FIG. 3. Steady cooperator density ρ_c as a function of the clustering coefficient C for $\rho_c(0) = 0.5$ and $k = 6$ (top) and $k = 8$ (bottom). In the legend (l) and (r) refer to lattice and ring networks, respectively.

impression that clustering is beneficial to cooperators, it must be noted that some curves are not monotonic with C, such as all curves corresponding to random ring networks. This is in correspondence with previously reported results [20,21], where the existence of an optimum disorder value for the prevalence of cooperation was observed.

Another interesting feature to note is that for ring networks the addition of a very small amount of disorder causes an abrupt decrease in the steady fraction of cooperators. This happens because of the one-dimensional nature of the ring: Rewiring very few links at each side of a cluster of cooperators can be very effective in stopping its expansion. When more links are rewired the dimensionality of the system begins to increase and these clusters find new directions to expand.

For all values of C curves for random ring networks are always below those for random lattice networks for the same values of t. This is probably related to the lower dimensionality of the substrate of the random ring network that may have an influence even for high values of the disorder. Notice that the curves only overlap for very small values of the clustering coefficient. This means that a large amount of disorder is needed for the network to "forget" the starting substrate.

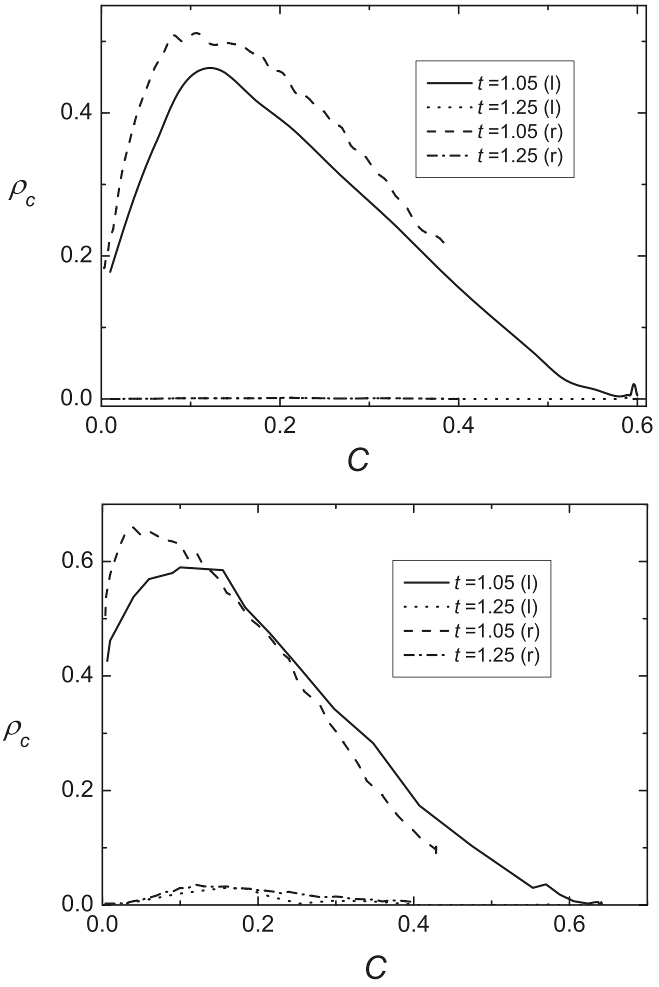

FIG. 4. Steady cooperator density ρ_c as a function of the clustering coefficient C for $\rho_c(0) = 0.1$ and $k = 6$ (top) and $k = 8$ (bottom). In the legend (l) and (r) refer to lattice and ring networks, respectively.

In Fig. 2 only one point is shown for random lattice networks because both the square lattice and the completely random network with $k = 4$ have a vanishing clustering coefficient. The large difference seen in Fig. 2 between the steady state fraction of cooperators could be attributed to the much shorter minimal distances between nodes in random regular networks (which have a diameter proportional to $\log_{10} N$ [32]) or to the presence of short loops in the square lattice (see the next section).

When the initial state is composed of few cooperators, the curves for the average fraction of cooperators are very different from what is obtained when the initial state has many cooperators. Figure 4 shows the steady cooperator density for $\rho_c = 0.1$. The same happens when cooperators are placed in more complex networks [16]. In our case, we see that for small values of t the success of cooperation, measured as the average of the fraction of cooperators in the steady state, depends strongly on the clustering coefficient. Furthermore, this dependence is not monotonic and seems to be similar for several networks, as Fig. 4 shows. Qualitatively, the same features are observed when the dynamics is replaced by a stochastic one, as Fig. 5 shows. In the stochastic dynamics we have used, the agents choose the strategy of a neighbor with

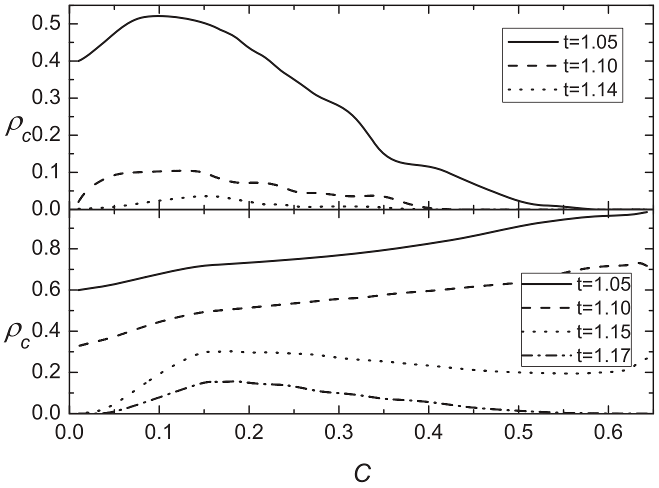

FIG. 5. Average value of the steady-state fraction of cooperators for a stochastic evolutionary dynamics for several values of t for random ring networks with $k = 8$ and for $\rho_c(0) = 0.1$ (top) and $\rho_c(0) = 0.5$ (bottom).

a probability proportional to the corresponding cumulative payoff, but only if it is larger than their own cumulative payoff [10]. Notice that in this case the number of final cooperators depends continuously on t.

To understand the differences that appear when different initial concentrations of cooperators are used, it is useful to go beyond mean values and have a look at the whole distribution of final states. Figure 6 shows the values obtained in each realization for a network with $k = 6$ for both initial conditions. For $\rho_c(0) = 0.5$ the steady fractions of cooperators obtained are clustered about their average values, as expected. However, for $\rho_c(0) = 0.1$ the values are clustered into two clearly separated groups. Furthermore, histograms reveal that the lower group has in fact two components, one centered at a low value of ρ_c and the other composed by realizations where the cooperators have become extinct. Thus the simple average is not representative of the steady states of the system. In fact, if the average is taken only over the realizations whose steady state has a significant fraction of cooperators, the curve is very similar to what is obtained for $\rho_c(0) = 0.5$ (compare the upper group of points for $\rho = 0.1$ with the points for $\rho = 0.5$ in Fig. 6). The same qualitative features are found for different networks with $k = 4$ and 8 and also when the deterministic dynamics is replaced by a stochastic one (compare the left and right panels of Fig. 6).

From the fact that the possible final states are divided into two groups and that the average number of cooperators of the second group is roughly independent of the initial condition, we infer that the relevant information regarding initial conditions and clustering lies in the probability of a

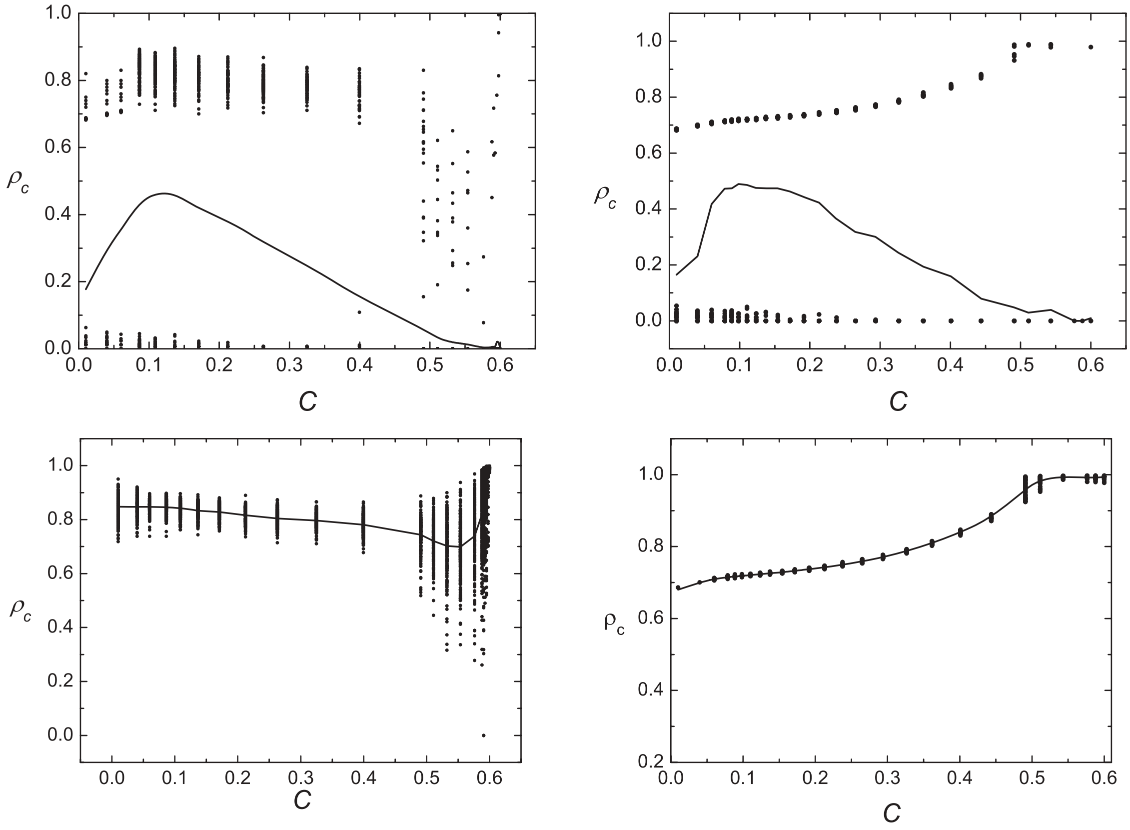

FIG. 6. Each point in the figures represents the fraction of cooperators in the steady state, starting from a state where each agent has a probability of 0.1 (top panels) or 0.5 (bottom panels) of being a cooperator, as a function of the clustering coefficient of otherwise random ring networks with $k = 6$. The dynamics used is both deterministic (left panels) and stochastic (right panels). The lower curve in each panel gives the average number of cooperators, whereas the upper curve represents the average of cooperators taken only over the points of the upper group.

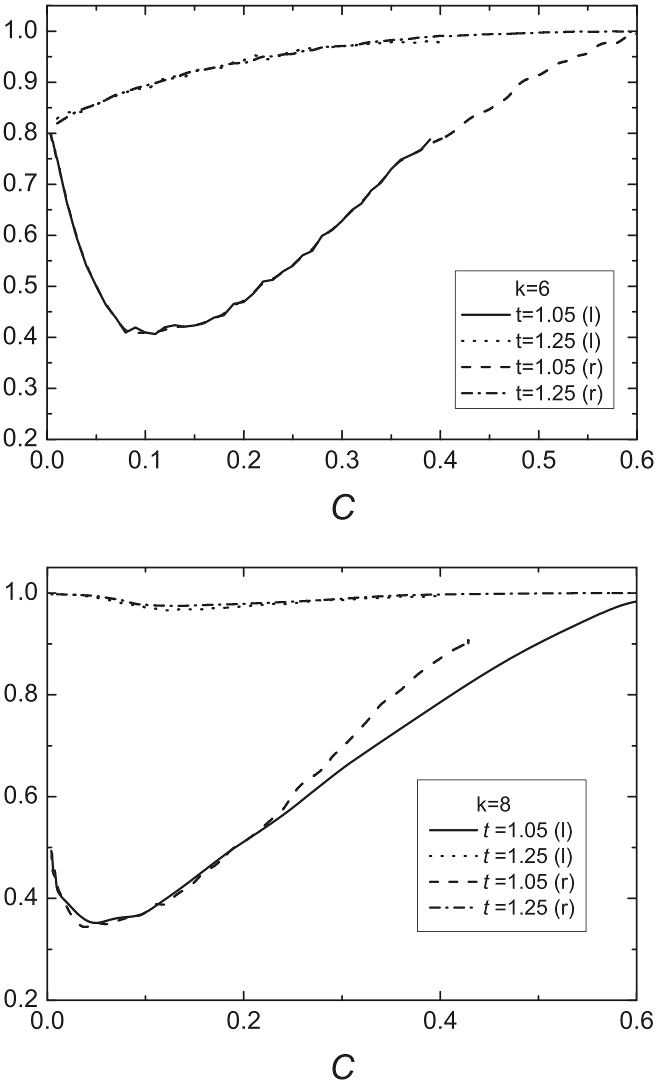

FIG. 7. Fraction of realizations that converge to a steady state dominated by defectors for networks with $k = 6$ (top panel) and $k = 8$ (bottom panel).

random system of ending up in each group. Figure 7 shows the probability of ending up in the lower group, that is, in a state dominated by defectors. The dependence on the clustering coefficient is again nonmonotonic and there is a value of C that is optimal for the success of cooperation. As shown in the next section, this nonmonotonicity can be understood as the competition between two mechanisms by which clustering either promotes or hinders cooperation.

IV. ROLE OF THE INITIAL FRACTION OF COOPERATORS

As shown in the preceding sections, there is no qualitative difference between the results for deterministic and stochastic dynamics. Thus, to understand the conflicting ways in which network clustering can affect cooperation, we concentrate in the following only on the deterministic dynamics, which is easier to analyze. We begin by considering what happens for networks at the two extremes: completely disordered (vanishing C) and completely ordered (large C) networks. For the latter class we concentrate on lattice networks with $k = 6$ and 8, which have clustering coefficients $C = 2/5$ ($k = 6$) and $3/7$ ($k = 8$).

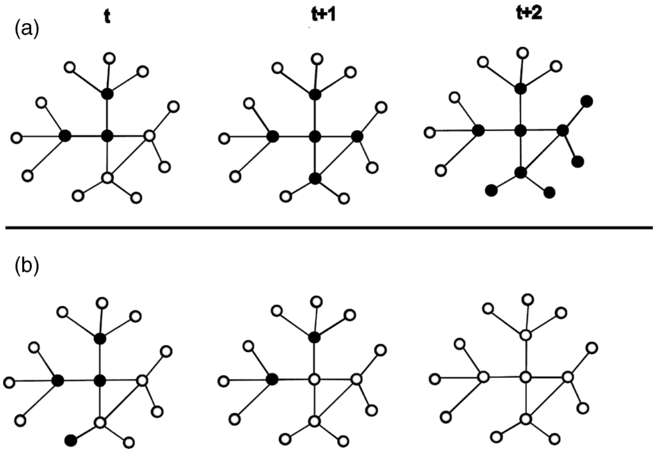

FIG. 8. Evolution of two different three-clusters in a random network with $k = 4$. Black circles represent cooperators and white circles represent defectors. (a) Expansion of a three-cluster. (b) Disappearance of a three-cluster.

First, we analyze the fate of a cluster of three cooperators. In the case of the lattice networks with $k = 6$ and 8, there are two and three possible configurations, respectively. However, all of them are unstable because, due to the large value of C, some defector neighbors of the cluster can be connected to two or three cooperators in the cluster, having thus a larger payoff than any of them. In contrast, in a random network there is a finite probability $[1 - 3k^2/N + O(k/N)]$ that all the neighbors of a three-cluster are not neighbors of more than one cooperator and thus the cluster is stable. Furthermore, if this cluster does not disappear it will grow to become a cluster of $k + 1$ cooperators (a central cooperator surrounded by cooperators), which we call a *star*. This cluster in turn has a nonvanishing probability of continuing its expansion. For example, if there is a link joining two of the new surrounding cooperators [which happens with probability $(k/N)k(k-1)/2$], the cluster grows by turning into cooperators the $2(k-2)$ noncooperating neighbors of the nodes that share the link [see Fig. 8(a)]. However, there is now a nonvanishing probability that there is also a link joining the nodes of the cluster "surface," which would lead to an increase of its size of $2(k-2)$. Further, in general, at any step of its growth it could keep growing with a probability roughly proportional to $(k/N)N_s 2(k-2)$, where N_s is the size of the cluster surface. Considering that most of the nodes of the clusters lie in its surface, this implies that once the cluster has reached a size of order $N/2k(k-2)$ it will keep growing until it spans the whole lattice. Thus, if we consider the evolution of all possible three-clusters, the distribution of final cluster sizes should be nonvanishing only for sizes $O(N)$ and for sizes smaller than $O(N/2k(k-2))$.

In the case of four-clusters, it is easy to see that the situation is qualitatively the same as for three-clusters in regular random networks. In contrast, for lattice networks the picture is completely different: Square four-clusters expand until they occupy the whole lattice because every cooperator is connected to two or three others, whereas noncooperating neighbors can only have at most two cooperating neighbors.

To understand the general case of a random initial state, we make the approximation of using the first two steps of the dynamics to predict whether the final state will be dominated by cooperators or defectors. For the sake of simplicity we consider only small values of the payoff t, but the analysis is basically the same for larger values of t. We begin by calculating the probability that a given cooperator survives the first step of a dynamics with an initial state with a probability p that each agent is a cooperator. In the range of values of t we are considering, a cooperator will survive the first step if it has more than $j \geqslant 2$ cooperating neighbors. Simultaneously, none of its neighbors can have more than $j-2$ cooperating neighbors (besides the cooperator whose fate we are analyzing). The probability of this is

$$P'(j,p) = \sum_{i=0}^{j-2} \binom{k-1}{i} p^i (1-p)^{k-1-i}. \qquad (1)$$

Using this, the probability of survival of a cooperator is

$$P_s(k,p,N) = \sum_{j=2}^{k} \binom{k}{j} p^j (1-p)^{k-j} [P'(j,p)]^{k-j}. \qquad (2)$$

Thus the probability that all cooperators disappear after the first step is

$$f_0(k,p,N) = [1 - p\, P_s(k,p,N)]^N. \qquad (3)$$

If at least some of the cooperators have survived the first step, it is most probable that they have turned their defecting neighbors into cooperators, thus becoming stars. If a couple of these newly generated cooperators happen to be joined by a link, they will most probably become stars themselves in the next step. As a first approximation we assume that if there exists at least one such link, the number of cooperators becomes large enough to guarantee that cooperators will dominate the population in the long run. Thus the fraction of realizations that lead to a state dominated by defectors, but with a few stable cooperators, is

$$f_1(p) = \sum_{i=1}^{pN} \binom{pN}{i} P_s^i (1-P_s)^{pN-i} \left(1 - \frac{k-1}{N}\right)^{ki(ki-1)/2}. \qquad (4)$$

Each addend gives the probability of having, after the first step, i stars such that the $k \times i$ *surface* cooperators are not connected among themselves. Figure 9 shows that Eqs. (3) and (4) are indeed good approximations for random networks with $k = 8$. For networks with a smaller number of neighbors the approximation of Eq. (4) is not so good and a term must be added to it to account for the possibility that having only one link joining two surface cooperators is not enough for the success of cooperation and that at least two such links are necessary. This approximation is very good even for networks with $k = 4$, as Fig. 10 shows. Using all this, the probability that the final state is dominated by defectors can be approximated by the sumof the two contributions mentioned above:

$$P(p) = f_0(p) + f_1(p). \qquad (5)$$

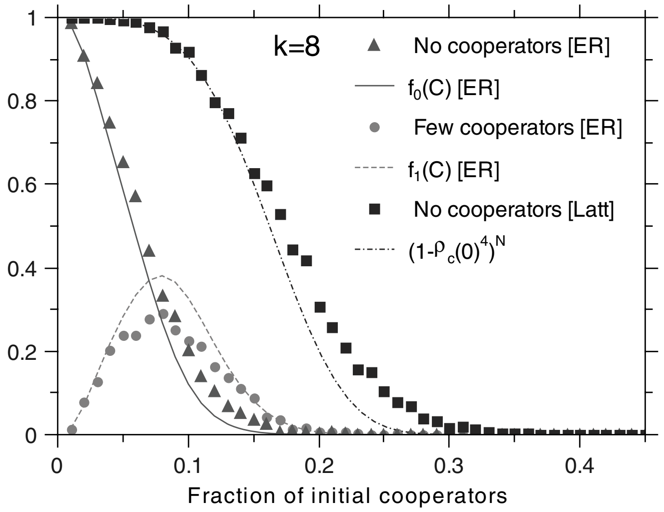

FIG. 9. Probability of having a steady state with no cooperators (solid symbols) or with a few stable cooperators (open symbols) as a function of the initial fraction of cooperators for regular random networks (circles) and lattice networks (triangles), with $k = 8$. The lines show the theoretical estimates, assuming independence (solid lines) or dependence (dashed lines) among all clusters of cooperators.

To make a comparison with ordered lattice networks, we assume that the steady state can only be dominated by defectors if there is no square four-cluster of cooperators. The probability of such a state is approximately $f_0(p) + f_1(p) = (1-p^4)^N$. For $k = 6$ and 8 we have seen that smaller clusters are not stable, which implies that $f_1(p) = 0$. In contrast, three-clusters are stable for $k = 4$ and thus $f_1(p) = [1 - (1-6p^3)^N](1-p^4)^N$. Figures 10 and 9 show that these expressions provide reasonable approximations.

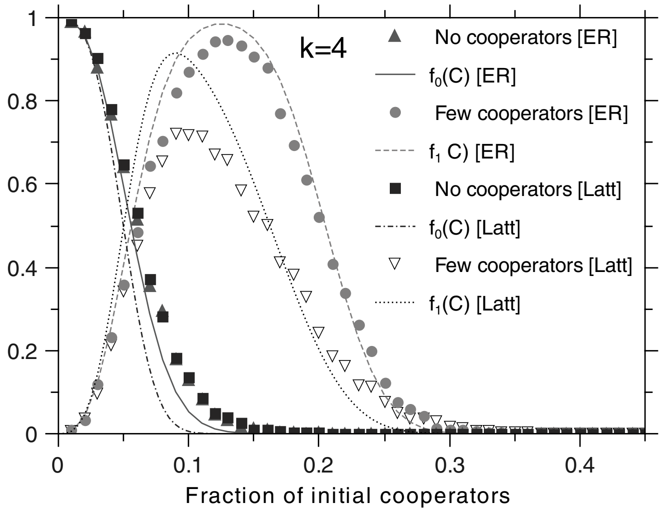

FIG. 10. Probability of having a steady state with no cooperators (solid symbols) or with a few stable cooperators (open symbols) as a function of the initial fraction of cooperators for random regular networks (circles) and lattice networks (triangles), with $k = 4$. The lines show the theoretical estimates assuming dependence among all clusters of cooperators.

V. APPROXIMATION FOR GRAPHS WITH FIXED CLUSTERING

In this section we generalize the approximations f_0 and f_1 to random networks with any given value of the clustering coefficient C. For this we pick a random node, which we call a *central* node, and consider the probability that a *fixed* neighbor is connected to nodes inside or outside the neighborhood. Considering that C is defined as the number of connections between neighbors of a given node divided by the number of possible connections between neighbors, we can make the approximation that the probability that the fixed neighbor is connected to another neighbor is C, whereas the probability of it being connected to a node outside the neighborhood of the central node is $(k-1)(1-C)$.

Thus the probability that the fixed neighbor is connected to l other neighbors of the central node is $\binom{l}{k-1}C^l(1-C)^{k-1}$. The probability of being connected to m of the j cooperators that are neighbors to the central node is $\binom{j}{m}\binom{k-1-j}{l-m}\binom{k-1}{l}^{-1}$. The probability that the remaining $k-1-l$ links are connected to $i-m$ cooperators is $p^{i-m}(1-p)^{k-1-l-(i-m)}\binom{k-1-l}{i-m}$. Thus, for graphs with fixed clustering the probability that each neighbor of a cooperator is connected to no more than $j-2$ other cooperators is

$$P'(j,C) = \sum_{i=0}^{j-2}\sum_{l=0}^{k-1}\sum_{m=0}^{\min i,j,l}\binom{j}{m}\binom{k-1-j}{l-m}\binom{k-1-l}{i-m} \times p^{i-m}(1-p)^{k-1-l-i+m}C^l(1-C)^{k-1-l}. \quad (6)$$

The probability that the population of initial cooperators becomes extinct after the first step is again given by Eq. (3), but with $P'(j)$ replaced by $P'(j,C)$. Figure 11 shows that for networks with $k=6$ and 8 and for $p=0.1$, the probability of extinction after the first step f_0 is an increasing function of C. To understand this, note that, given that the central node has at least two cooperating neighbors, its defecting neighbors are more likely to be connected to a cooperator when at least some of their connections go to a member of the central node neighborhood (which is favored by increasing values of C) and thus they are more likely to destabilize the central node.

To calculate the probability of reaching a steady state dominated by defectors but with a few stable cooperators, we make again the approximation that a population of stars will remain stable (i.e., it will not grow) only if none of the cooperators on the surface of stars is connected to another cooperator, either in the same or in a different star. Thus we get

$$f_1(p) = \sum_{i=1}^{pN}\binom{pN}{i}P_s^i(1-P_s)^{pN-i}\left(1-\frac{k-1}{N}\right)^{k^2 i(i-1)/2} \times(1-C)^{ik(k-1)/2}. \quad (7)$$

It is easy to see that the growth of the star population is favored by clustering. After the first evolution step, all the neighbors of the surviving initial cooperators have become cooperators themselves. Thus a large clustering coefficient makes it more likely that these neighboring cooperators are connected among themselves, which is the condition for the success of the star population.

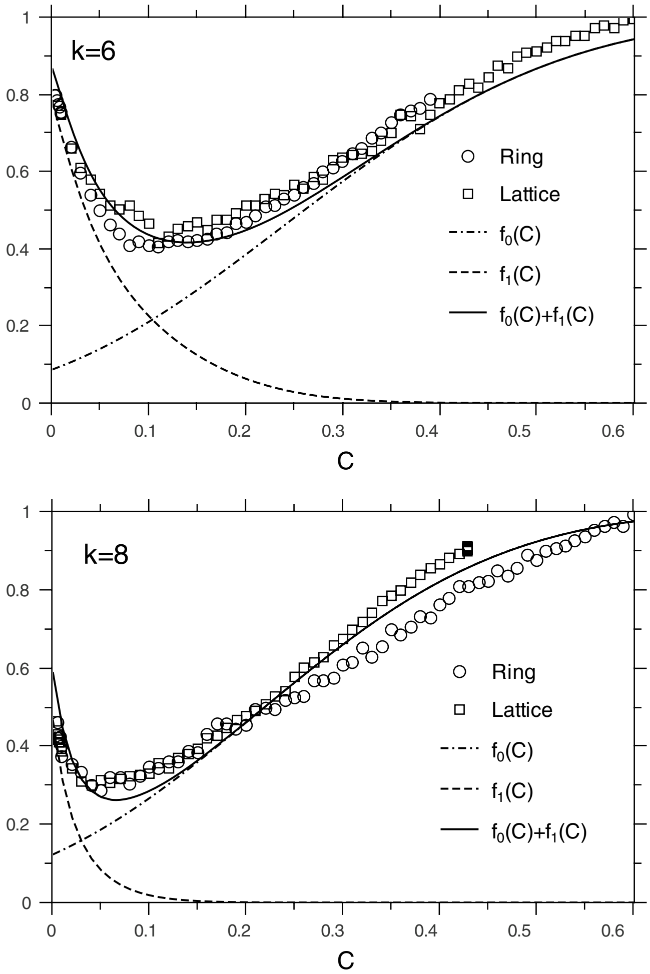

FIG. 11. Probability of having a steady state dominated by defectors as a function of the clustering coefficient C for networks with $k=6$ (top panel) and $k=8$ (bottom panel) for $\rho_c(0)=0.1$. The solid line represents the theoretical prediction $f_0(C)+f_1(C)$. The other lines represent the theoretical predictions for the probability of having a steady state with only a few cooperators (dashed line) or with no cooperators at all (dash-dotted line). In the legend (l) and (r) refer to lattice and ring networks, respectively.

Figure 11 shows that the approximations described above give reasonably accurate predictions. Note that even though they are being compared against networks that are obtained by rewiring substrates that are not random, the results of simulations seem to depend rather weakly on the substrate used.

VI. CONCLUSION

The most amazing result revealed by works dealing with the prisoner's dilemma is the spontaneous emergence and survival of cooperation, even in cases when common sense would have dictated otherwise. The findings of these surprising outcomes has sometimes discouraged the search for initial conditions that could lead to less surprising results, such as the prevalence of defection. This could be one of the reasons why in most of the literature dealing with the evolutionary prisoner's dilemma the initial condition of the simulation is a state where every agent has the same probability of being a defector or a cooperator (i.e., the initial fraction of cooperators is approximately equal to 0.5). However, in real situations the number of individuals of an invading species is much smaller than the aboriginal population. It thus makes sense to study what happens when the initial number of cooperators is relatively small. Our work shows that in this case the possible steady states are clustered in two groups: one in which the number of cooperators is much smaller than N and another in which the number of cooperators is "macroscopic" (i.e., proportional to N). This fraction does not seem to depend on the initial fraction of cooperators ρ_0. Instead, it is the probability of ending up in each group that depends on ρ_0. Thus we show that for these initial conditions, taking the average of the fraction of cooperators over all steady states gives a number that is not representative of the possible outcomes of the simulations. We have checked that this happens both for a stochastic and a deterministic rule and for several different networks. This suggests that our results could in principle be valid also for some other networks and update rules. Needless to say, it would not be reasonable to hope for universal validity of our results for practical settings, as the modeling of these often demand very specific networks and update rules.

The analysis of the influence of the initial conditions was complemented with the study of the effect of the clustering on the emergence of cooperation. It has been sometimes suggested that one of the possible reasons for the success of cooperating strategies in spatially structured populations is the possibility of forming globular clusters. In this way, cooperators inside the cluster are "protected" by the ones on the border. At the same time, the border cooperators need to get enough support from the inner cooperators to resist the defector invasion. When the populations is placed on a graph, the globularity of the possible clusters is proportional to the clustering coefficient. Therefore, cooperating strategies should be more successful in networks with large C than in networks with small C, with some pathological exceptions. For the evolutionary dynamics studied here we have seen that this is the case when we compare the steady fractions of cooperators in random regular networks (low C) and lattice networks (large C) having the same degree distributions. However, for graphs where the value of the clustering coefficient can be tuned we find that the equilibrium fraction of cooperators is not a monotonic function of C. While the occurrence of this effect has been previously noticed, its causes remained unknown.

The nonmonotonicity of the curves, together with the dependence on the initial condition, suggests that there might be several mechanisms that determine the success or failure of cooperation. In the preceding section we have shown that this is indeed the case: We have identified two different mechanisms whose dependence on C is very different. This is based on the observation that the systems that end up dominated by defectors can be divided into two families: the states in which all cooperators have disappeared and the states in which a few stable cooperators remain. We have found that in general the probability of ending up in the first group increases with C, whereas the probability of ending up in the second group decreases with C. In many of the cases analyzed this leads to a nonmonotonic dependence on C of the probability of having a steady state dominated by defectors, which is the sum of the two probabilities mentioned above. The numerical simulations presented in this work were backed up by analytical studies of the limiting cases. There is reasonably good agreement between the results obtained in both cases.

Recently, some experiments have been conducted with groups of people playing the prisoner's dilemma game [33,34]. As in the cases analyzed in this paper, the participants of these experiments were located in different lattices (in the sense that their interactions were restricted to a given set of neighbors). The results obtained in these works suggest that the adoption of a new strategy is not necessarily determined by an imitative behavior. Furthermore, it has been shown that sometimes the strategies of the individuals are actually randomly chosen [33], which in turn leads to results that seem to be network independent. However, it is not clear that the strategies used by a group of people playing a computerized game with a payoff of a few cents of euro can be generalized to other situations [35] in which food, land, or other resources might be at stake. Moreover, the prisoner's dilemma is considered to be a useful approach to study animal behavior [36], and experiments have been conducted with many species, such as rats [37], blue jays [38], zebra finches [39], and pigeons [40]. In animal species imitative behavior is very common and random choices are likely to play a less important role. These caveats, however, do not diminish the importance of the experiments mentioned, which provide much insight into alternative strategies.

In order to analyze how our results are affected by the presence of nonimitative behavior, we repeated our simulations allowing each player, with a probability p_r, to abandon the strategy of imitating the best and to choose a random strategy instead. Figure 12 shows the average number of cooperators for several values of p_r as a function of the clustering coefficient of the network. In the preceding sections we have shown that when starting from a small number of cooperating agents, purely imitative behavior leads to two groups of clearly

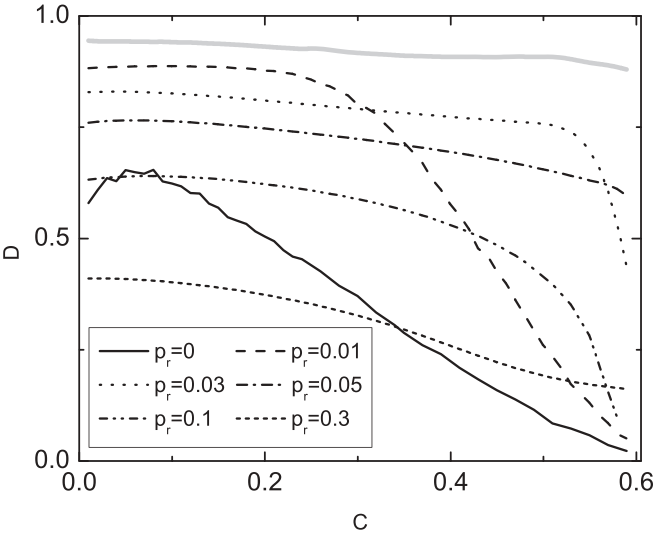

FIG. 12. Steady cooperator density ρ_c as a function of the clustering coefficient C for ringlike networks with $k = 8$ and $\rho_c(0) = 0.1$. Each curve corresponds to a different value of the probability of adopting a random strategy. The thick upper curve corresponds to the number of the cooperators for $p_r = 0$ but restricted to those realizations that end up with a significant number of cooperators.

different steady states: one in which the number of cooperators is large and one in which only a few stable cooperators remain. Figure 12 shows that when p_r is very small its only effect on the dynamics is to ensure that the steady state always belong to the group with a significant fraction of cooperators. For larger values of C the curve drops because p_r is not large enough to take the system out of the steady state with few cooperators. Evidently, this drop depends on the time limit that has been set for this simulation: If this time is increased the drop happens for larger values of C. Thus, for fixed times random strategies are less effective in networks with relatively large C (i.e., in more ordered networks).

For larger values of p_r ($p_r \gtrsim 0.07$) the number of cooperators in the steady state has a very marked dependence on C. Even though it is evident that for large values of ρ the curve should become flat (because the strategy becomes essentially random and therefore independent from the neighborhood), an important drop is still observed for values of p_r as large as 0.3, for which a system on an ordered network ends up with half the number of cooperators as on a completely random network.

It should be emphasized that when some noise is included in the dynamics, we are in the presence of a different actualization rule. It has been already shown that the addition of noise increased the cooperation levels observed in purely deterministic dynamics [41] and our results point in the same direction.

In summary, our results show that even under the presence of a random component in the behavior of the players, the topology of the network still plays an important role. Thus the results of the experiments of Traulsen *et al.* and Grujić *et al.* could probably be better explained by different deviations from the imitative behavior, such as the existence of "moody" strategists (who tend to repeat their strategies) and "pure" strategists (who tend to follow the same strategy most of the time) [34]. Nevertheless, the debate around the evidence for an imitative behavior is not conclusive inasmuch other works [42,43] do not find traces of such behavior. This fact opens an interesting debate about not only the character of the evolutionary dynamics of the strategies of the players but also about the validity of abstract mathematical models of the system.

---

[1] J. Maynard Smith, *Evolution and the Theory of Games* (Cambridge University Press, Cambridge, 1982).
[2] R. Axelrod and W. D. Hamilton, Science **211**, 1390 (1981).
[3] M. A. Nowak and R. M. May, Nature (London) **359**, 826 (1992).
[4] M. A. Nowak and R. M. May, Int. J. Bifurcat. Chaos **3**, 35 (1993).
[5] M. A. Nowak, Science **314**, 1560 (2006).
[6] G. Abramson and M. N. Kuperman, Phys. Rev. E **63**, 030901(R) (2001).
[7] M. N. Kuperman and S. Risau-Gusman, Eur. Phys. J. B **62**, 233 (2008).
[8] G. Szabó and G. Fáth, Phys. Rep. **446**, 97 (2007).
[9] E. Lieberman, C. Hauert, and M. A. Nowak, Nature (London) **433**, 312 (2005).
[10] H. Ohtsuki, C. Hauert, E. Lieberman, and M. A. Nowak, Nature (London) **441**, 502 (2006).
[11] C. P. Roca, J. A. Cuesta, and A. Sánchez, Phys. Rev. E **80**, 046106 (2009).
[12] M. Doebeli and C. Hauert, Ecol. Lett. **8**, 748 (2005).
[13] C. Hauert and G. Szabó, Am. J. Phys. **73**, 405 (2005).
[14] R. Cong, Y. Qiu, X. Chen, and L. Wang, Chin. Phys. Lett. **27**, 030203 (2010).
[15] G. Assenza, J. Gómez-Gardeñes, and V. Latora, Phys. Rev. E **78**, 017101 (2008).
[16] C. Lei, J. Jia, X. Chen, R. Cong, and L. Wang, Chin. Phys. Lett. **26**, 080202 (2009).
[17] M. Ifti, T. Killingback, and M. Doebeli, J. Theor. Biol. **231**, 97 (2004).
[18] J. Vukov, G. Szabó, and A. Szolnoki, Phys. Rev. E **77**, 026109 (2008).
[19] F. C. Santos, J. F. Rodrigues, and J. M. Pacheco, Phys. Rev. E **72**, 056128 (2005).
[20] G. Wu, K. Gao, H. Yang, and B. Wang, Chin. Phys. Lett. **25**, 2307 (2008).
[21] J. Ren, W. X. Wang, and F. Qi, Phys. Rev. E **75**, 045101(R) (2007).
[22] X. J. Chen, F. Fu, and L. Wang, Phys. Lett. A **372**, 1161 (2008).
[23] A. Rapoport and A. M. Chammah, *Prisoner's Dilemma* (University of Michigan Press, Ann Arbor, 1965).
[24] M. M. Flood, RAND Corporation, Report No. RM-789, 1952 (unpublished); Manage. Sci. **5**, 5 (1958).
[25] G. Szabó and C. Tőke, Phys. Rev. E **58**, 69 (1998).
[26] M. A. Nowak, and K. Sigmund, Science **303**, 793 (2004).
[27] L. G. Moyano and A. Sánchez, J. Theor. Biol. **259**, 84 (2009).
[28] C. Hauert and M. Doebeli, Nature (London) **428**, 643 (2004).
[29] O. Durán and R. Mulet, Physica D **208**, 257 (2005).
[30] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[31] S. Wasserman and K. Faust, *Social Network Analysis: Methods and Applications* (Cambridge University Press, Cambridge, 1994).
[32] B. Bollobás and W. F. de la Vega, Combinatorica **2**, 125 (1982).
[33] A. Traulsen, D. Semmann, R. D. Sommerfeld, H. J. Krambeck, and M. Milinski, Proc. Natl. Acad. Sci. U.S.A. **107**, 2962 (2010).
[34] J. Grujić, C. Fosco, L. Araujo, J. A. Cuesta, and A. Sánchez, PLoS ONE **5**, e13749 (2010).
[35] D. Helbing and W. Yu, Proc. Natl. Acad. Sci. U.S.A. **107**, 5265 (2010).
[36] N. J. Raihani and R. Bshary, J. Evol. Biol. **24**, 1628 (2011).
[37] D. S. Viana, I. Gordo, E. Sucena, and M. A. P. Moita, PLoS ONE **5**, 1 (2010).
[38] D. W. Stephens, C. M. McLinn, and J. R. Stevens, Science **298**, 2216 (2002).
[39] A. St-Pierre, K. Larose, and F. Dubois, Proc. R. Soc. London Ser. B **276**, 4223 (2009).
[40] L. Green, P. C. Price, and M. E. Hamburger, J. Exp. Anal. Behav. **64**, 1 (1995).
[41] C. P. Roca, J. A. Cuesta, and A. Sánchez, Europhys. Lett. **87**, 48005 (2009).
[42] O. Kirchkamp and R. Nagel, Games Econ. Behav. **58**, 269 (2007).
[43] A. Cassar, Games Econ. Behav. **58**, 209 (2007).