Enhanced response of regular networks to local signals in the presence of a fast impurity

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We consider an array of inductively coupled Josephson junctions with a fast impurity (a junction with a smaller value of the critical current) and study the consequences of imposing a small amplitude periodic signal at some point in the array. We find that when the external signal is imposed at the impurity, the response of the array is boosted and a small amplitude signal can be detected throughout the array. When the signal is imposed elsewhere, minor effects are seen on the dynamics of the array. The same results have also been seen in the presence of a single fast-spiking neuron in a chain of diffusively coupled FitzHugh-Nagumo neurons.

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I. INTRODUCTION

When a subthreshold periodic signal is imposed on a nonlinear dynamical system with an energetic activation barrier, a source of noise (inherent or external) can enhance the response of the system by preparing a floor for the external signal. This phenomenon, known as stochastic resonance, is an intriguing example where a source of disorder enhances order in the behavior of a dynamical system by enhancing the response of the system to the external signal as a resonance-like behavior [1,2]. It has been shown that when identical systems are coupled, whether in an all-to-all or an array configuration, the response of the system to external signals is further amplified in the presence of noise [3]. Coupling the systems together into a network introduces two other possible sources of diversity: a quenched disorder in the parameters of the coupled systems (nodes) and heterogeneity of the topology of the network (links). Even in the absence of noise both of these sources can also bring spatiotemporal order into the dynamics of the extended system by amplifying the response of the system to an external signal [4,5]. Scale-free networks are the prototypical form of heterogeneous networks, which can amplify the external signals far better than regular networks [5]. Interestingly, while disorder obviously acts against synchrony in the regular networks of coupled autonomous oscillators, in the presence of external (periodic) signals synchrony can be enhanced by disorder through a collective resonant behavior [4].

A special form of disorder-induced spatiotemporal synchronization is seen when disorder is imposed by embedding a single *fast* impurity in an otherwise homogeneous array consisting of identical oscillators [6,7]. A fast impurity in such a system serves as the leading component and drives other oscillators in the array. The dominant role of a fast oscillator is well known in the prominent example of the oscillations of electrical excitations in the human heart [8]. Two periodically spiking groups of different intrinsic periods as well as an excitable tissue oscillate with equal frequency, which is the higher of two natural frequencies. High-frequency locking situations also appear in the case of two adjacent limit-cycle regions [9], and rare inverse situations in which low frequencies are dominant are called abnormal locking [10].

In this study we investigate the response to a local external signal of a homogeneous array of similar coupled oscillators with a single impurity (a fast oscillator). While proper functioning of the signaling devices always implies high sensitivity to external signals, most of the studies on diversity-assisted amplification of signal responses have been done considering (globally) an extended source of the external signal [3–5]. Local signals, on the other hand, may act as a pacemaker for the whole network and guide the functioning of the whole ensemble by dictating their rhythm [11,12], and the signal amplification in many natural and artificial systems may use only local information [13]. Perc has studied the effect of such a pacemaker on a topologically inhomogeneous network [11]; here our network is topologically homogeneous, and instead, inhomogeneity is imposed on the nodes by introducing a fast impurity. We show that a resonance-like effect is seen as the position of the external signal is changed in the array.

In the present study we consider Josephson junctions as the model elements. An array of Josephson junctions is a prototype nonlinear system with many degrees of freedom [14,15]. An external periodic signal that would entrain the dynamics of a single junction [causing plateaus in the current-voltage (I-V) characteristic of the junction [16]] leads to more complex dynamics in the case of the array [17–19]. Here we investigate the influence of a local periodic signal on the dynamics of a chain of linearly coupled Josephson junctions. The equations describing this system can be reduced to the well-known Frenkel-Kontorova (FK) model, which has applications that range from the pedagogical example of a mechanical transmission line consisting of linearly coupled pendula [20] to the dislocation dynamics in metals [21,22], DNA dynamics [23], and strain waves in earthquakes [24]. Disorder, whether induced by a single impurity or by the inhomogeneity of the parameters of the components, has been exploited to remove chaos in a FK model consisting of chaotic components [25]. It has been shown previously that a single *fast* impurity in the FK model can serve as the source of solitary waves, giving it a leading role in the dynamics of the whole array [7]. Here we show that such an arrangement shows different responses to the locally imposed external signal, depending on where the signal is imposed. We also check that the idea is also valid in a chain of coupled FitzHugh-Nagumo (FHN) oscillators [26].

This paper is organized as follows: in the subsequent section the results are given in a chain of linearly coupled Josephson junctions. In Sec. III we show that similar results can also be seen when the model elements are FHN oscillators. The conclusions are given in Sec. IV.

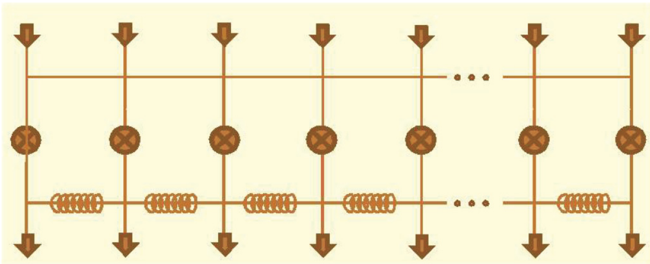

FIG. 1. (Color online) An array of parallel Josephson junctions. The solid line at the top is a superconducting bus bar, which sets a common phase (chosen to be 0) and supplies current as necessary. The element in the vertical wires represents a Josephson junction with resistive and capacitative shunts. The voltage across the junction is proportional to the time derivative of the phase difference, as described by the Josephson relation. These parallel elements are coupled by identical inductors. The arrows at the bottom represent externally imposed currents: the signals that drive the array.

II. THE CHAIN OF LINEARLY COUPLED JOSEPHSON JUNCTIONS

We consider a parallel array of Josephson junctions which are coupled inductively [27], as shown in Fig. 1. The network dynamics is described by the set of equations

$$\frac{\hbar C_j}{2e}\ddot{\theta}_j + \frac{\hbar}{2eR_j}\dot{\theta}_j + I_{cj}\sin\theta_j = I_j + E_j\sin\omega t + \frac{\Phi_0}{2\pi}\left[\frac{1}{L_j}(\theta_{j+1}-\theta_j) - \frac{1}{L_{j-1}}(\theta_j-\theta_{j-1})\right], \tag{2.1}$$

where C_j, R_j, and I_{cj} are the capacitance, resistance, and critical current of the jth junction, respectively; θ_j is the phase difference across the jth junction; and L_j is the inductance of the jth plaquette. $\Phi_0 = hc/2e$ is the flux quantum, and I_j and E_j are the constant current and the amplitude of the periodic current of jth junction, respectively.

We scale the parameters of the junctions by C_0, R_0, and I_{c0} and the inductance by L_0 and introduce a dimensionless time $\tau = \omega_p t$, where $\omega_p = \sqrt{2eI_{c0}/\hbar C_0}$ is the plasma frequency. Assuming the inductances to be equal, Eq. (2.1) becomes

$$\ddot{\theta}_j + \beta_c^{-1/2}\dot{\theta}_j + \alpha_j\sin\theta_j = i_j + \epsilon_j\sin\omega\tau + k_0(\theta_{j+1} - 2\theta_j + \theta_{j-1}). \tag{2.2}$$

Here $\beta_c = 2eR_0^2 I_{c0}C_0/\hbar$ is the McCumber parameter, which characterizes the relative importance of the damping, and $k_0 = \Phi_0/2\pi I_{c0}L_0$ is the coupling constant. The normalized values of the constant input and the amplitude of the periodic input are i_j and ϵ_j, respectively. The normalized critical current of the junctions is $\alpha_j = I_{cj}/I_{c0}$, ω is the drive frequency rescaled by the plasma frequency, and overdots indicate derivatives with respect to τ.

When i_j are small, θ_j undergo bounded oscillations, and the average voltage across each junction is zero. For sufficiently large i_j, θ_j increase with time with an average rate of increase that is proportional to the voltage difference across the junction. In this case in analogy with pendula we will say that the junctions are rotating. As a consequence of the inductive coupling, the average voltages across all the junctions in the array are equal in the steady state. Hence the I-V characteristic of all the junctions in the array is the same for all the junctions.

Equation (2.2) describes the damped driven FK model [20]. We consider an almost-homogeneous array into which a single junction (the impurity) that has a critical current relatively lower than that of the other junctions has been introduced. All the junctions then are driven by a constant current which is larger than their critical current. In the absence of the inductive coupling all the junctions would be in the rotating state, and the impurity would rotate faster than the rest of the array.

We place the fast impurity at the site $i = 50$ of a chain of 99 junctions and impose the periodic signal at site m. The critical currents of the other junctions are chosen from a uniform distribution in the range [0.98, 1.02]. The response of the system to the pacemaker is probed via the Fourier coefficients $Q_n = \sqrt{R_n^2 + W_n^2}$ according to Ref. [11]

$$R_n = \frac{2}{T}\int_0^T \sin(\omega t)v_n(t)dt, \tag{2.3}$$

$$W_n = \frac{2}{T}\int_0^T \cos(\omega t)v_n(t)dt. \tag{2.4}$$

Here $v_n = d\theta_n/dt$ is the voltage of a sample junction (we take $n = 70$ throughout thus paper), and T is the integration time. We then vary the location at which the periodic signal is imposed and record Q_n as a measure of how the time course of the sample node is correlated to the external signal.

As seen in Fig. 2 [28], the system responds to the presence of a fast impurity by a sharp resonance when the nodes near the fast impurity host the pacemaker. (There is also a boosted response when the pacemaker is located on the sample junction, but this is trivial.) The results show only a minor variation for different trials. We also observe that changing the frequency affects the magnitude of the correlation maximum, but the behavior of the system is the same, and a resonance is seen when the signal is located on the impurity. Without an impurity the response of the system shows considerably more variability from trial to trail, and upon averaging over the trials the response shows a maximum only when the signal is imposed on the sample node and its neighbors.

Studying the dependence of the response of the model on the frequency reveals why the resonances in Fig. 2 appear with different magnitudes. In Fig. 3 the Fourier coefficient Q_n is plotted for a range of the frequency of the periodic signal, with the other parameters being the same as in Fig. 2. Again two cases are considered: when the signal is imposed on the fast impurity and when it is placed on another junction. Figure 3 shows a resonant effect for certain values of the frequency (multiples of approximately 0.42 for both cases), but the resonance amplitude is considerably larger when the signal is located on the fast impurity. A comparison of the resonant frequency with the intrinsic frequency of the array in the absence of a periodic signal (shown by dashed lines) shows that the first maximums belong to the harmonic (1:1) locking and the next maximums appear due to the superharmonic (n : 1) locking. The larger width of the first maximum can be related to the wider Arnold tongue for the main locking zone [29]. Note that since the critical currents of the impurity and the other junctions are different, in isolation they would

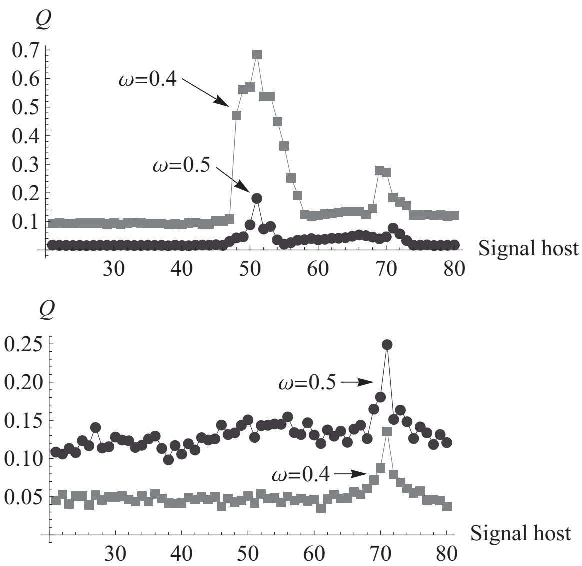

FIG. 2. (top) When an impurity with a relatively lower critical current (fast impurity) is introduced in an almost-homogeneous array, the Fourier coefficient dependence on the position of the external signal shows a resonance when the host node is near the impurity. The total number of junctions is 99, the critical currents α_j are chosen from a uniform distribution in the range [0.98,1.02] except for the impurity $\alpha_i = 0.8$ with $i = 50$. The sample junction here is chosen with $n = 70$, but qualitatively, the result is independent of the size of the array and the number of sample junctions if the transients have been elapsed. External dc current is $i_j = 1.02$ for all j. The damping parameter and coupling constant are $\beta_c^{-1/2} = 0.75$ and $k_0 = 0.25$, respectively. The external periodic input with amplitude $a_m = 0.1$ with two different frequencies is imposed only on the mth junction, and m is varied from 20 to 80. The boundary conditions are absorbing; i.e., the dc drive of the boundary junctions is switched off to prevent the waves from reentering the array. (bottom) Results are shown for the array without impurity.

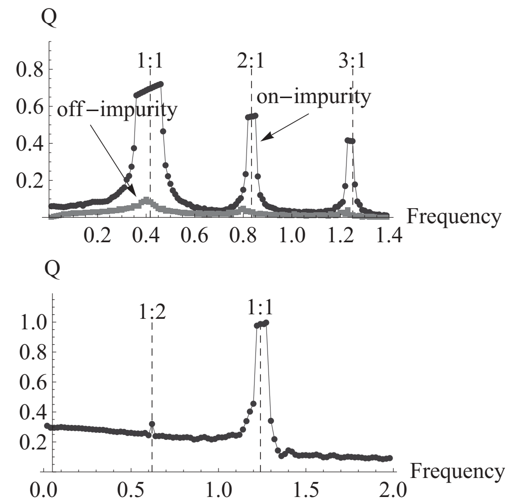

FIG. 3. (top) Fourier coefficient vs frequency of the external signal when the local periodic signal with amplitude $a_m = 0.1$ is imposed on the impurity (black circles) and on 60th junction (gray squares). Vertical dashed lines are plotted in the multiples of the intrinsic frequency of the array in the absence of a periodic signal. The number of junctions is 99, and the sample junction is chosen with $n = 70$, but the result is independent of the position of the sample junction and the size of the array. Critical currents of all the junctions are chosen from a uniform distribution from the range [0.98, 1.02], except for the middle junction, which has a smaller critical current $\alpha_i = 0.8$ with $i = 50$. The external dc current is $i_j = 1.02$ for all j. The damping parameter and coupling constant are $\beta_c^{-1/2} = 0.75$ and $k_0 = 0.25$, respectively. (bottom) The response of the impurity when isolated.

have different resonant frequencies. But for the array (with linear couplings), whether the external signal is imposed on the impurity or on the other junctions, the resonant frequencies would be identical, and for all the values of frequencies the response of the array is larger if the signal is imposed on the impurity. The response of the isolated impurity junction shown in the bottom panel of Fig. 3 shows similar qualitative behavior but with a higher resonant frequency. Also for the isolated junction a small amplitude resonance can be seen due to the subharmonic (1:2) locking, which is absent in the response of the array.

The role of the fast impurity in creating a *location sensitive response* to the local inputs can be better understood by studying the current-voltage (I-V) characteristics of the array. Average (dc) voltage of a junction in the normalized units introduced in Eq. (2.2) can be found from $V = \langle\dot{\theta}\rangle$, where $\langle\rangle$ shows averaging over time. For a single junction it is known that the width of the Shapiro steps on which the average rate of the evolution of the phase of the junction is locked to the external signal shows a Bessel-function-type relation to the amplitude of the signal [30]. For the small amplitude signals which we consider in this study, the width of the main Shapiro step has a linear relation with the amplitude of the periodic current. As noted before, in the array with inductive couplings, the I-V characteristic for all the junctions would be the same in the steady state regime (after transients have died out). We have shown the width of the main Shapiro step in the characteristic of the sampling junction when the signal is imposed on the impurity and when imposed on another junction in the array (Fig. 4). The width of the main step for the junctions in the array shows a linear relation with signal amplitude when signal is imposed on the impurity. On the other hand, if the signal is located on another junction, no locking region is seen for small amplitude signals.

Figure 4 further shows that an off-impurity signal can entrain the array dynamics if its amplitude is larger than a critical value. For off-impurity signals with an amplitude larger than a threshold, the critical current of the junction which receives the signal is decreased, so that the impurity loses its role as the fastest junction (the junction with the largest value of voltage when isolated). In this case the entrainment of the array by a large amplitude off-impurity signal would be possible.

The behavior of the system described above originates from the role of the fast component in generating solitary pulses consisting of a kink-antikink pair (see Ref. [7]). When the periodic signal is imposed on the fast component, the rate of the nucleation of the kinks (which is proportional to the rate of the change of the phase of the junction) can be locked to the external frequency. The solitary pulses move along the array and entrain the whole array after a transient time which grows

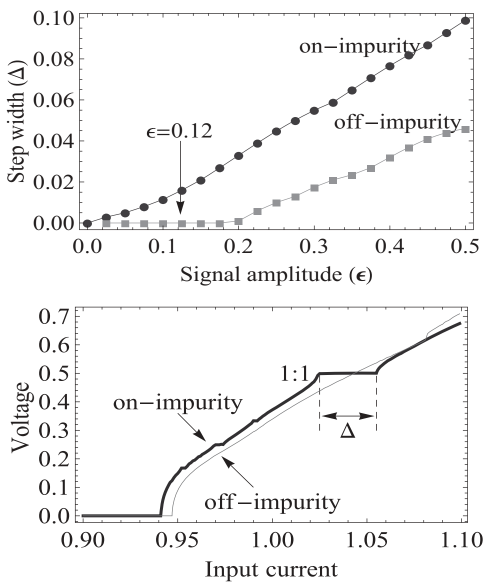

FIG. 4. (top) The width of the main 1:1 Shapiro step for an array when the local signal is imposed on the fast impurity (black circles) and on the 60th junction (gray squares). Total number of junctions is 99, and sample junction here is chosen with $n = 70$, but the results are independent of the index of the sample junction and the size of the array. Critical currents of all the junctions are chosen from a uniform distribution in the range [0.98,1.02], except for the middle junction $\alpha_i = 0.8$, with $i = 50$. The damping parameter and coupling constant are $\beta_c^{-1/2} = 0.75$ and $k_0 = 0.25$, respectively. (bottom) Current-voltage (I-V) characteristics of the array are plotted for a sample value of the amplitude of the periodic signal $\epsilon = 0.12$ depicted by the vertical arrow in the top plot. The thick black line and thin gray line show the characteristics for on-impurity and off-impurity signals, respectively. For the on-impurity signal the main Shapiro step on which the voltage of the junction (in normalized units) is locked to the frequency of the external signal is shown. For the off-impurity signal no locking zone is seen for this value of the amplitude of the periodic signal.

with the size of the array. A signal that is imposed on the other junctions can entrain the nearby junctions but has no effect on the rate of solitary pulses which are being produced by the impurity; then a long range influence is not expected by off-impurity signals. We also note that the existence of solitary excitations of the FK model is crucial for the existence of the behavior seen above: a ladder arrangement of Josephson junctions is a counterexample in which the long range effect of local signals cannot be seen. As we will show in the subsequent section, the same behavior can be seen in other models, so long as the model supports excitations which do not decay in space.

III. ARRAY OF COUPLED FITZHUGH-NAGUMO OSCILLATORS

While the FitzHugh-Nagumo model was originally proposed as a simplification of the Hodgkin-Huxley equations [26], it has been widely used as a generic model for excitable systems and media and can be applied to a variety of systems [31]. Here we consider an array of diffusively coupled

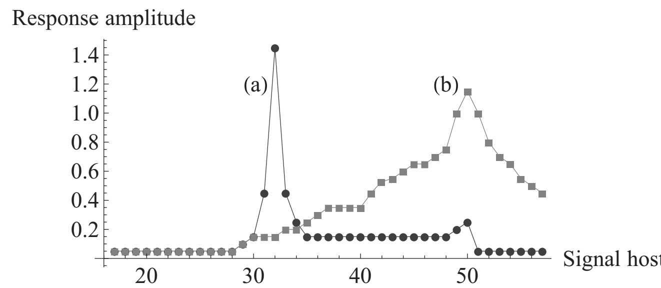

FIG. 5. Amplitude of variation of interspike intervals plotted vs the position of the neuron on which a periodic signal is imposed. The chain consists of 63 neurons. For line (a) a fast spiking impurity is in the middle (black circles). For line (b) the experiment is repeated in an array without impurity (gray squares). External currents for all the neurons are chosen from a uniform distribution in the range [1.18,1.22] except for the impurity which receives a larger input $I_{32} = 2.2$ to fire with a higher rate. The frequency and the amplitude of the external periodic signal are $a = 0.2$ and $\omega = 0.005$, respectively. The sample neuron is chosen with $n = 50$.

FHN-type oscillators:

$$\begin{aligned} \frac{dv_i}{dt} &= v_i - v_i^3/3 - w_i + I_i - g(v_{i+1} + v_{i-1} - 2v_i), \\ \frac{dw_i}{dt} &= 0.08[2.5 + 2.5\tanh(\eta v_i) - w_i], \end{aligned} \tag{3.1}$$

where v_i and w_i are the fast (voltage) and slow (recovery) variables, respectively. I_i is the external current, and g is the coupling constant (synaptic strength). The model oscillators show type-I excitability and undergo an infinite period bifurcation in $I_{\rm ext} = 2/3$ when $\eta \gg 1$ [32]. Interspike intervals (ISIs) are defined as the interval between two successive spikes when the neurons are spiking repeatedly, i.e., for $I_{\rm ext} > 2/3$, and they decease with increasing external current.

In an almost homogeneous array, we introduce again an impurity with a higher rate of activity and impose a small amplitude periodic signal on one of the neurons in the array. We then record ISIs in a sample neuron. If the period of the external signal is large compared to the ISI of the neurons, the response of the system to the external signal can be probed by the extent of the variation of the ISIs in a sample neuron. The maximum and the minimum of the ISIs of a sample neuron in a period of the external signal are recorded in our experiment, and their difference is reported as the amplitude of the response of the system. The results shown in Fig. 5 indicate a very sharp resonance when the impurity (and its neighbors) hosts the signal. In the absence of the impurity the response of the array is trivial and shows a maximum when the signal is imposed on the sample neuron. As for the FK model, the result is independent of the size of the array and location of the sample neuron while the system is in the steady state.

IV. CONCLUSION

In conclusion, we have shown that when a fast impurity is present in an array of inductively coupled Josephson junctions, locally imposed periodic signals show different levels of influence depending on whether they are imposed on the impurity or not. In such an array the fast impurity has a leading role as the source of solitary waves and even a small amplitude periodic signal when it is imposed on the impurity can entrain the dynamics of the array. Otherwise, if the signal is imposed on the other junctions, the junctions of the array do not lock to the signal unless the amplitude of the signal is large.

The results are independent of the size of the array, and for the large arrays, only transients are longer. Similar results are expected to be observed in arrays which support nondecaying excitations. As an example which could have application in studies of neuronal networks, we have considered an array of diffusively coupled FitzHugh-Nagumo oscillators. It has been shown that, in such an array, a small amplitude signal can affect the firing rate of all the neurons in the array if it is imposed on the fast impurity.

The results of the present study can be checked experimentally by investigating the current-voltage curve of a junction in a parallel array of Josephson junctions. We have predicted that, for small amplitude signals, no Shapiro step can be seen for the off-impurity signals. For larger amplitude signals wider Shapiro steps would appear in the characteristics of the junctions in the array if the signal is imposed on the impurity. A recent experimental preparation of an annular parallel array can be found in Ref. [33]. With the typical values of a junction's critical current $I_c \sim 1\ \mu$A, normal state resistance $R_N \sim 100\ \Omega$, and the junction capacitance $C \sim 0.1$ pF, the damping parameter would be $\beta_c^{-1/2} \sim 1/\sqrt{\pi}$, and the coupling (discreteness) constant $k_0 \sim 0.2$ can be achieved with cells having an inductance of $L \sim 100$ pH [33].

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[28] The numerical integration was done using the fourth order Runge-Kutta algorithm. We checked several cases using the methods built into MATHEMATICA, with good agreement. To construct the results shown in Figs. 2–4 integration was carried out for 10^6 time steps $dt = 0.01$ after discarding the first 10 0000 steps as transient. Averaging then was performed over 50 trials with random initial conditions. Different values for the time increment were also tried to ensure the reproducibility of the results.

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