High-temperature expansions of the higher susceptibilities for the Ising model in general dimension d

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The high-temperature expansion coefficients of the ordinary and the higher susceptibilities of the spin-1/2 nearest-neighbor Ising model are calculated exactly up to the 20th order for the general d-dimensional (hyper)simple-cubical lattices. These series are analyzed to study the dependence of critical parameters on the lattice dimensionality. Using the general d expression of the ordinary susceptibility, we have more than doubled the length of the existing series expansion of the critical temperature in powers of $1/d$.

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I. INTRODUCTION

We have derived high-temperature (HT) expansions of the ordinary and the higher susceptibilities (see the definitions in Sec. II) of the spin-1/2 Ising model exactly up to the 20th order for the general d-dimensional (hyper)simple-cubical (hsc) lattices.

These expressions are obtained by interpolation of the HT series expansion coefficients of the susceptibilities over the integer values of the lattice dimensionality, and not by analytic continuation. Thus they have no obvious uniqueness properties when d is allowed to take noninteger values. *A priori* a different dependence of physical quantities on the dimensionality might result from different possible interpolations, such as that obtained by formulating the Ising model on a fractal lattice [1,2], whose Hausdorff dimension can be varied continuously, or also in other ways [3]. An analogous situation is known to occur for the N-vector model, whose HT series coefficients can be interpolated [4] by rational functions of N. Of course, noninteger values of d (or similarly of N) might in some cases have no physical meaning [5].

The expansions [6–8] of the physical quantities in powers of $1/d$, i.e., around the mean-field (MF) approximation (or in powers [9] of $1/N$, i.e., the expansions around the spherical model limit), are related with these analytic representations in terms of d (or of N).

Our results not only provide reference data in a compact form for the higher susceptibilities, which are generally difficult to compute by methods different from series expansions, but also make a variety of other investigations possible. For example, in discussing [10–16] how the finite-size-scaling behavior [17] changes for systems above the upper critical dimensionality, accurate estimates of the critical parameters are needed as benchmarks. Our data can also help to assess the accuracy of estimates of physical parameters obtained from approximations of a different nature, such as the the $\epsilon = 4 - d$ expansion [9,18] of the renormalization group theory, the fixed-dimension renormalization group [19], the $1/d$ expansion [6,8], the Monte Carlo (MC) simulations, etc. It is appropriate at this point to observe that the MC simulations become increasingly time and memory demanding as the lattice dimensionality d grows, whereas the nonanalytic corrections to scaling in the asymptotic critical behavior of physical quantities, which usually make the HT series analyses delicate matters and are the main source of their uncertainties, become simpler and smaller [20] with increasing d. Thus even moderately long HT expansions can lead to very accurate estimates already for not very large d.

It is also worth mentioning that extremely long, although approximate HT expansions of the ordinary susceptibility have been recently generated [21] by a MC method for the hsc lattices of dimensionalities $d = 5, \ldots, 8$ and used to test the accuracy of the "extended scaling" ideas [22–24] above the upper critical dimension. The results of this investigation can now be compared with those from the analysis of our far shorter, but exact expansions.

The expressions presented here are obtained from recently derived [25–27] HT and low-field series expansions of the magnetization in presence of an external magnetic field for the spin-1/2 Ising model with nearest-neighbor interactions. Actually, we have produced a wider ranging set of data including as well other spin systems in the Ising model universality class, such as the general spin-s Ising model and the lattice scalar-field theories with polynomial self-interaction and thus, also for these models we are able to write exact expressions valid for general d-dimensional hsc lattices.

Our derivation of the HT and low-field series, which significantly extend the longest known results, even in zero field [4,28–32], has been performed for the lattice dimensionalities $d = 1, 2, \ldots, 10$. The expansions are carried to the 24th order in the case of the (hyper)body-centered-cubical lattices, but they are slightly shorter in the case of the hsc lattices: We have obtained the 24th order for $d < 5$, the 22nd for $d = 5$, the 21st for $d = 6$, and the 20th for $7 \leqslant d \leqslant 10$.

The layout of the paper is as follows. In the second section, we set our notations and tabulate the additional coefficients obtained in our work for the expansion of the ordinary

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susceptibility as closed-form polynomials in the coordination number $q = 2d$ [33]. In the third section, we discuss some results of an analysis of these series. We conclude in the last section.

II. ISING MODEL IN GENERAL DIMENSION

The partition function of a spin-1/2 Ising system with nearest-neighbor interactions, in the presence of an external magnetic field H, on a finite d-dimensional lattice of N sites can be written as

$$Z_N(H,T;d) = \sum_{\rm conf} \exp\left[J/k_BT \sum_{\langle ij\rangle} s_i s_j + H/k_BT \sum_i s_i \right]. \tag{1}$$

Here $s_i = \pm 1$ denotes an Ising spin variable associated to the lattice site i. The first sum extends to all configurations of the spins, the second to all distinct pairs $\langle ij\rangle$ of nearest-neighbor spins and the third to all spins. We shall set $K = J/k_BT$, with T the temperature, J the exchange coupling, k_B the Boltzmann constant, and $h = H/k_BT$ the reduced magnetic field. In terms of the variable $v = \tanh K$, the HT expansion coefficients are simple integers and so this variable is more convenient for the data tabulation.

In the thermodynamic limit $N \to \infty$, the specific free-energy $\mathcal{F}(h,K;d)$ is defined by

$$-\frac{K}{J}\mathcal{F}(h,K;d) = \lim_{N\to\infty} \frac{1}{N} \ln Z_N(h,K;d). \tag{2}$$

The specific magnetization $\mathcal{M}(h,K;d)$ is defined by

$$\mathcal{M}(h,K;d) = -\frac{K}{J}\frac{\partial \mathcal{F}(h,K;d)}{\partial h}. \tag{3}$$

Our calculation of the field-dependent magnetization, has yielded significant extensions of the existing HT expansions in zero field for the $2n$-spin connected-correlation functions at zero wave number and zero field $\chi_{2n}(K;d)$ (usually called higher susceptibilities). These quantities are defined by the successive field derivatives of the specific magnetization

$$\chi_{2n}(K;d) = \left(\partial^{2n-1}\mathcal{M}(h,K;d)/\partial h^{2n-1}\right)_{h=0}$$
$$= \sum_{s_2,s_3,\ldots,s_{2n}} \langle s_1 s_2 \ldots s_{2n}\rangle_c \tag{4}$$

at zero field. The even field derivatives of the magnetization vanish at zero field in the symmetric HT phase, while all derivatives are nontrivial in the broken-symmetry low-temperature phase.

The HT expansion coefficients of the susceptibilities at a given order l in K are usually computed as sums of contributions classified in terms of graphs having l edges. To each graph we associate a weight depending on the symmetry of the graph and on its free multiplicity (i.e., the number of distinct ways per lattice site in which the graph can be embedded in the lattice) associating each vertex to a site and each line to a nearest-neighbor bond [34]. Only the latter quantity, technically denoted as "free-lattice-embedding number" within the linked-cluster HT expansion method, depends on the lattice dimensionality. An analysis of these numbers for the various classes of graphs, like that performed in Refs. [6,7], leads to the conclusion that, at any given expansion order l, the HT series coefficients of the ordinary and the higher susceptibilities can be written as simple polynomials in the lattice dimensionality d of degree at most l. It is then clear that a straightforward prescription to represent the HT series coefficients of a susceptibility as polynomials in d up to the order $K^{l_{\max}}$, consists in repeating the computation of the HT series for lattices of dimensionalities $d = 0,1,\ldots,l_{\max}$ and then in interpolating each series coefficient with respect to d. Unfortunately, this straightforward strategy works only for relatively small orders of expansion, since in the case of the hsc lattice the combinatorial complexity of the computation of the graph-embedding numbers grows large exponentially with the dimensionality.

Here we take advantage of a well known result that is helpful in mitigating this difficulty. It was shown long ago [35] that the HT expansions of the successive derivatives of the magnetic field with respect to the magnetization $\partial^{2n+1}h/\partial\mathcal{M}^{2n+1}$ at zero magnetization, for $n = 0,1\ldots$ contain only star graphs (i.e., connected graphs having no articulation vertex).

This property was sometimes used to restrict the number of graphs contributing to the HT expansion of higher susceptibilities. What is interesting for our aims is the fact that the lattice-embedding number of a star graph with l edges is a polynomial in d of degree $[l/2]$ at most. The higher susceptibilities are simply related to the quantities $\partial^{2n+1}h/\partial\mathcal{M}^{2n+1}$

$$\frac{\partial h}{\partial \mathcal{M}}(K;d) = \frac{1}{\chi_2(K;d)}, \tag{5}$$

$$\frac{\partial^3 h}{\partial \mathcal{M}^3}(K;d) = \frac{\chi_4(K;d)}{\chi_2(K;d)^4}, \tag{6}$$

$$\frac{\partial^5 h}{\partial \mathcal{M}^5}(K;d) = \frac{\chi_6(K;d)}{\chi_2(K;d)^6} - 10\frac{\chi_4(K;d)^2}{\chi_2(K;d)^7}, \tag{7}$$

$$\frac{\partial^7 h}{\partial \mathcal{M}^7}(K;d) = \frac{\chi_8(K;d)}{\chi_2(K;d)^8} - 56\frac{\chi_4(K;d)\chi_6(K;d)}{\chi_2(K;d)^9} + 280\frac{\chi_4(K;d)^3}{\chi_2(K;d)^{10}}, \tag{8}$$

$$\frac{\partial^9 h}{\partial \mathcal{M}^9}(K;d) = \frac{\chi_{10}(K;d)}{\chi_2(K;d)^{10}} - 120\frac{\chi_4(K;d)\chi_8(K;d)}{\chi_2(K;d)^{11}} - 126\frac{\chi_6(K;d)^2}{\chi_2(K;d)^{11}} + 4620\frac{\chi_4(K;d)^2\chi_6(K;d)}{\chi_2(K;d)^{12}} - 15400\frac{\chi_4(K;d)^4}{\chi_2(K;d)^{13}}, \tag{9}$$

$$\frac{\partial^{11}h}{\partial\mathcal{M}^{11}}(K;d) = \frac{\chi_{12}(K;d)}{\chi_2(K;d)^{12}} - 220\frac{\chi_4(K;d)\chi_{10}(K;d)}{\chi_2(K;d)^{13}} - 792\frac{\chi_6(K;d)\chi_8(K;d)}{\chi_2(K;d)^{13}} + 17160\frac{\chi_4(K;d)^2\chi_8(K;d)}{\chi_2(K;d)^{14}} + 36036\frac{\chi_4(K;d)\chi_6(K;d)^2}{\chi_2(K;d)^{14}} - 560560\frac{\chi_4(K;d)^3\chi_6(K;d)}{\chi_2(K;d)^{15}} + 1401400\frac{\chi_4(K;d)^5}{\chi_2(K;d)^{16}}, \tag{10}$$

and so on.

Then it transpires that it is sufficient to interpolate the HT series expansion coefficients of the combinations of (higher) susceptibilities on the rhs of Eqs. (5)–(10), etc. only over the dimensionalities $1 \leqslant d \leqslant [l/2]$, to obtain the representations of these quantities for general d through the lth order in K. Finally, from these results the representations of the single higher susceptibilities can be easily recovered. This simple remark leads to a decisive reduction of the combinatorial complexity of the necessary calculations. In our case, only the knowledge of the HT expansions of the (higher) susceptibilities for all hsc lattices with $d \leqslant 10$ up to the 20th order is sufficient to obtain the expression of these susceptibilities for general d up to the same order. The fact that the coefficient of order K^l in $\chi_{2n}(K;d)$ is a polynomial in d of degree l, while the corresponding coefficient at the same order in the expansion of $\partial^{2n-1}h/\partial\mathcal{M}^{2n-1}$ is a polynomial of degree $[l/2]$, provides a simple consistency check of our computations.

A brief technical comment on a detail of the calculation is appropriate at this point, since most of the computing time goes into counting the number of lattice embeddings of each graph for relatively large lattice dimensionality. The first step of the computation consists in ordering appropriately the vertices of the graph and in placing the first of them at the lattice origin. The second step consists in counting the possible positions of coordinates $(x_1,\ldots,x_d)$ for the second vertex. It is crucial to optimize this step by using the hypercubical symmetry to restrict the possible positions of the second vertex to the fundamental region $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_d \geqslant 0$. Separating each $\geqslant$ case into a $>$ and an $=$, one gets 2^d cases. The case with all $>$ corresponds to the inside of the fundamental region, whose sites are representatives of a group orbit of length $d!2^d$. A small program precomputes the length of the orbits for the each of the 2^d cases, which is then used in the embedding program. After fixing the first two points of the embedding, the possible positions of the remaining vertices are restricted to relatively few sites by the constraints given by the distances from the first two points.

The timings for computing the HT expansion of the d-dimensional Ising model at order n have been roughly $O(5.5^n 2.5^d)$. In particular, the 10-dimensional Ising model at order 20 took 42 days of single-core time on a quad-core computer with a CPU-clock frequency of 2.8 GHz.

III. HIGH-TEMPERATURE EXPANSIONS

Expressions like those obtained here have already appeared at lower orders in the literature for the spin-1/2 Ising model [6,36–38] with nearest-neighbor interaction. In the case of the ordinary susceptibility $\chi_2(K;d)$, they reached at most [37] the 15th order, while for $\chi_4(K;d)$ and $\chi_6(K;d)$ they were computed [38] only up to the 11th order.

For brevity, we shall report our results in Table I only for $\chi_2(K;d)$ in the case of the spin-1/2 Ising model. We shall tabulate only the coefficients of order > 15, since the

TABLE I. The coefficients $c_n^{(2)}(d)$ of the HT series expansion in powers of $v = \tanh K$ for the ordinary susceptibility $\chi_2(v;d) = \sum_{n=0}^{\infty} c_n^{(2)}(d)v^n$ of the spin-1/2 Ising model with nearest-neighbor interaction, on a (hyper)-simple-cubical lattice of general dimensionality d. We have not reproduced here the expressions of the coefficients $c_n^{(2)}(d)$ with $n < 16$ that were already tabulated in Ref. [37] in terms of the variable d, whereas for convenience, here we have used the variable $q = 2d$. It should also be stressed that also in Ref. [37] the series coefficients refer to the expansion variable v like in this Table and not to the variable K as erroneously stated.

$$c_{16}^{(2)}(d) = q^{16} - 15q^{15} + 92q^{14} - 317q^{13} + 699q^{12} - 2879/3q^{11} + 7663/3q^{10} + 62404/3q^9 + 951902/3q^8 - 29340047/5q^7 - 1222190629/15q^6 + 1842802906q^5 - 40303287247/3q^4 + 727440333881/15q^3 - 436050363522/5q^2 + 61362596609q$$

$$c_{17}^{(2)}(d) = q^{17} - 16q^{16} + 106q^{15} - 398q^{14} + 963q^{13} - 1526q^{12} + 3198q^{11} + 53594/3q^{10} + 872213/3q^9 + 62822606/15q^8 - 1721035544/5q^7 + 24837980998/5q^6 - 103041998423/3q^5 + 658765700268/5q^4 - 1406657809016/5q^3 + 1518632487582/5q^2 - 124150140027q$$

$$c_{18}^{(2)}(d) = q^{18} - 17q^{17} + 121q^{16} - 492q^{15} + 1298q^{14} - 6931/3q^{13} + 12916/3q^{12} + 14580q^{11} + 803611/3q^{10} + 58164043/15q^9 - 6630346477/45q^8 - 18614162023/15q^7 + 2490068122951/45q^6 - 9120432843283/15q^5 + 153498549866917/45q^4 - 160125507535387/15q^3 + 791246681603369/45q^2 - 35100413743831/3q$$

$$c_{19}^{(2)}(d) = q^{19} - 18q^{18} + 137q^{17} - 600q^{16} + 1716q^{15} - 10124/3q^{14} + 18163/3q^{13} + 31678/3q^{12} + 746245/3q^{11} + 10791224/3q^{10} + 155233339/3q^9 - 119093401726/15q^8 + 797131628934/5q^7 - 7738214717002/5q^6 + 26108335469563/3q^5 - 441756642770324/15q^4 + 870753648372433/15q^3 - 893908363283714/15q^2 + 23668071765201q$$

$$c_{20}^{(2)}(d) = q^{20} - 19q^{19} + 154q^{18} - 723q^{17} + 2230q^{16} - 4792q^{15} + 8677q^{14} + 15815/3q^{13} + 701276/3q^{12} + 50129237/15q^{11} + 48537033q^{10} - 18335572678/5q^9 - 109608216238/9q^8 + 8227897945731/5q^7 - 1179114186108353/45q^6 + 1059691462407483/5q^5 - 9089741189180104/9q^4 + 2855597663272273q^3 - 6597100697141436/15q^2 + 2798133827599029q$$

lower-order ones reproduce those already listed in Ref. [37]. A simplification of the expressions of the coefficients is obtained by using the variable $q = 2d$ rather than the variable d used in Ref. [37]. As already pointed out, we can produce analogous formulas also for the other models in the Ising universality class for which we have computed the HT expansions of the magnetization, but they will not be presented here [33].

By solving recursively the equation

$$1/\chi_2(v_c; d) = 0 \tag{11}$$

with the Ansatz $v_c(d) = \tanh K_c(d) = a_1/q + a_2/q^2 + \cdots$ an expansion of the critical temperature in inverse powers of q was carried in Ref. [6] to the fifth order. We confirm the results of Ref. [6], (which are expressed in terms of the variable K rather than v) and also, taking advantage of our data, we are able to carry that expansion to the 12th order,

$$\begin{aligned}\frac{1}{qK_c(d)} &= 1 - \frac{1}{q} - \frac{4}{3q^2} - \frac{13}{3q^3} - \frac{979}{45q^4} - \frac{2009}{15q^5} - \frac{176749}{189q^6}\\ &\quad - \frac{6648736}{945q^7} - \frac{765907148}{14175q^8} - \frac{5446232381}{14175q^9}\\ &\quad - \frac{829271458256}{467775q^{10}} + \frac{164976684314}{22275q^{11}}\\ &\quad + \frac{6495334834824112}{638512875q^{12}}\cdots.\end{aligned} \tag{12}$$

A fifth order expansion of $K_c(d)$ was also obtained [7] for the N-vector model. All these expansions are presumably of asymptotic character, but so far this property has been established [7] only in the case of the spherical model (i.e., in the large N limit).

IV. SERIES ANALYSES

We shall now very briefly discuss the numerical estimates of some nonuniversal critical parameters of the spin-1/2 Ising models for $d > 4$. In particular, we shall use the expansion of $\chi_2(K; d)$ to locate the critical points $K_c(d)$. We shall also estimate the critical amplitudes of the five lowest-order susceptibilities and a few universal ratios of these. In our analysis, we have simply assumed that all (higher) susceptibilities show MF exponents of divergence, as also our recent work [25,26] has contributed to confirm.

The critical parameters are defined by the asymptotic critical behaviors of the susceptibilities

$$\chi_{2n}(K; d) \approx A_{2n}(d)\tau(d)^{-\gamma_{2n}}[1 + a_{2n}(d)\tau^{\theta(d)}\ldots] \tag{13}$$

as $K \to K_c(d)^-$. Here $\tau(d) = (1 - K/K_c(d))$, $\gamma_{2n} = \gamma_2 + 3(n-1)$ is the MF exponent, (assumed to depend on the order $2n$ of the susceptibility, but not on the lattice dimensionality for $d > 4$) and $\gamma_2 = 1$. $A_{2n}(d)$ denotes the amplitude of the leading singularity, $a_{2n}(d)$ the amplitude of the leading correction to scaling, and $\theta(d)$ is the exponent of the leading correction to scaling. The value of $\theta(d)$ is expected [20] to be 1/2 for $d = 5$, while for $d = 6$ it should be 1 with a possible logarithmic multiplicative correction. Generally, for $d > 6$ it is expected that $\theta(d) = (d-4)/2$.

We can only briefly outline the now standard numerical approximation techniques that we have used for these analyses, since a more detailed discussion was already given in Refs. [25,26,29]. We have mainly employed the differential approximant (DA) method, that generalizes [39] the elementary well known Padé approximant method, to resum the HT expansions up to the border of their convergence disks. This technique estimates the values of the finite quantities or the singularity parameters for the divergent quantities from the solution, called *differential approximant*, of an initial value problem for an ordinary linear inhomogeneous differential equation of the first or of a higher order. Various differential equations can be formed from a given series expansion. For each of them, the coefficients are polynomials in the expansion variable, defined in such a way that the series expansion of the solution of the equation equals, up to some appropriate order, the series to be approximated.

Sometimes, to determine the location of the critical points, it is more convenient to use a smoother and faster converging modification [29,39,40], called *modified-ratio approximant*(MRA) of the traditional methods of extrapolation of the series coefficient ratio sequence. The MRAs produce sequences $[K_c^{(r)}(d)]$ of approximations of the critical point that can be easily extrapolated to large orders r of expansion and therefore in some cases they may yield more accurate estimates than the DAs for which the analogous extrapolation is somewhat arbitrary. Let us finally stress that when using the DAs the evaluation of the uncertainties has not the same meaning as for MCs, but remains subjective to some extent and only indicates a small multiple of the spread of the values of a conveniently large sample of the highest-order approximants, formed from all or most expansion coefficients. If the sample averages remain stable as the order of the series grows and it can be assumed that stability indicates convergence, then the spread can be seen as a reasonable measure of the uncertainty of the results.

For the critical inverse temperatures $K_c(d)$ of the systems under study, we consider our best estimates those reported in Table II. They are obtained from the HT expansion of the ordinary susceptibility $\chi_2(K; d)$ by extrapolating to large order of expansion, a few (from four to six) highest-order terms of the MRA sequence of estimates $[K_c^{(r)}(d)]$ of the critical inverse temperature, basing on a fit to their simple asymptotic behavior [29]

$$K_c^{(r)}(d) = K_c(d) - \frac{\Gamma(\gamma_2)}{\Gamma(\gamma_2 - \theta)}\frac{\theta^2(1-\theta)a_2(d)}{r^{1+\theta}} + o\left(\frac{1}{r^{1+\theta}}\right). \tag{14}$$

A small multiple of the fit error is taken as the uncertainty of the final estimate.

In the case of six-dimensional lattices, we expect $\theta = 1$. Therefore the second term on the right-hand side of Eq. (14) vanishes and it must be replaced by a higher-order term reflecting the exponent of the next-to-leading correction to scaling in Eq. (13). In the $d = 5$ case, in which one expects $\theta = 1/2$, the coefficient of $1/r^{1+\theta}$ in Eq. (14) appears to be numerically negligible, so that the situation is similar to that of the six-dimensional case. In general, to avoid making assumptions on the values of the exponents of the next-to-leading correction to scaling, we have assumed an asymptotic form $K_c^{(r)} = K_c + w/r^{1+\epsilon}$ and determined K_c, w, and the effective exponent $\epsilon = \epsilon(d)$ by a best fit to the few highest-order terms

TABLE II. Our estimates of the critical inverse temperatures $K_c(d)$, obtained from the ordinary susceptibility expansions, for several hsc lattices of dimensionality $d > 4$. We have marked by an asterisk the estimates in the cases in which expansions extend beyond the 20th order [26]. In particular for $d = 5$ our series extend to the 22nd order, and for $d = 6$ to the 21st order.

Source	$K_c(5)$	$K_c(6)$	$K_c(7)$	$K_c(8)$	$K_c(9)$	$K_c(10)$
HT This work	0.113920(1)*	0.092298(1)*	0.0777094(2)	0.067155(1)	0.059148(1)	0.052858(1)
HT [37,41]	0.113935(15)	0.092295(3)	0.077706(2)			
HT [42]	0.113915(3)					
MC [21]	0.113925(12)	0.092290(5)	0.077706(2)	0.067144(4)		
MC [12,13]	0.11391(?)	0.09229(4)				
MC [14–16]		0.09229(4)	0.0777(1)	0.06712(4)		
MC [10]	0.1139152(4)					
MC [11]	0.1139139(5)					

of the sequence $(K_c^{(r)})$. We thus obtain the values $\epsilon = 1.1(2)$ for $d = 5$ and $\epsilon = 1.5(2)$ for $d = 6$. For $d > 6$, the values of ϵ thus obtained are larger. Therefore our estimates consistently confirm the above mentioned expectations about $\theta(d)$, and suggest that in $d = 5$ and $d = 6$ the asymptotic behavior of Eq. (14) is actually determined by the next-to-leading, rather than the leading, correction to scaling. On the other hand, for $d = 5$ and $d = 6$, a measure of the exponent $\theta(d)$ of the leading corrections to scaling, whose amplitudes $a_{2n}(d)$ are probably not negligible in spite of the fact that they are not seen by the MRAs, can be performed studying by DAs the critical behavior of appropriate universal ratios of higher susceptibilities, such as those introduced below in Eqs. (16)–(18). In these ratios the dominant critical singularities cancel, while the leading corrections to scaling generally survive and thus can be detected by DAs. This was already discussed in Ref. [26]. In conclusion, all these results are in reasonable agreement with the expectations [20,21] indicated above.

In the Table II, we have also reported a few of the most recent estimates of the critical inverse temperatures obtained in the literature either from shorter HT series or from MC simulations, for the values of d considered in our study. Since for $d > 4$ no logarithmic factors are expected to modify the leading MF behavior of the physical quantities, our series analyses are likely to yield estimates of a high accuracy, which moreover seem to improve with increasing lattice dimensionality, both because of the decreasing influence of the corrections to scaling and of the increasing lattice coordination number. All the results obtained from MRAs are consistent, within their uncertainties, also with the analyses employing DAs. In $d = 5$ dimensions, our estimate is slightly larger than other estimates [10,11] of similar nominal accuracy, but can be essentially considered compatible with those of Refs. [21,37,41,42]. It is of interest to quote here also the estimate $K_c(5) = 0.113919(2)$ obtained from second-order quasidiagonal DAs that use all series coefficients up to order $20 \leqslant l \leqslant 22$. The same value, with a slightly larger uncertainty, is obtained from DAs using all series coefficients up to order $18 \leqslant l \leqslant 20$. Generally, for higher values of d, our estimates of the critical inverse temperatures do not differ much from the old values, but they show a greater accuracy. For $d > 7$, the estimates of $K_c(d)$ obtained from the $1/d$ expansion of Eq. (12) reproduce our MRA values within the errors.

Let us now turn to the critical amplitudes $A_{2n}(d)$ of the susceptibilities $\chi_{2n}(K;d)$ with $n = 1,2,\ldots,5$, that can be determined, in terms of the previously estimated values of $K_c(d)$, by extrapolating the effective amplitudes

$$A_{2n}^{\mathrm{eff}}(K;d) = [1 - K/K_c(d)]^{\gamma_{2n}} \chi_{2n}(K;d) \qquad (15)$$

to $K = K_c(d)$, namely $A_{2n}(d) = A_{2n}^{\mathrm{eff}}(K_c;d)$. Our analysis uses first- and second-order DAs of the HT expansion of $A_{2n}^{\mathrm{eff}}(K;d)$.

In the Table III, we have reported our series estimates of $A_2(d)$, $A_4(d)$, $A_6(d)$, $A_8(d)$, and $A_{10}(d)$ normalized to their values in the MF approximation [43]: $A_2^{\mathrm{MF}} = 1$, $A_4^{\mathrm{MF}} = -2$, $A_6^{\mathrm{MF}} = 40$, $A_8^{\mathrm{MF}} = -2240$, and $A_{10}^{\mathrm{MF}} = 246400$. As expected, these ratios tend to unity as $d \to \infty$. For comparison, we have also reported the corresponding MC results of Ref. [21].

TABLE III. Our estimates of the critical amplitudes $A_{2n}(d)$ of the susceptibilities $\chi_{2n}(K;d)$, normalized to their values A_{2n}^{MF} in the MF approximation for several hypersimple-cubical lattices of dimensionality $d > 4$. We have marked by an asterisk the estimates obtained from series extending beyond the 20th order. (For $d = 5$ our series extend to the 22nd order, and for $d = 6$ the 21st order.)

Amplitude	Source	$d = 5$	$d = 6$	$d = 7$	$d = 8$	$d = 9$	$d = 10$
$A_2(d)/A_2^{\mathrm{MF}}$	This work	1.32(1)*	1.179(2)*	1.124(2)	1.096(2)	1.079(1)	1.067(3)
$A_4(d)/A_4^{\mathrm{MF}}$	This work	1.40(1)*	1.20(1)*	1.138(2)	1.105(2)	1.085(1)	1.068(3)
$A_6(d)/A_6^{\mathrm{MF}}$	This work	1.49(1)*	1.22(1)*	1.147(3)	1.114(2)	1.091(2)	1.077(3)
$A_8(d)/A_8^{\mathrm{MF}}$	This work	1.57(1)*	1.25(1)*	1.165(2)	1.122(2)	1.097(3)	1.081(4)
$A_{10}(d)/A_{10}^{\mathrm{MF}}$	This work	1.65(2)*	1.28(1)*	1.19(1)	1.13(1)	1.103(4)	1.085(4)
$A_2(d)/A_2^{\mathrm{MF}}$	MC [21]	1.291(3)	1.1606(17)	1.1008(5)	1.0836(5)		
$A_2(d)A_2^{\mathrm{MF}}$	HT [20]	1.311(9)	1.168(8)				

The results of this simulation are slightly, but systematically smaller than our HT estimates. This minor disagreement cannot be due to the very small difference in the estimates of $K_c(d)$ used in the two calculations, but must probably be ascribed to an underestimate of the uncertainties inherent in the MC generation of the HT series. On the other hand, the old estimates [20] of the same amplitudes obtained from an analysis of the 11th order HT series of Ref. [6] are compatible with ours. Estimates for $A_2(d)$ have also been obtained [8] from a third-order expansion in $1/d$, but they are not accurate enough. No comparison at all was possible for $A_4(d),\dots,A_{10}(d)$ for which we do not know of other estimates in the literature.

Finally, we have computed also for $d>6$, the critical values of a few universal ratios of higher susceptibilities such as the lowest-order terms in the sequences $\mathcal{I}^+_{2r+4}(d)$, $\mathcal{A}^+_{2r+4}(d)$, and $\mathcal{B}^+_{2r+8}(d)$, defined by

$$\mathcal{I}^+_{2r+4}(d)=\lim_{K\to K_c^-}\frac{\chi_2(K;d)^r\chi_{2r+4}(K;d)}{\chi_4(K;d)^{2r+1}}=\frac{A_2(d)^rA_{2r+4}(d)}{A_4(d)^{2r+1}} \tag{16}$$

$$\mathcal{A}^+_{2r+4}(d)=\lim_{K\to K_c^-}\frac{\chi_{2r}(K;d)\chi_{2r+4}(K;d)}{\chi_{2r+2}(K;d)^2}=\frac{A_{2r}(d)A_{2r+4}(d)}{A_{2r+2}(d)^2} \tag{17}$$

$$\mathcal{B}^+_{2r+8}(d)=\lim_{K\to K_c^-}\frac{\chi_{2r}(K;d)\chi_{2r+8}(K;d)}{\chi_{2r+4}(K;d)^2}=\frac{A_{2r}(d)A_{2r+8}(d)}{A_{2r+4}(d)^2} \tag{18}$$

for $r>0$. These universal ratios were introduced in Ref. [44] and were studied in detail for $d=4,5,6$ in Refs. [25,26]. For the first few values of $r=1,2,3$, we have checked that as expected, also for $d>6$, they take the MF values: $\mathcal{I}^{+\mathrm{MF}}_6=10$, $\mathcal{I}^{+\mathrm{MF}}_8=280$, $\mathcal{I}^{+\mathrm{MF}}_{10}=15400$, $\mathcal{A}^{+\mathrm{MF}}_8=14/5$, $\mathcal{A}^{+\mathrm{MF}}_{10}=55/28$, and $\mathcal{B}^{+\mathrm{MF}}_{10}=154$ within a relative accuracy generally higher than 10^{-3}, although the single amplitudes entering into the ratios reach their MF value only in the $d\to\infty$ limit.

V. CONCLUSION

We have represented in a compact form, as simple polynomials in the lattice dimensionality, the HT expansion coefficients of the (higher) susceptibilities in the case of the spin-1/2 Ising model on the hsc lattices of general dimensionality d. Our calculations add five more orders to the existing expansions of the ordinary susceptibility for general d and nine more orders to those of the fourth- and sixth-order susceptibilities. For the susceptibilities of order greater than the sixth no such data already exist in the literature.

An analysis of the series for lattice dimensionality $d>4$ yields estimates of nonuniversal parameters that compare well with the previous results whenever available, but are generally more accurate. Our estimates of a few universal ratio amplitudes provide a high-accuracy check that, unsurprisingly, they take MF values for $d>4$.

Finally, using the general d expression of the ordinary susceptibility, we have been able to expand up to the 12th order the critical temperature in powers of $1/d$ more than doubling the length of the result already known in the literature.

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