Stochastic resonance in multistable systems: The role of dimensionality

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The theory of stochastic resonance in multistable systems is extended to account for both direct transitions between all stable states present and indirect ones involving intermediate states. It is shown that to satisfy these requirements the dynamics needs to be embedded in phase spaces of dimension equal to at least two. Under well defined conditions, the conjunction of the presence of intermediate states and the multidimensional character of the process leads to an enhancement of the response of the system to an external periodic forcing.

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I. INTRODUCTION

In recent work by the present author the theory of stochastic resonance was extended to account for the situation where fluctuation-induced transitions between an initial and a final stable state occur through at least one intermediate stable state [1]. In addition to the existence of an optimal noise strength, it was found that there also exists a driving frequency-dependent optimal number of intermediate states for which the response to a weak external periodic forcing is maximized.

As for the majority of works on stochastic resonance reported in the literature, the above study was limited to systems amenable to a single variable, for which there necessarily exists a potential function generating the dynamics. From the standpoint of transitions between states, the principal limitation of this setting is that transitions between (initial) state 1 and (final) state n necessarily occur through successive visiting of the (intermediate) states i in a prescribed order. The objective of the present work is to extend this approach to account for both direct transitions between the stable states present and indirect ones involving intermediate states. Such mixed transitions constitute the rule rather than the exception in many situations of interest, from the different modes of atmospheric circulation [2] to the nucleation of self-assembly of nanosize materials [3].

Technically, the topological constraints needed to accommodate such transitions entail that the evolution is embedded in phase spaces of at least two dimensions. In what follows we consider the first nontrivial case of three simultaneously stable steady states in systems involving two variables x and y. We also require that the evolution of these variables derives from a potential function $U(x,y)$, which in the absence of periodic forcing should possess three minima located on the stable states $\overline{x}_i$ and $\overline{y}_i$ $(i = 1,2,3)$ and should allow for all possible transitions between these states.

The general formulation is presented in Sec. II. Section III is devoted to the long-time linear response. Transient behavior is considered in Sec. IV and the main conclusions are summarized in Sec. V.

II. FORMULATION

The evolution of a two-variable potential system subjected to additive periodic and stochastic forcings can be cast in the form

$$\frac{dx}{dt} = -\frac{\partial U(x,y,t)}{\partial x} + F_1(t),$$
$$\frac{dy}{dt} = -\frac{\partial U(x,y,t)}{\partial y} + F_2(t), \tag{1}$$

where the stochastic forcings F_i are assimilated to uncorrelated Gaussian white noises of variances q_i^2,

$$\langle F_i(t)\rangle = 0,$$
$$\langle F_i(t)F_j(t')\rangle = q_i^2\delta_{ij}^{\mathrm{kr}}\delta(t-t'), \quad i,j = 1,2. \tag{2}$$

We decompose the potential U in the following way, compatible with its additive character:

$$U(x,y,t) = U_0(x,y) - \epsilon g(x,y)\sin(\omega t + \phi), \tag{3}$$

where U_0 is the potential in the absence of the periodic forcing, $g(x,y)$ is a coupling function, and ϵ, ω, and ϕ stand for the amplitude, frequency, and phase of the forcing, respectively.

It is by now well established [4] that if the noise strength is sufficiently weak, Eqs. (1) can be mapped onto a discrete-state rate process describing the transfer of probability masses p_i $(i = 1,2,3)$ between the attraction basins of the stable states i $(i = 1,2,3)$ as depicted in Fig. 1. The corresponding kinetic equations read

$$\frac{dp_i}{dt} = \sum_{j=1}^{3} M_{ij}(t)p_j, \quad i,j = 1,2,3. \tag{4}$$

The configuration in Fig. 1(a) is the relevant one for the purposes of the present study. The transfer matrix M is then a full 3×3 matrix

$$M = \begin{pmatrix} -[k_{12}(t)+k_{13}(t)] & k_{21}(t) & k_{31}(t) \\ k_{12}(t) & -[k_{21}(t)+k_{23}(t)] & k_{32}(t) \\ k_{13}(t) & k_{23}(t) & -[k_{31}(t)+k_{32}(t)] \end{pmatrix}, \tag{5}$$

where k_{ij} are the individual transition rates between state i and state j. Notice that in the one-dimensional contraction of the problem the scheme in Fig. 1(a) collapses to that in Fig. 1(b) and the matrix M becomes tridiagonal ($k_{13} = k_{31} = 0$). This brings us to the case studied in previous work by the present author.

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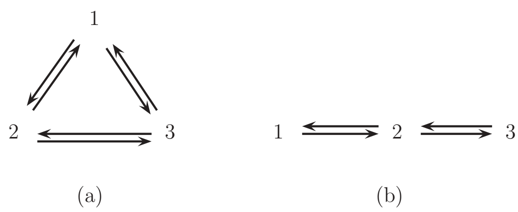

FIG. 1. Schematic representations of two types of configurations of a discrete-state process describing the transfer of probability masses between the attraction basins of three coexisting stable steady states.

As is well known, in an autonomous system, in the weak noise limit and under the conditions of existence of a potential the most probable path of transition between two stable states $\bar{x}_i,\bar{y}_i$ and $\bar{x}_j,\bar{y}_j$ goes through an intermediate unstable state of the saddle point type $\bar{x}_{ij},\bar{y}_{ij}$ lying on the manifold separating the corresponding attraction basins. Taking for simplicity $q_1^2 = q_2^2 = q^2$, one may then write the transition rate $k_{ij}^{(0)}$ in the form [5,6]

$$k_{ij}^{(0)} = \frac{1}{2\pi}\left(\sigma_i^{(1)}\sigma_i^{(2)}\right)^{1/2}\left(\frac{\sigma_{ij}^{+}}{|\sigma_{ij}^{-}|}\right)^{1/2}\exp\left(-\frac{\Delta U_{ij}^{(0)}}{q^2/2}\right). \quad (6)$$

Here $\Delta U_{ij}^{(0)}$ is the potential barrier, i.e., the difference of values of U_0 on the unstable state ij and the initial stable state i, and is necessarily a positive quantity; $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$ are the eigenvalues of the Hessian of U_0 on the stable state i; and $\sigma_{ij}^{\pm}$ stand, respectively, for the unstable and the stable eigenvalue of the Hessian evaluated on the saddle point ij.

The presence of an external time-dependent forcing will in principle modify the kinetics of the transitions as given by Eq. (6). In what follows we will adopt the adiabatic approximation [7,8], according to which individual transitions from state i to state j adjust instantaneously to the forcing in the sense that they are given by an extended form of Eq. (6) in which U_0 is replaced by the full time-dependent potential as given by Eq. (3). This is expected to hold as long as the forcing frequency ω is sufficiently small. The corresponding time-dependent potential barrier then becomes

$$\Delta U_{ij} = \Delta U_{ij}^{(0)} - \epsilon\Delta g_{ij}\sin(\omega t + \phi), \quad (7)$$

where Δg_{ij} is the difference of the values of the coupling function between the saddle point and the initial state. Furthermore, we will be interested in the linear response to the forcing. This will allow us to expand the ϵ-dependent terms in Eq. (5) and keep the first nontrivial order

$$k_{ij}(t) = k_{ij}^{(0)} + \epsilon\delta k_{ij}\sin(\omega t + \phi), \quad (8)$$

where $k_{ij}^{(0)}$ is given by Eq. (6) and

$$\delta k_{ij} = \frac{2}{q^2}k_{ij}^{(0)}\Delta g_{ij}. \quad (9)$$

Notice that by virtue of Eq. (3) and inasmuch as the Hessian involves second derivatives of the potential, the pre-exponential factors involving the eigenvalues σ_i and σ_{ij} are ϵ (and thus also time) independent.

III. LINEAR RESPONSE: ASYMPTOTIC BEHAVIOR

Within the framework of a linear response we seek solutions of Eq. (4) in the form

$$p_i = p_i^{(0)} + \epsilon\delta p_i(t), \quad (10)$$

where $p_i^{(0)}$ is evaluated in the absence of the forcing and $\epsilon\delta p_i(t)$ is the response induced by the forcing. Substituting into Eq. (4) and taking into account Eqs. (5), (8), and (9) we obtain, to the first nontrivial order in ϵ,

$$\frac{d\delta p_i}{dt} = \sum_j M_{ij}^{(0)}\delta p_j + f_i\sin(\omega t + \phi), \quad i = 1,2,3, \quad (11a)$$

where $M^{(0)}$ is the matrix in Eq. (5) evaluated in the absence of the forcing [with $k_{ij}(t)$ replaced by $k_{ij}^{(0)}$] and

$$f_i = \sum_i \Delta_{ij}p_j^{(0)}, \quad (11b)$$

with

$$\Delta = \frac{2}{q^2}\begin{pmatrix} -\left(k_{12}^{(0)}\Delta g_{12} + k_{13}^{(0)}\Delta g_{13}\right) & k_{21}^{(0)}\Delta g_{23} & k_{31}^{(0)}\Delta g_{31} \\ k_{12}^{(0)}\Delta g_{12} & -\left(k_{21}^{(0)}\Delta g_{21} + k_{23}^{(0)}\Delta g_{23}\right) & k_{32}^{(0)}\Delta g_{32} \\ k_{13}^{(0)}\Delta g_{13} & k_{23}^{(0)}\Delta g_{23} & -\left(k_{31}^{(0)}\Delta g_{31} + k_{32}^{(0)}\Delta g_{32}\right) \end{pmatrix}. \quad (11c)$$

Notice the conservation conditions

$$\sum_i p_i^{(0)} = 1, \quad \sum_i \delta p_i^{(0)} = 0. \quad (12)$$

Equation (11) admits, in the long time limit, solutions of the form

$$\delta p_i(t) = A_i\cos(\omega t + \phi) + B_i\sin(\omega t + \phi) \quad (13a)$$

or

$$\delta p_i(t) = R_i\sin(\omega t + \phi + \psi_i), \quad (13b)$$

where the amplitude R_i and phase shift ψ_i are given by

$$R_i = \left(A_i^2 + B_i^2\right)^{1/2}, \qquad \psi_i = \arctan\left(\frac{A_i}{B_i}\right). \quad (14)$$

The functions A_i and B_i are evaluated by substituting Eq. (13a) into Eq. (11) and by equating the coefficients of the sine and cosine terms. One obtains in this way

$$A_i = -\sum_{k=1}^{3} \frac{\omega}{\lambda_k^2+\omega^2}\gamma_k u_{k,i},$$
$$B_i = -\sum_{k=1}^{3} \frac{\lambda_k}{\lambda_k^2+\omega^2}\gamma_k u_{k,i}, \tag{15}$$

where λ_k and $\mathbf{u}_k$ are the eigenvalues and eigenfunctions of $M^{(0)}$ and γ_k are the expansion coefficients of $\mathbf{f}$ in the basis of $\mathbf{u}_k$,

$$f_i = \sum_{k=1}^{3} \gamma_k u_{k,i}. \tag{16}$$

Notice that by virtue of the conservation condition (12), one of the eigenvalues of $M^{(0)}$, say, λ_1, is zero.

These relations are rather intractable for the most general form of $M^{(0)}$ and Δ. As an example. considering state 2 to be the final state, with 1 and 3 being the initial and intermediate ones, one obtains the following expression for the amplitude R_2:

$$R_2^2 = \frac{a_0^2 + a_2^2\omega^2}{b_0^2 + b_2\omega^2 + \omega^4}, \tag{17}$$

with

$$\begin{aligned}
a_0 &= \left(k_{12}^{(0)} + k_{13}^{(0)} + k_{31}^{(0)}\right)\left[\delta k_{32} + (\delta k_{12} - \delta k_{32})p_1^{(0)}\right.\\
&\quad \left. - (\delta k_{21} + \delta k_{23} + \delta k_{32})p_2^{(0)}\right]\\
&\quad + \left(k_{12}^{(0)} - k_{32}^{(0)}\right)\left[\delta k_{31} + (\delta k_{21} - \delta k_{31})p_2^{(0)}\right.\\
&\quad \left. - (\delta k_{12} + \delta k_{13} + \delta k_{31})p_1^{(0)}\right],\\
a_2 &= \delta k_{32} + (\delta k_{12} - \delta k_{32})p_1^{(0)} - (\delta k_{21} + \delta k_{23} + \delta k_{32})p_2^{(0)},\\
b_0 &= \left(k_{12}^{(0)} + k_{13}^{(0)} + k_{31}^{(0)}\right)\left(k_{21}^{(0)} + k_{23}^{(0)} + k_{32}^{(0)}\right)\\
&\quad - \left(k_{21}^{(0)} - k_{31}^{(0)}\right)\left(k_{12}^{(0)} - k_{32}^{(0)}\right),\\
b_2 &= \left(k_{12}^{(0)} + k_{13}^{(0)} + k_{31}^{(0)}\right)^2 + \left(k_{21}^{(0)} + k_{23}^{(0)} + k_{32}^{(0)}\right)^2\\
&\quad + 2\left(k_{21}^{(0)} - k_{31}^{(0)}\right)\left(k_{12}^{(0)} - k_{32}^{(0)}\right).
\end{aligned} \tag{18}$$

As expected, the response vanishes in the limit where the driving frequency ω is much higher than the inverse of the system's intrinsic time scales. The existence of an optimal frequency $\omega_{\rm opt}$ between the low and high frequency limits depends in contrast on the system's parameters and in particular on the sign of the expression $a_0^2 b_2 - a_2^2 b_0$. If this expression is positive, R_2 possesses a minimum equal to a_0^2/b_0^2 at $\omega = 0$ and a maximum at a finite frequency $\omega_{\rm opt}$ and tends subsequently to zero as $\omega \to \infty$. If in contrast the expression is negative, R_2 is maximum at $\omega = 0$ and tends to zero monotonically as $\omega \to \infty$.

Checking the limits of validity of the above conditions as well as other more specific properties of R_2 requires the *a priori* computation of the coefficients $k_{ij}^{(0)}$ and δk_{ij}. This depends in turn on the structure of the unperturbed potential $U^{(0)}(x,y)$ and of the coupling function $g(x,y)$. It can be shown that the minimal setting necessary to accommodate three simultaneously stable states communicating with each other both directly and indirectly implies also the existence of three unstable states of the saddle point type and of an extra unstable state of the unstable node type (cf. Fig. 2). This type of phase space portrait can be generated by potentials of a polynomial form corresponding to a double cusp catastrophe [9,10]

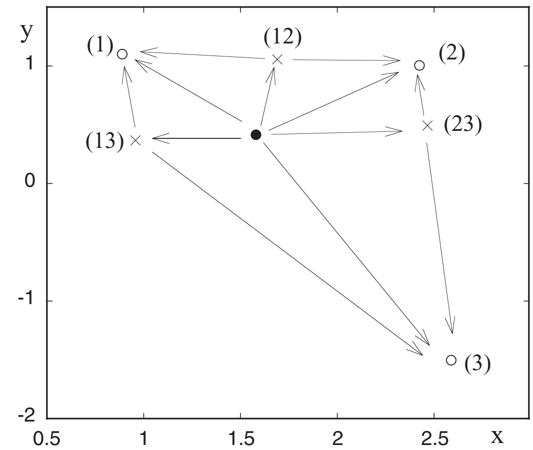

FIG. 2. Location of steady states and the unstable manifolds in the two-dimensional phase space of a double cusp catastrophe (x, y) [Eq. (19)] for $a = 0$, $c = 0$, $b = -5$, $\lambda = 7.75$, $\mu = -7/4$, $u = -3.75$, $v = 1/2$, and $\nu = 0.1$.

$$U^{(0)}(x,y) = \frac{x^4+y^4}{4} + a\frac{x^2y^2}{2} + b\frac{x^3}{3} + c\frac{x^2y}{2} + \lambda\frac{x^2}{2} + \mu\frac{y^2}{2} + \nu xy + ux + vy. \tag{19}$$

Depending on the way this potential is unfolded, i.e., on the values of the control parameters a, b, c, λ, μ, ν, u, and v, different values of the parameters in Eqs. (17) and (18) will be generated, which will determine the relative stability of the three stable states and the kinetics of transitions between them. To proceed further we consider hereafter some representative examples in which $k_{ij}^{(0)}$ and $\delta k_{ij}^{(0)}$ display particular symmetry properties.

A. Equal unperturbed rates and symmetrically disposed states

We set $k_{ij}^{(0)} = k$ and thus $p_i^{(0)} = 1/3$. To determine δk_{ij} we need to specify further the potential $U(x,y)$ and in particular the coupling function $g(x,y)$ in Eqs. (3), (9b), and (11c). We consider here the most symmetric situation in which the stable and unstable states are located, at equal distances from each other, on the circumference of a circle. This corresponds to reducing the x-y dependence of $U^{(0)}$ and g to a single angular variable α. The minimal setting for accommodating the desired number of (three) stable states and unstable ones of the saddle type is then provided by a periodic potential of the form

$$U = -\frac{\mu}{3}\cos(3\alpha) - \epsilon\alpha\sin(\omega t + \phi) \quad (\alpha \bmod 2\pi). \tag{20}$$

The unperturbed barriers $\Delta U_{ij}^{(0)}$ and those associated with the action of the forcing Δg_{ij} can be evaluated straightforwardly, yielding

$$\Delta U_{ij}^{(0)} = \frac{2\mu}{3}, \quad \Delta g_{ij} = -\frac{\pi}{3} \tag{21}$$

and thus

$$\delta k_{12} = \delta k_{23} = \delta k_{31} = \frac{2\pi}{3q^2}k, \quad \delta k_{21} = \delta k_{13} = \delta k_{32} = -\frac{2\pi}{3q^2}k. \tag{22}$$

Substituting into Eq. (17), one then finds $R_2 = 0$; in other words, the linear response to the forcing vanishes under the above conditions. This is a consequence of the symmetries imprinted on the system.

B. Two distinct unperturbed rates

We choose

$$k_{12}^{(0)} = k_{21}^{(0)} = k_{13}^{(0)} = k_{31}^{(0)} = k, \quad k_{23}^{(0)} = k_{32}^{(0)} = k'. \tag{23}$$

This choice is incompatible with a periodic potential as in Eq. (20) but can be realized by an appropriate unfolding of $U^{(0)}(x,y)$ in Eq. (19). Since the forward and backward rates are equal we still have a uniform invariant probability $p_i^{(0)} = 1/3$. To evaluate δk_{ij} we choose a coupling function $g(x,y) = x$. Furthermore, we focus on the configuration where the saddle type of unstable states are in the middle of the segments joining the stable states 1, 2, and 3. This yields

$$\delta k_{21} = -\delta k_{12} = -\delta k_{31} = \delta k_{13} \equiv \delta > 0, \quad \delta k_{23} = -\delta k_{32} \equiv \delta' > 0, \tag{24}$$

entailing that the prefactors $\beta_{ij} = \beta$ in Eq. (6) as well as the potential barriers $\Delta U_{ij}^{(0)} = \Delta U$ for the transitions between states 1 and 3 are identical. Substituting into Eq. (17) one finds

$$R_2 = \frac{4}{3q^2}\frac{|k\delta - k'\delta'|}{k+2k'}\frac{1}{\sqrt{1+\frac{\omega^2}{(k+2k')^2}}}. \tag{25}$$

This expression has a structure similar to that found in classical stochastic resonance [6,11], albeit with different combinations of rate constants. A detailed analysis leads to the following conclusions.

(i) The response is enhanced as the forcing frequency ω becomes less than $k + 2k'$.

(ii) For given ω there is an optimal noise strength $q_{\rm opt}^2$ for which the response is maximized. One can evaluate analytically this optimal strength in the case where k' is an integer multiple of k, $k' = nk$. One finds then

$$q_{\rm opt}^2 = \frac{4\Delta U}{W\left(\frac{2\beta^2(2n+1)^2e^{-2}}{\omega^2}\right)+2}, \tag{26}$$

where the Lambert W function $W(z)$ is defined by $W(z)\exp[W(z)] = z$ [12]. We notice that $q_{\rm opt}^2$ gets smaller as n increases.

(iii) The response takes a minimum (zero) value for parameter values satisfying the equality $k\delta = k'\delta'$. Moving one of these parameters, e.g., k, around the value k^* determined by this relation while keeping the other ones fixed yields the behavior depicted in Fig. 3. We observe an increase of the response as k moves both toward values larger and smaller than k^*, although there exists a marked asymmetry in the magnitude and the sensitivity of the response. This is in contrast with classical stochastic resonance where the response increases monotonically with the (unique) rate constant and constitutes the signature of the role of both the intermediate state and the two-dimensional character of the process. Notice that under the conditions of Eqs. (23) and (24) the response R_1 of the initial state 1 to the forcing vanishes for reasons of symmetry.

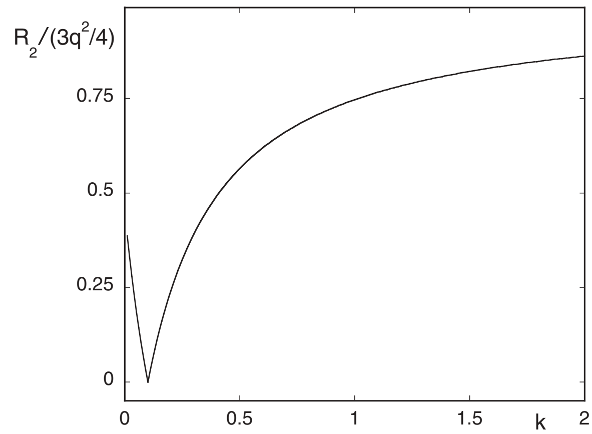

FIG. 3. Response R_2 [Eq. (25)] normalized by $3q^2/4$ as a function of the parameter k for $k' = 0.1$, $\delta = \delta' = 1$, and $\omega = 0.1$.

IV. LINEAR RESPONSE: TRANSIENT BEHAVIOR

In practice, the effectiveness of the response of a system to an external forcing is manifested not only through the forcing-induced change of the final state reached in the long time limit but also through the delay needed until the effects of the forcing begin to be apparent. To address this question a full scale time-dependent analysis of Eq. (4) and its linearized version in Eq. (11) is necessary.

To set the stage and get a feeling of the type of questions that may be raised in this context we first consider as a reference the (irreversible) transition from an initial state 1 to a final state 2,

$$\frac{dp_1(t)}{dt} = -k(t)p_1(t) = -\frac{dp_2(t)}{dt}, \tag{27}$$

with

$$p_1(0) = 1, \quad p_2(0) = 0, \tag{28a}$$

and [cf. Eq. (1b)]

$$k(t) = k\left\{1 + \frac{2\epsilon}{q^2}\delta\sin(\omega t + \phi)\right\}, \tag{28b}$$

where the value of δ is determined by the coupling function g. The unperturbed ($\epsilon = 0$) and perturbed solutions of Eq. (27) read

$$p_2^{(0)} = 1 - \exp(-kt) \tag{29a}$$

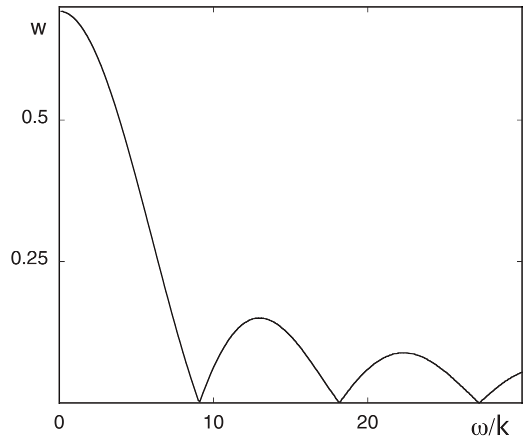

FIG. 4. Maximum effectiveness of the accelerating effect of a periodic forcing to reach half of the asymptotic probability mass in a transient two-state process [Eq. (31)] as a function of ω/k. These optimal values are achieved with different phase values of the forcing.

and [cf. Eq. (1b)]

$$p_2 = p_2^{(0)} + \epsilon\delta p_2 = 1 - \exp(-kt)\left(1 - k\frac{2\epsilon\delta}{q^2}\frac{\cos\phi - \cos(\omega t + \phi)}{\omega}\right), \tag{29b}$$

entailing that both $p_2^{(0)}$ and p_2 tend to 1 as $t \to \infty$, the corresponding asymptotic values of $p_1^{(0)}$ and p_1 being zero.

We regard as a primary indicator of the effectiveness of the forcing the time at which the probability mass in state 2 reaches half of its asymptotic value $p_2(t^*) = 1/2$. For the unperturbed system t^* can be evaluated straightforwardly from Eq. (29a),

$$t_0^* = \frac{1}{k}\ln 2. \tag{30}$$

The forcing will thus be deemed effective if the value t^* deduced from Eq. (29b) turns out to be significantly less than the above reference value t_0^*. Alternatively, one may seek conditions such that $\delta p_2(t)$ in Eq. (29b) at $t = t_0^*$, i.e., the expression

$$w = -\frac{\cos\left(\frac{\omega}{k}\ln 2 + \phi\right) - \cos\phi}{(\omega/k)} \tag{31}$$

has an appreciable positive value. In particular, for $\omega/k \to 0$, $w \to \sin\phi \ln 2$, entailing that the optimal phase needed to accelerate the crossing is $\pi/2$. In contrast, for $\omega/k \to \infty$ one obtains $w \to 0$. Figure 4 depicts the maximum of w with respect to the adimensionalized frequency ω/k, achieved for different forcing phase values. We observe that, typically, the forcing tends to accelerate the crossing of level $p_2 = 1/2$. Furthermore, the magnitude of this maximal response depends on the phase ϕ. In particular, for moderate values of ω/k, as this latter quantity is increased the phase ϕ needed to achieve the maximum of w decreases, as does the magnitude of the maximal response itself. These conclusions carry through for reversible transitions

$$\frac{dp_1}{dt} = -k_{12}p_1 + k_{21}p_2 = -\frac{dp_2}{dt} \tag{32}$$

in the limit $k_{12}^{(0)} = k_{21}^{(0)}$ (the analog of the case in Sec. III A), the only difference being that the factor ω/k in the argument of the cosine in Eq. (31) is now replaced by $\omega/2k$.

We turn next to the three-state case, the main object of our study. Let T be the matrix whose columns are the eigenvectors of matrix $M^{(0)}$ in Eq. (11a). Operating on both sides of this equation by matrix T^{-1} and defining new variables through

$$\mathbf{z} = T^{-1}\delta\mathbf{p}, \tag{33}$$

we obtain

$$\frac{dz_i}{dt} = \lambda_i z_i + (T^{-1}\Delta\mathbf{p}^{(0)})_i, \quad i = 1,2,3, \tag{34}$$

where λ_i are the eigenvalues of $M^{(0)}$ and Δ is defined by Eq. (11c). These equations can be solved separately by simple quadrature. Inverting Eq. (33), one can then obtain the response vector $\delta\mathbf{p}$ and in particular the amplitude of the response of the final state 2,

$$\delta p_2 = \sum_{j=1}^{3} T_{2j}z_j. \tag{35}$$

For clarity we hereafter give the explicit form of δp_2 in the case where state 2 can be regarded as practically absorbing $k_{21}^{(0)} = k_{23}^{(0)} = 0$ (the three-state analog of the scheme at the beginning of the present section). Furthermore, we take $k_{13}^{(0)} = k_{31}^{(0)} = k$ with $k_{12}^{(0)}$ and $k_{32}^{(0)}$ much smaller than k. One obtains then, to the dominant order in the ratios $k_{12}^{(0)}/k$ and $k_{32}^{(0)}/k$, remembering that one of the eigenvalues λ_i (say, λ_2) vanishes,

$$\delta p_2 = -2z_1, \tag{36}$$

with

$$z_1 = \frac{a_1 k}{\omega}[\cos\phi - \cos(\omega t + \phi)]\exp(\lambda_1 t) + \frac{b_1}{4 + (\omega^2/k^2)}\left[\exp(\lambda_1 t)\left(\frac{\omega}{k}\cos\phi + 2\sin\phi\right) - \exp(\lambda_3 t)\left(\frac{\omega}{k}\cos(\omega t + \phi) + 2\sin(\omega t + \phi)\right)\right]. \tag{37}$$

The quantities λ_1,λ_3 and a_1,b_1 in this expression are given, to the dominant order, by

$$\lambda_1 = -\frac{1}{2k}\left(k_{12}^{(0)} + k_{32}^{(0)}\right), \quad \lambda_3 = -2 \tag{38}$$

and

$$a_1 = -\frac{1}{4}\left(\frac{k_{12}^{(0)}}{k}\delta k_{12} + \frac{k_{32}^{(0)}}{k}\delta k_{32}\right), \quad b_1 = -\frac{1}{4}\left(\frac{k_{12}^{(0)}}{k}\delta k_{12} - \frac{k_{32}^{(0)}}{k}\delta k_{32}\right). \tag{39}$$

Under the same conditions the unperturbed value $p_2^{(0)}$ is given by

$$p_2^{(0)} = 1 - \exp(\lambda_1 t), \tag{40a}$$

independent of λ_3. Arguing as in the two-state case in the beginning of this section, we conclude that $p_2^{(0)}$ reaches the

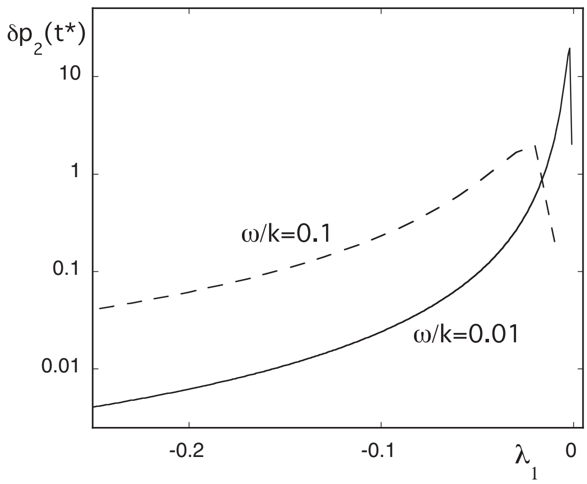

FIG. 5. Excess probability mass at time t^* [Eq. (40b)] induced by the presence of a periodic forcing in a transient three-state process [Eq. (41)] as a function of the slowest eigenvalue λ_1. The parameters are $a_1 = 1$, $b_1 = 1$, and $\phi = -\pi$.

value 1/2 at

$$t^* = -\frac{\ln 2}{\lambda_1} = \frac{2k \ln 2}{k_{12}^{(0)} + k_{32}^{(0)}}. \tag{40b}$$

The presence of the forcing will accelerate the crossing of level 1/2 of the full response $p_2 = p_2^{(0)} + \epsilon \delta p_2$ if $\delta p_2(t^*)$ turns out to be positive. Substituting Eq. (40) into Eqs. (36) and (37) and noticing that the contribution of eigenvalue λ_3 is negligible, one obtains

$$\delta p_2(t^*) = \frac{a_1 k}{\omega}\left[\cos\left(\frac{\omega \ln 2}{-\lambda_1} + \phi\right) - \cos\phi\right] - \frac{b_1}{4 + (\omega^2/k^2)}\left(\frac{\omega}{k}\cos\phi + 2\sin\phi\right). \tag{41}$$

A detailed study shows that this quantity displays the following properties.

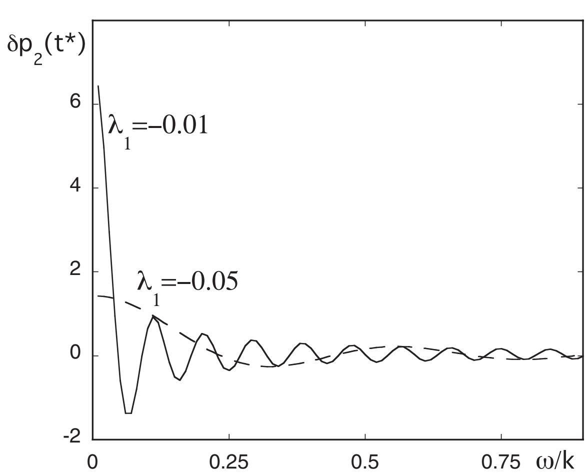

FIG. 6. Same as in Fig. 5 but as a function of ω/k for two different values of λ_1. The parameters are $a_1 = b_1 = 0.1$ and $\phi = -\pi/2$.

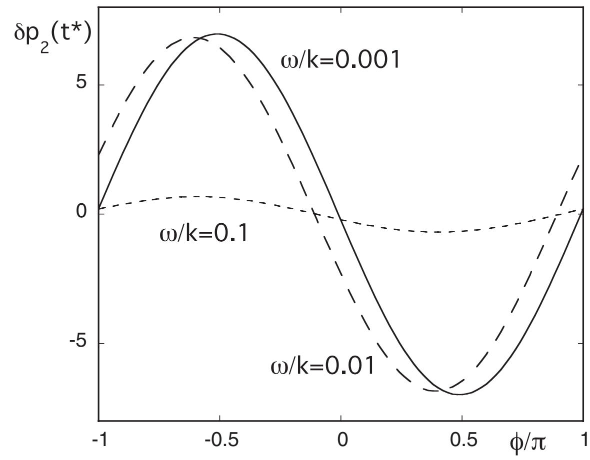

FIG. 7. Same as in Fig. 5 but as a function of ϕ/π for three different values of ω/k. The other parameters are $a_1 = b_1 = 0.1$ and $\lambda_1 = -0.01$.

(i) It possesses an extremum with respect to $-\lambda_1$ or equivalently with respect to the rates $k_{12}^{(0)}/k$ or $k_{32}^{(0)}/k$,

$$(-\lambda_1)_{\text{opt}} = \frac{\omega \ln 2}{2\pi n - \phi}, \tag{42}$$

where n is an integer. The extremum is a maximum if $a_1 > 0$ in Eq. (39) and minimum if $a_1 < 0$. The former property, which is of more interest for our purposes, is in turn satisfied for appropriate coupling functions $g(x,y)$ and for phase space configurations where the saddle points 12 and 32 and the initial states 1 and 3 are such that $\Delta g_{12} < 0$ and $\Delta g_{32} < 0$ [cf. Eq. (9b)]. These requirements are fulfilled by the choice $g(x,y) = x$ and the unfolding of potential $U^{(0)}(x,y)$ in Eq. (19) generating the phase space portrait of Fig. 2. Here again the roles of the intermediate state and the two-dimensional character of the process are to be stressed. Notice that, as stressed earlier, δp_2 itself needs to be positive for a whole range of values of $-\lambda_1$ including $(-\lambda_1)_{\text{opt}}$, implying that b_1 in Eq. (39) also has to satisfy certain conditions. Figure 5 depicts $\delta p_2(t^*)$ as a function of $-\lambda_1$ and for values of the other parameters such that all these properties hold true. As can be seen, the maximum becomes more pronounced as ω/k is decreased and can attain values corresponding to a tenfold or so amplification of the forcing.

(ii) For fixed $-\lambda_1$ and for values of the phase ϕ other than an integer multiple of π, $\delta p_2(t^*)$ starts with a finite value for $\omega/k = 0$ and decreases to zero nonmonotonically as $\omega/k \to \infty$ (Fig. 6). The values attained in the low frequency region can again be positive and quite substantial provided that $-\lambda_1$ is sufficiently small and ϕ is close to $\pi/2$.

(iii) For fixed $-\lambda_1$ and ω/k, $\delta p_2(t^*)$ varies periodically with ϕ with period 2π and amplitude that increases as ω/k is decreased. It reaches a finite limiting value as ω/k tends to zero (Fig. 7), which can be positive and substantial within an appropriate range of ϕ values.

V. CONCLUSION

In this work the theory of stochastic resonance has been extended to include transitions between an initial and a final state occurring both directly as well as indirectly via intermediate stable states. We have shown that in the minimal setting of just three stable states a dynamics compatible with these requirements needs to be embedded in phase spaces of dimension equal to at least two. For systems deriving from a potential and in the absence of an external periodic forcing, this can in turn be fully accounted for by a potential involving fourth-order nonlinearities corresponding to the universal unfolding of the double cusp catastrophe [9,10]. In particular, varying the unfolding parameters controls the relative stability of the states and the rates of transitions between them.

The effects induced by the external forcing have been analyzed in the asymptotic limit of long times and during the transient evolution toward the final state. The efficiency of the forcing was quantified through the amplitude of the long time response and through the value of the time of crossing of a certain prescribed level (chosen to be equal to $1/2$) attained by the probability of the final state, starting from a configuration where the entire probability mass is centered on the initial state. Our main conclusion has been that in both cases the presence of an intermediate state and the two-dimensional character of the process leave a clear-cut signature in the response. The latter becomes as a rule more flexible and can be further enhanced, as compared to classical stochastic resonance or to its one-dimensional extension allowing for the presence of intermediate states [1].

The analysis carried out in this paper can be extended straightforwardly to four stable states, as this is part of the full unfolding of the double cusp catastrophe [9,10]. Its extension to five or more stable states in systems deriving from a potential constitutes in contrast an open problem, since there is no known potential generating all possible phase space configurations compatible with this situation.

Our work has been concerned with generic classes of dynamical systems satisfying the aforementioned properties. It would undoubtedly be interesting to address concrete systems of concern in physical, environmental, and life sciences in which evidence for classical stochastic resonance is by now well established [6,13] and identify situations in which intermediate states are present and may interfere with the process. The goal would be to link the structure of the potential to the structure and dynamics of the underlying system and seek conditions under which the intermediate states can switch on new pathways optimizing the overall response.

Throughout this work the transitions between the stable states were mediated by unstable states of the saddle point type, while an additional state of the unstable node type was securing the structurally stable subdivision of the phase space into the attraction basins of the stable states. In recent work on winnerless competition dynamics [14,15], a transient evolution between unstable states prior to the establishment of a final stable state mediated by structurally stable heteroclinic connections with a predefined sequence of transitions was identified. While the topological configuration associated with this scenario is different from that of Fig. 2, our formulation can mimic an analogous situation by (i) choosing one of the stable states, say, α, to be absorbing (in the sense of $k_{\alpha i} = 0$ for all $i \neq \alpha$) and at the same time to be reached from the other states i ($i \neq \alpha$) at a very slow rate $k_{i\alpha} \ll k_{ij}$ ($i,j \neq \alpha$) and (ii) adopting for the transitions among states i the asymmetric configuration considered in Sec. III B, where direct transitions for visiting these states in a certain order proceed at a rate k much higher than the rate k' of their reverse counterparts. Conversely, winnerless competition dynamics under the influence of both noise and an external periodic forcing could give rise to new properties in line with those highlighted in the present work, in the form of additional control mechanisms in the succession of the unstable states and their associated lifetimes.

[1] C. Nicolis, Phys. Rev. E **82**, 011139 (2010).
[2] R. G. Barry and A. H. Perry, *Synoptic Climatology: Methods and Applications* (Methuen, London, 1973).
[3] J. F. Lutsko and G. Nicolis, Phys. Rev. Lett. **96**, 046102 (2006).
[4] C. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1983).
[5] B. Matkowsky and Z. Schuss, SIAM J. Appl. Math. **33**, 365 (1977).
[6] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[7] M. Evstigneev and P. Reimann, Phys. Rev. E **72**, 045101(R) (2005).
[8] M. Evstigneev, Phys. Rev. E **78**, 011118 (2008).
[9] G. Nicolis and C. Nicolis, Adv. Chem. Phys. **151**, 1 (2012).
[10] J. Callahan, Proc. London Math. Soc. **45**, 227 (1982).
[11] C. Nicolis, Tellus **34**, 1 (1982); R. Benzi, G. Parisi, A. Sutera, and A. Vulpiani, *ibid.* **34**, 10 (1982).
[12] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, Adv. Comput. Math. **5**, 329 (1996).
[13] F. Moss, L. Ward, and W. Sannita, Clin. Neurophysiol. **115**, 267 (2004).
[14] V. Afraimovich, I. Tristan, R. Huerta, and M. I. Rabinovich, Chaos **18**, 043103 (2008).
[15] M. I. Rabinovich, R. Huerta, P. Varona, and V. Afraimovich, PLoS Comput. Biol. **4**, e1000072 (2008).