Phase transitions in the q-voter model with two types of stochastic driving

Piotr Nyczka, Katarzyna Sznajd-Weron, and Jerzy Cisło
Institute of Theoretical Physics, University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland
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We study a nonlinear q-voter model with stochastic driving on a complete graph. We investigate two types of stochasticity that, using the language of social sciences, can be interpreted as different kinds of nonconformity. From a social point of view, it is very important to distinguish between two types nonconformity, so-called anticonformity and independence. A majority of work has suggested that these social differences may be completely irrelevant in terms of microscopic modeling that uses tools of statistical physics and that both types of nonconformity play the role of so-called social temperature. In this paper we clarify the concept of social temperature and show that different types of noise may lead to qualitatively different emergent properties. In particular, we show that in the model with anticonformity the critical value of noise increases with parameter q, whereas in the model with independence the critical value of noise decreases with q. Moreover, in the model with anticonformity the phase transition is continuous for any value of q, whereas in the model with independence the transition is continuous for $q \leqslant 5$ and discontinuous for $q > 5$.

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I. INTRODUCTION

Recently, various microscopic models of opinion dynamics have been proposed and widely studied by physicists and social scientists (for reviews see [1–4]). In the world of social studies this kind of modeling is known as agent-based modeling (ABM). It has been noted recently that despite the power of ABM in modeling complex social phenomena, widespread acceptance in the highest-level economic and social journals has been slow due to the lack of commonly accepted standards of how to use ABM rigorously [2,5]. As has been pointed out by Macy and Willer [3], one of the main problems in the field of social simulations is that there has been "little effort to provide analysis of how results differ depending on the model designs."

A similar problem is visible in a field of sociophysics. For example, to study opinion dynamics under conformity (one of the major paradigms of social response), a whole large class of models based on binary opinions $S = \pm 1$ has been proposed, among them the voter model [6,7], majority rule [8,9], the Sznajd model [10], and nonlinear voter models [11,12]. For all these models the ferromagnetic state is an attractor [1]. On one hand, this is expected since the conformity is the only factor influencing opinion dynamics in these models. On the other hand, this is obviously not realistic for real social systems. To make models of opinion dynamics more realistic several modifications has been proposed, among them the introduction of contrarians [13,14], inflexibles [15], and zealots [16]. From the social point of view all these modifications describe another major paradigm of social response—so-called nonconformity [17]. There are two widely recognized types of nonconformity: anticonformity and independence. From a social point of view, it is very important to distinguish between these two types of nonconformity [17,18]. The term "independence" implies the failure of attempted group influence. Independent individuals evaluate situations independently of the group norm. From this point of view both zealots, introduced by Mobilia [16], as well as inflexibles, introduced by Galam [15], describe a particular type of independent behavior. In contrast, anticonformists are similar to conformers in the sense that both take cognizance of the group norm—conformers agree with the norm, while anticonformers disagree. Therefore, the contrarians introduced by Galam in [13] or the stochastic driving proposed by de la Lama *et al.* [14] describe anticonformity.

Although differences between two types of nonconformity are very important for social scientists, the results obtained so far indicate that differences may be irrelevant from the physical point of view. Both contrarian and independent behaviors play the role of social temperature, which induces an order-disorder transition [13,14,19,20]. However, addressing the problem posed by Macy and Willer [3] we would like to check rigorously the differences between these two types of nonconformity under the framework of a possibly general model of opinion dynamics. In a class of models with binary opinions such a general model has been recently introduced in Ref. [12] under the name of the "q-voter model." As special cases this model consists of both the linear voter model as well as the Sznajd model. In this paper we investigate this model in the presence of different types of nonconformity and check whether results for anticonformity and independence are qualitatively the same, according to our first expectation. It should be mentioned that another general class of opinion dynamics, known as majority rule [8,9], would also be a good candidate to test the differences between these two types of nonconformity. However, introducing a general type of independence is not so straightforward in this case.

The paper is organized as follows. In the next section we introduce the generalized model with two types of nonconformity on a complete graph (topology, which is particularly convenient for analytical calculations). In Sec. III we analyze the time evolution of the system described by the master equation. In this section we will already see differences between models with anticonformity and independence, in contrast to our first prediction. In Sec. IV we calculate analytically the stationary values of public opinion in a case of an infinite system. The results presented in Secs. III and IV indicate clearly a phase transition between phases with and without majority. Therefore in Sec. V we find the phase diagram and calculate the transition point as a function of the model's parameters. The results presented in this section show clearly important qualitative differences between the two types of nonconformity. In Sec. VI we apply the approach that has been used to study nonequilibrium systems with two (Z_2) symmetric absorbing states in Refs. [21,22] to understand more deeply these differences and in particularly the origin of the discontinuous phase transition in the case of nonconformity. We conclude the paper in the last section.

II. MODEL

We consider a set of N individuals on a complete graph, which are described by the binary variables $S = \pm 1$. At each elementary time step q individuals $S_1,\dots,S_q$ (denoted by $\uparrow$ for $S_i = 1$ or $\downarrow$ for $S_i = -1$, where $i = 1,\dots,q$) are picked at random and form a group of influence, called a q lobby. Then the next individual ($\Uparrow$ or $\Downarrow$) that the group can influence, called the voter, is randomly chosen.

The part of a model described above is a special case of the nonlinear q-voter model introduced in Ref. [12]. In the original q-voter model, if all q individuals are in the same state, the voter takes their opinion; if they do not have a unanimous opinion, still a voter can flip with probability ϵ. For $q = 2$ and $\epsilon = 0$ the model is almost identical with Sznajd's model on a complete graph [23]. The only difference is that in the q-voter model repetitions in choosing neighbors are possible. In Ref. [24] the q-voter model with $\epsilon = 0$ and without repetition has been considered on a one-dimensional lattice. In this paper we also deal with a q-voter model with $\epsilon = 0$ and without repetition, but additionally we introduce a certain type of noise to the model. The original voter model describes only conformity, whereas noise is introduced to describe nonconformity.

In our model conformity and anticonformity take place only if the q lobby is homogeneous i.e., all q individuals are in the same state. In the case of conformity the voter takes the same decision as the q lobby (like in the original q-voter model), whereas in a case of anticonformity the voter takes the opposite opinion to that of the group. In the case of independent behavior, the voter does not follow the group but acts independently—with probability 1/2 it flips to the opposite direction, i.e., $S_{q+1} \to -S_{q+1}$.

To check the differences in results that are caused by different types of nonconformity we consider three versions of the model:

Anticonformity I: With probability p_1 the voter behaves like a conformist and with p_2 like an anticonformist. This type of anticonformity has been investigated in a case of the Sznajd model on a complete graph in Ref. [20]. Because results depend only on the ratio $p = p_1/p_2$, in this paper we consider $p_1 = 1$ and $p_2 = p$. In this version of the model the following changes are possible:

$$\underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Downarrow \xrightarrow{p_1=1} \underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Uparrow,$$
$$\underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Uparrow \xrightarrow{p_1=1} \underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Downarrow,$$
$$\underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Uparrow \xrightarrow{p_2=p} \underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Downarrow,$$
$$\underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Downarrow \xrightarrow{p_2=p} \underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Uparrow. \tag{1}$$

In other cases nothing changes.

Anticonformity II: With probability p the voter behaves like an anticonformist and with $1 - p$ like a conformist. This type of anticonformity has been investigated in a case of the Sznajd model on several networks in Ref. [14] and results were qualitatively the same as in Ref. [20]. Indeed it is quite easy to notice that anticonformity II is a special case of anticonformity I. However, for the record we consider here both cases and show that indeed differences are only quantitative. In this case the following changes are possible:

$$\underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Downarrow \xrightarrow{1-p} \underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Uparrow,$$
$$\underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Uparrow \xrightarrow{1-p} \underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Downarrow,$$
$$\underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Uparrow \xrightarrow{p} \underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Downarrow,$$
$$\underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Downarrow \xrightarrow{p} \underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Uparrow. \tag{2}$$

In other cases nothing changes.

Independence: With probability p the voter behaves independently and with $1 - p$ like a conformist. In the case of independent behavior an individual changes to the opposite state with probability 1/2. The following changes are possible:

$$\underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Downarrow \xrightarrow{1-p} \underbrace{\uparrow\uparrow\dots\uparrow}_{q}\Uparrow,$$
$$\underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Uparrow \xrightarrow{1-p} \underbrace{\downarrow\downarrow\dots\downarrow}_{q}\Downarrow,$$
$$\underbrace{\dots}_{q}\Downarrow \xrightarrow{p/2} \underbrace{\dots}_{q}\Uparrow,$$
$$\underbrace{\dots}_{q}\Uparrow \xrightarrow{p/2} \underbrace{\dots}_{q}\Downarrow. \tag{3}$$

In other cases nothing changes.

III. TIME EVOLUTION

In a single time step Δ_t, three events are possible: the number of "up" spins, $N_\uparrow$, increases, decreases by 1, or remains constant. Of course, all three events can be rewritten for the number of "down" spins $N_\downarrow$ as $N_\uparrow + N_\downarrow = N$. Also the concentration $c = N_\uparrow/N$ of spins up increases or decreases by $\Delta_N = 1/N$ or remains constant:

$$\begin{aligned}
\gamma^+(c) &= \text{Prob}\{c \to c + \Delta_N\},\\
\gamma^-(c) &= \text{Prob}\{c \to c - \Delta_N\},\\
\gamma^0(c) &= \text{Prob}\{c \to c\} = 1 - \gamma^+(c) - \gamma^-(c).
\end{aligned} \tag{4}$$

The time evolution of the probability density function of c is given by the master equation [7]

$$\begin{aligned}\rho(c,t+\Delta_t) &= \gamma^+(c-\Delta_N)\rho(c-\Delta_N,t) \\ &\quad + \gamma^-(c+\Delta_N)\rho(c+\Delta_N,t) \\ &\quad + [1-\gamma^+(c)-\gamma^-(c)]\rho(c,t). \end{aligned} \tag{5}$$

Of course, an analogous formula can be written for $N_\uparrow$. The exact forms of the probabilities $\gamma^+(c)=\gamma^+(N_\uparrow)=\gamma^+$ and $\gamma^-(c)=\gamma^-(N_\uparrow)=\gamma^-$ depend on the version of the model; for a finite system they are the following: For anticonformity I,

$$\begin{aligned}\gamma^+ &= \frac{N_\downarrow \prod_{i=1}^{q}(N_\uparrow - i + 1) + p\prod_{i=1}^{q+1}(N_\downarrow - i + 1)}{\prod_{i=1}^{q+1}(N-i+1)}, \\ \gamma^- &= \frac{N_\uparrow \prod_{i=1}^{q}(N_\downarrow - i + 1) + p\prod_{i=1}^{q+1}(N_\uparrow - i + 1)}{\prod_{i=1}^{q+1}(N-i+1)};\end{aligned} \tag{6}$$

for anticonformity II,

$$\begin{aligned}\gamma^+ &= \frac{(1-p)N_\downarrow \prod_{i=1}^{q}(N_\uparrow - i + 1) + p\prod_{i=1}^{q+1}(N_\downarrow - i + 1)}{\prod_{i=1}^{q+1}(N-i+1)}, \\ \gamma^- &= \frac{(1-p)N_\uparrow \prod_{i=1}^{q}(N_\downarrow - i + 1) + p\prod_{i=1}^{q+1}(N_\uparrow - i + 1)}{\prod_{i=1}^{q+1}(N-i+1)};\end{aligned} \tag{7}$$

and for independence,

$$\begin{aligned}\gamma^+ &= \frac{(1-p)N_\downarrow \prod_{i=1}^{q}(N_\uparrow - i + 1)}{\prod_{i=1}^{q+1}(N-i+1)} + \frac{pN_\downarrow}{2N}, \\ \gamma^- &= \frac{(1-p)N_\uparrow \prod_{i=1}^{q}(N_\downarrow - i + 1)}{\prod_{i=1}^{q+1}(N-i+1)} + \frac{pN_\uparrow}{2N}.\end{aligned} \tag{8}$$

For an infinite system the above formulas take much simpler forms: For anticonformity I,

$$\begin{aligned}\gamma^+ &= (1-c)c^q + p(1-c)^{q+1}, \\ \gamma^- &= c(1-c)^q + pc^{q+1};\end{aligned} \tag{9}$$

for anticonformity II,

$$\begin{aligned}\gamma^+ &= (1-p)(1-c)c^q + p(1-c)^{q+1}, \\ \gamma^- &= (1-p)c(1-c)^q + pc^{q+1};\end{aligned} \tag{10}$$

and for independence,

$$\begin{aligned}\gamma^+ &= (1-p)(1-c)c^q + p(1-c)/2, \\ \gamma^- &= (1-p)c(1-c)^q + pc/2.\end{aligned} \tag{11}$$

Solving analytically master equation (5) is not an easy task, but exact formulas for γ^+ and γ^- allow for a numerical solution of the equation. For an arbitrary initial state the system reaches the same steady state. In the case of an infinite system the probability density function is a sum of delta functions, $\rho_{st}(c)=\delta(c-c_1)+\delta(c-c_2)+\cdots+\delta(c-c_k)$, whereas for a finite system $\rho_{st}(c)$ has maxima (peaks) for the $c=c_j$ $j=1,\ldots,k$, which are getting higher and more narrow with the system size, approaching the deltas for the infinite system. The number of peaks, k, and values $c_1,\ldots,c_k$ depend on the version of the model, as well as on the model's parameters p and q.

Examples of the stationary probability density functions for the q lobby of size $q=7$ and a system size of $N=200$ are presented in Figs. 1 (anticonformity I) and 2 (nonconformity). As seen from Figs. 1 and 2, for small values of noise p (whether the noise is introduced as independence or anticonformity) the system is polarized, whereas for large values of p there is no majority in the system. However, the transition from the phase with majority to the phase without majority is very different for each type of noise. In the case with anticonformity we observe a continuous phase transition for arbitrary values of q, whereas in the case with nonconformity there is a continuous phase transition for $q \leqslant 5$ and a discontinuous phase transition for $q > 5$.

In the case with anticonformity two states with majority, represented by two equally high peaks, are stable below the critical value of p^*. As $p < p^*$ increases the two peaks approach each other and eventually for $p = p^*$ they form a single peak, which is a typical picture for a continuous phase transition (see Fig. 1) [25,26]. In the case with independence this picture is valid only for the lobby $q \leqslant 5$. For $q > 5$

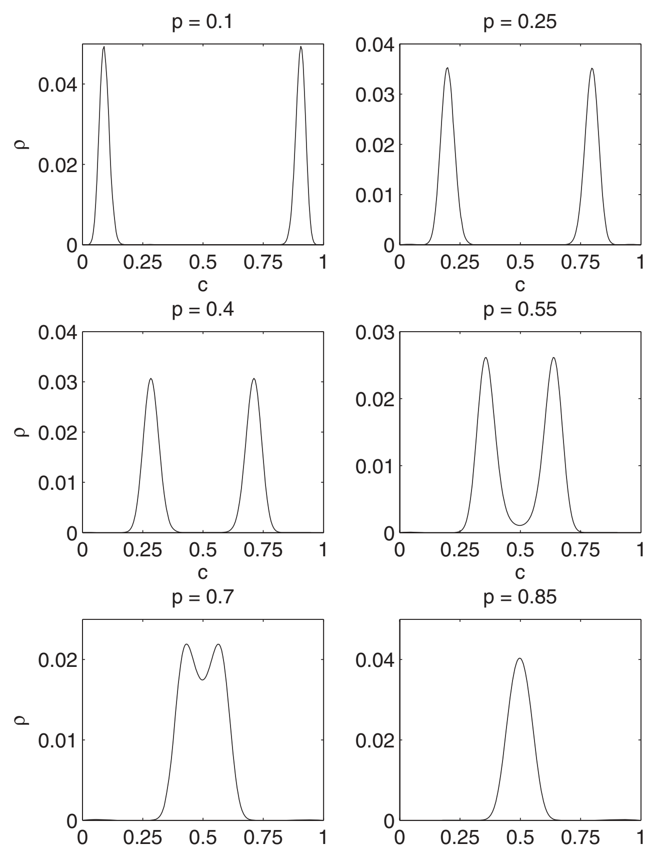

FIG. 1. Stationary probability density function of the concentration of up spins for the q-voter model with anticonformity I for a system of $N = 200$ individuals and a lobby size of $q = 7$. As seen for small values of anticonformity p the system is polarized, but for large values of p there is no majority in the system. For $p = 0$ (the case without anticonformity) the system consists of all spins up or all spins down. With increasing p maxima are getting lower and approaching each other. Eventually they form a single maximum. This is typical behavior for a continuous phase transition. The critical value of p can be found analytically (see Sec. V) and depends on q.

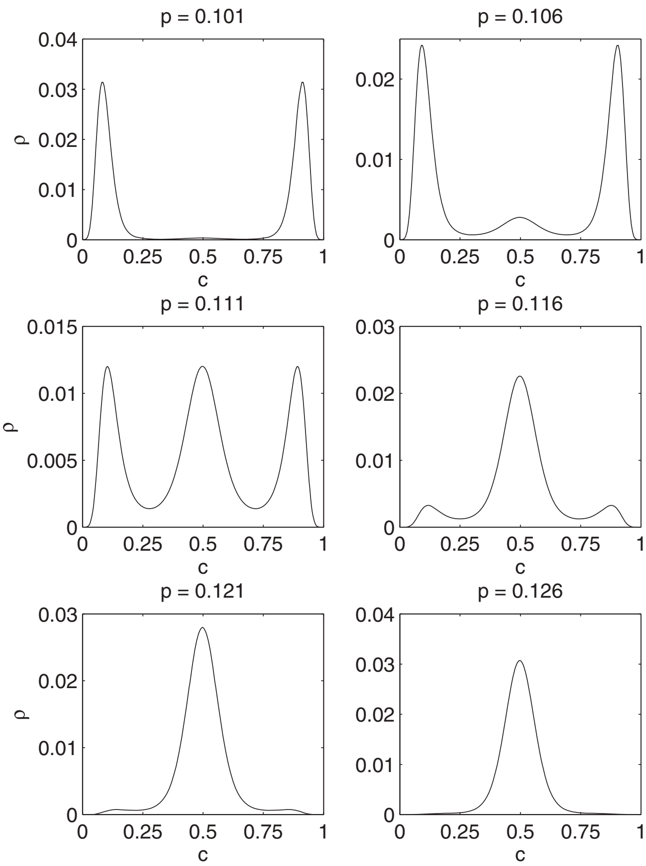

FIG. 2. Stationary probability density function of the concentration of up spins for the q-voter model with independence for a system of $N = 200$ individuals and a lobby size of $q = 7$. As seen for small values of independence p the system is polarized, but for large values of p there is no majority in the system. For $p = 0$ (the case without independence) the system consists of all spins up or all spins down. For larger values of p the third maximum appears at $c = 1/2$ (no majority). This maximum increases with p while the remaining two maxima are decreasing. Above a certain value of p there is only one maximum for $c = 1/2$. This is typical behavior for a discontinuous phase transition for which we can observe the phase coexistence.

the transition is very different. Again for small values of p there are two peaks but with increasing p they are not approaching each other. Instead, for $p = p_1^*$ the third peak appears at $c = 1/2$ (see Fig. 2). The third peak is initially lower than the remaining two peaks, which means that it represents a metastable state. As $p > p_1^*$ increases the third peak grows and for $p = p_2^*$ all three peaks have the same height. For $p > p_2^*$ the central peak dominates over the other two, which means that the state $c = 1/2$ is stable and the remaining two are metastable. Finally, for $p = p_3^*$ the side peaks disappear and only the center peak remains. This is a typical picture for a discontinuous phase transition, which takes place at $p = p_2^*$ [25,26]. Two values of the independence parameter, $p = p_1^*$ and $p = p_3^*$, demarcate the existence of metastability (spinodal lines) [27,28]. Values p_1^*, p_2^*, and p_3^* depend on the size of the lobby q, which will be shown exactly in Sec. V.

Before moving on to the analytical results for the infinite system and determining the points of phase transitions, let us

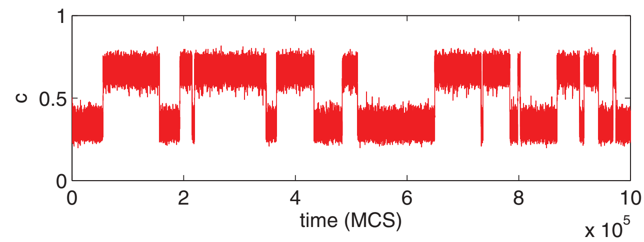

FIG. 3. (Color online) Time evolution of the concentration of up spins for the model with anticonformity with lobby $q = 7$ and level of anticonformity $p = 0.5$. The system size is $N = 200$. Spontaneous transitions between two stationary states are visible.

present the time evolution. We stop for a moment to focus on the case of a finite system. Having exact formulas for transition probabilities γ^+ and γ^- we are able not only to calculate numerically the stationary density function $\rho_{st}(c)$ but also to generate sample trajectories of concentration (Figs. 3–6). In the case of a finite system spontaneous transitions between states are possible. In the case with anticonformity transitions between two states, which correspond to peaks in the probability density function $\rho_{st}(c)$, are possible below a critical value of p. Because both peaks are equally high the system spends the same amount of time on average in each state.

This is also true in the case with independence and $q \leqslant 5$ (see Fig. 4). However, as we have already written for $q > 5$ there is a discontinuous phase transition between states with and without majority and for $p \in (p_1^*, p_3^*)$ there are three possible states. Therefore for $q > 5$ we expect spontaneous transitions among three states.

Such transitions are indeed observed. For $p \in (p_1^*, p_2^*)$ the state with majority is stable and the state without majority is metastable. Therefore, the system spends more time in states with majority. For $p \in (p_2^*, p_3^*)$ the situation is exactly the opposite—the state without majority is stable. At a transition point $p = p_2^*$ all three states are stable and the system spends the same time on average in each of three states (see Figs. 5 and 6).

IV. STATIONARY CONCENTRATION

In the stationary state we expect that the probability of growth, γ^+, should be equal to the probability of loss, γ^-, and

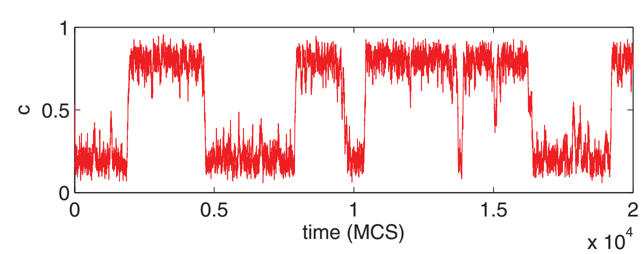

FIG. 4. (Color online) Time evolution of the concentration of up spins for the model with independence with lobby $q = 5$ and level of anticonformity $p = 0.175$. The system size is $N = 200$. Spontaneous transitions between two stationary states are visible.

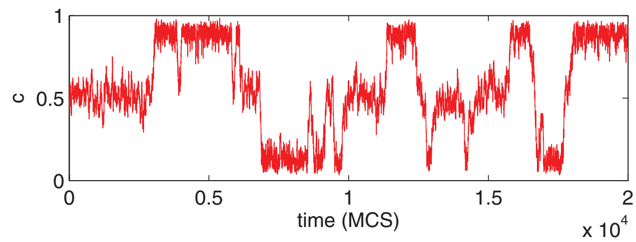

FIG. 5. (Color online) Time evolution of the concentration of up spins for the model with independence with lobby $q = 7$ and level of anticonformity $p = 0.111$ (where for this value all three states are stable). The system size is $N = 200$. Spontaneous transitions among three stationary states are visible.

therefore

$$F(c,q,p) = \gamma^+(c,q,p) - \gamma^-(c,q,p) = 0, \qquad (12)$$

where $F(c,q,p)$ can be treated as an effective force, γ^+ drives the system to the state spins up, and γ^- drives them to spins down. Therefore we can easily calculate also an effective potential:

$$V(c,q,p) = -\int F(c,q,p)\,dc. \qquad (13)$$

To calculate stationary values of concentration we simply solve the equation

$$F(c,q,p) = 0, \qquad (14)$$

or, alternatively, find the minima of the potential V. Although the first possibility is more straightforward, we will see in the next section that knowing the form of the potential will help us to calculate the transition points.

The exact forms of the force F and the potential V for an infinite system are as follows: For anticonformity I,

$$\begin{aligned} F &= (1-c)c^q + p(1-c)^{q+1} - c(1-c)^q - pc^{q+1}, \\ V &= -\frac{1}{q+1}(c^{q+1} + (1-c)^{q+1}) \\ &\quad + \frac{p+1}{q+2}(c^{q+2} + (1-c)^{q+2}); \end{aligned} \qquad (15)$$

for anticonformity II,

$$\begin{aligned} F &= (1-p)(1-c)c^q + p(1-c)^{q+1} \\ &\quad - (1-p)c(1-c)^q - pc^{q+1}, \\ V &= -\frac{1-p}{q+1}(c^{q+1} + (1-c)^{q+1}) \\ &\quad + \frac{1}{q+2}(c^{q+2} + (1-c)^{q+2}); \end{aligned} \qquad (16)$$

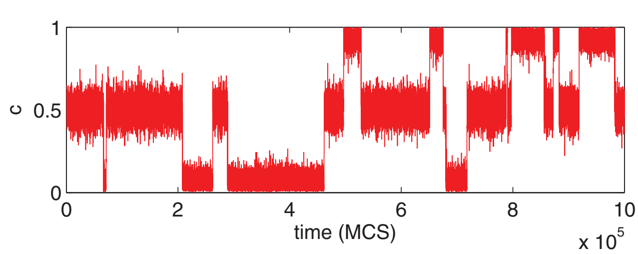

FIG. 6. (Color online) Time evolution of the concentration of up spins for the model with independence with lobby $q = 9$ and level of anticonformity $p = 0.0685$. The system size is $N = 200$. Spontaneous transitions among three stationary states are visible.

and for independence,

$$\begin{aligned} F &= (1-p)(1-c)c^q + \frac{p(1-c)}{2} \\ &\quad - (1-p)c(1-c)^q - \frac{pc}{2}, \\ V &= -\frac{1-p}{q+1}(c^{q+1} + (1-c)^{q+1}) \\ &\quad + \frac{1-p}{q+2}(c^{q+2} + (1-c)^{q+2}) - \frac{p}{2}c(1-c). \end{aligned} \qquad (17)$$

Solving analytically Eq. (14), i.e., finding c_{st} as a function of p for an arbitrary value of q, is impossible, but we can easily derive the opposite relations satisfying Eq. (14): For anticonformity I,

$$p = \frac{c_{st}(1-c_{st})^q - (1-c_{st})c_{st}^q}{(1-c_{st})^{q+1} - c_{st}^{q+1}}, \qquad (18)$$

for anticonformity II,

$$p = \frac{c_{st}(1-c_{st})^q - (1-c_{st})c_{st}^q}{(1-c_{st})^{q+1} + c_{st}(1-c_{st})^q - (1-c_{st})c_{st}^q - c_{st}^{q+1}}, \qquad (19)$$

and for independence,

$$p = \frac{c_{st}(1-c_{st})^q - (1-c_{st})c_{st}^q}{(1-c_{st})/2 + c_{st}(1-c_{st})^q - (1-c_{st})c_{st}^q - c_{st}/2}. \qquad (20)$$

We have used the above formulas to plot the dependence between steady value of concentration c_{st} and the level of noise p for several values of q (see Fig. 7). Although only the relation $p(c_{st})$ is calculated analytically and the opposite relation is unknown, we plot $c_{st}(p)$ by simply rotating the figure with the relation $p(c_{st})$. Clear differences between the two types of noise are visible—in a case with anticonformity the transition value of p increases with q and in a case with independence it decreases with p. Moreover, the type of transition is the same for arbitrary values of q in the case with anticonformity, whereas in the case with independence the transition between phases with and without majority changes its character for $q > 5$.

It should be also noticed that formulas (18)–(20) have been obtained from condition (14); i.e., they correspond to extreme values of potentials (15)–(17). However, only the minima of the potential correspond to the stable value of concentration. Therefore, in Figs. 7 and 8 we have denoted unstable values that correspond to the maxima of potentials by dotted lines. Moreover, we have presented the flow diagram for chosen values of q to show precisely which state is reached from given initial conditions. Particularly interesting behavior is related with independence (bottom panel in Fig. 8). Starting from two different initial concentrations disorder or order can be reached as a steady state (hysteresis).

In the next section we derive analytically transition points using knowledge of the effective potentials V in Eqs. (15)–(17).

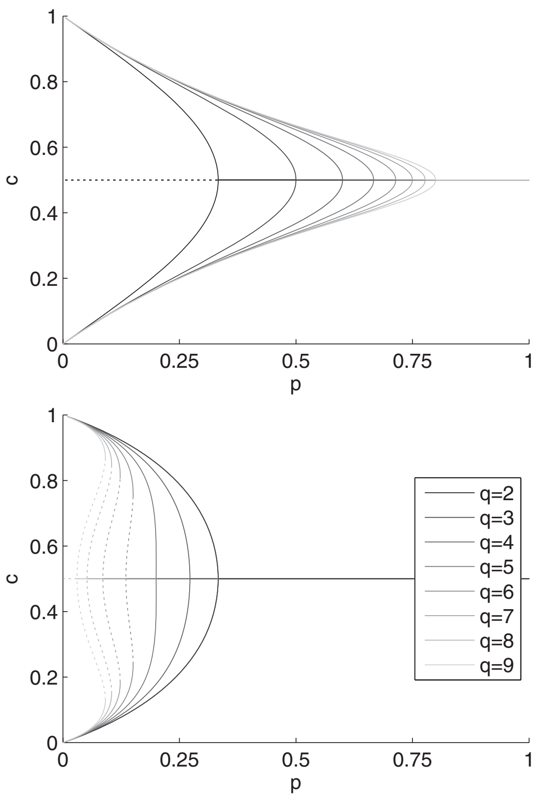

FIG. 7. Phase diagram for the models with anticonformity (top panel) and independence (bottom panel). Dependencies between steady values of concentration c_{st} and the level of noise p for several values of q are plotted using formulas (18)–(20). Although only relation $p(c_{st})$ is calculated analytically and the opposite relation is unknown, we plot $c_{st}(p)$ simply by rotating the figure. Dotted lines have been used to mark instability. Although both types of line (solid and dotted) are obtained from Eq. (14), i.e., correspond to extreme values of potentials (15) and (17), only solid lines denotes stable values, i.e., correspond to the minima of potentials (see also Fig. 8). A clear difference between two types of noise is visible—in a case with anticonformity the transition value of p increases with q and in a case with independence it decreases with p. Moreover, the type of transition is the same for arbitrary values of q in a case with anticonformity, whereas in a case with independence the transition between phases with and without majority changes its character for $q > 5$.

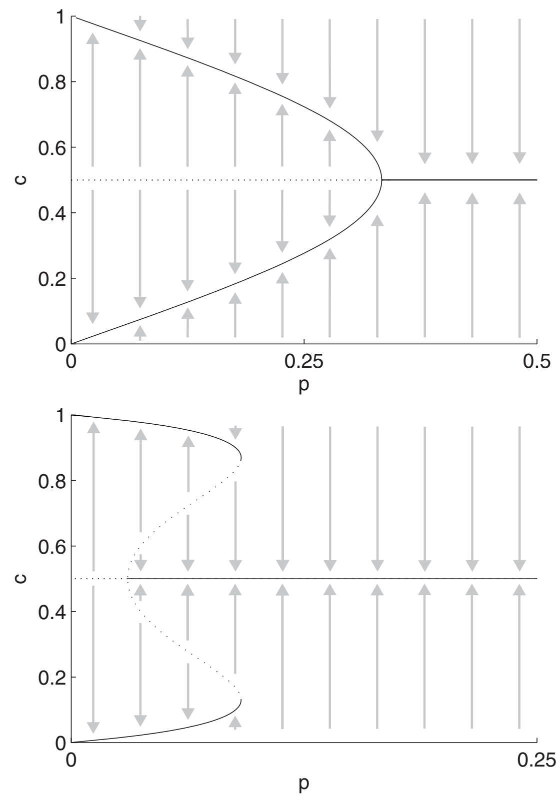

FIG. 8. Flow diagrams for the models with anticonformity for $q = 2$ (top panel) and independence for $q = 9$ (bottom panel). Particular values of q have been chosen just as examples and the dependencies between stationary values of c and parameter p for other values of q are seen in Fig. 7. Here solid lines denote stable (attracting) steady values of concentration that correspond to the minima of potentials (15)–(17), whereas dotted lines denote unstable values of c that correspond to maxima of potentials. Arrows denote the direction of flow, i.e., how the concentration changes in time. Particularly interesting behavior is related with independence (bottom panel). Starting from two different initial concentrations disorder or order can be reached as a steady state (hysteresis).

V. PHASE TRANSITIONS

As already noticed there is a continuous phase transition for the model with anticonformity I and II for arbitrary values of q. Below a critical value $p = p^*(q)$ the effective potential has two minima and above the critical value it has only one. Consequently, the stationary probability density function $\rho_{st}(c)$ for $p < p^*$ has two maxima and for $p > p^*$ only one at $c = 1/2$ (i.e., there is no majority in the system). Analogous behavior is observed for the model with nonconformity but only for $q \leqslant 5$. In all these cases we can easily calculate the critical value p^* by making a simple observation concerning the behavior of the effective potentials (15)–(17) for $q \leqslant 5$ at $c = 1/2$ (see also Fig. 1 for clarity): For $p < p^*$ potentials $V(c,p,q)$ have the maximum values for $c = 1/2$ and therefore

$$\left.\frac{\partial^2 V(c,p,q)}{\partial c^2}\right|_{c=\frac{1}{2}} < 0. \tag{21}$$

For $p > p^*$ potentials $V(c,p,q)$ have the minimum values for $c = 1/2$ and therefore

$$\left.\frac{\partial^2 V(c,p,q)}{\partial c^2}\right|_{c=\frac{1}{2}} > 0. \tag{22}$$

This means that for $p = p^*$ the maximum changes to the minimum at $c = 1/2$:

$$\left.\frac{\partial^2 V(c,p,q)}{\partial c^2}\right|_{c=\frac{1}{2}} = 0 \Rightarrow \left.\frac{\partial F(c,p,q)}{\partial c}\right|_{c=\frac{1}{2}} = 0. \tag{23}$$

Hence, the critical values are as follows: For anticonformity I,

$$p^*(q) = \frac{q-1}{q+1}, \tag{24}$$

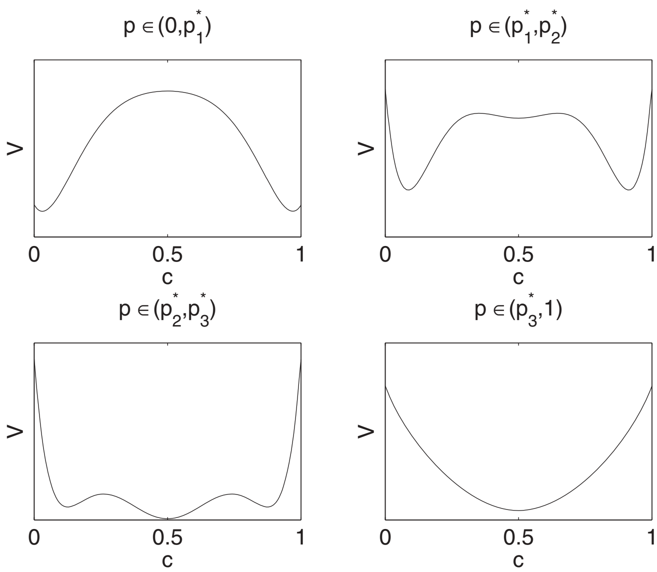

FIG. 9. Schematic plot of a potential for the model with independence and $q > 5$. For $p \in (0,p_1^*)$, potential $V(c,p,q)$ has two minima that correspond to the states with majority. For $p \in (p_1^*,p_2^*)$, the potential $V(c,p,q)$ has three minima and the state without majority is metastable. For $p \in (p_2^*,p_3^*)$, the potential $V(c,p,q)$ has three minima and the states with majority are metastable. Finally, for $p \in (p_3^*,1)$, the potential $V(c,p,q)$ has only one minimum that corresponds to the state without majority. The exact form of the potential is given by Eq. (17).

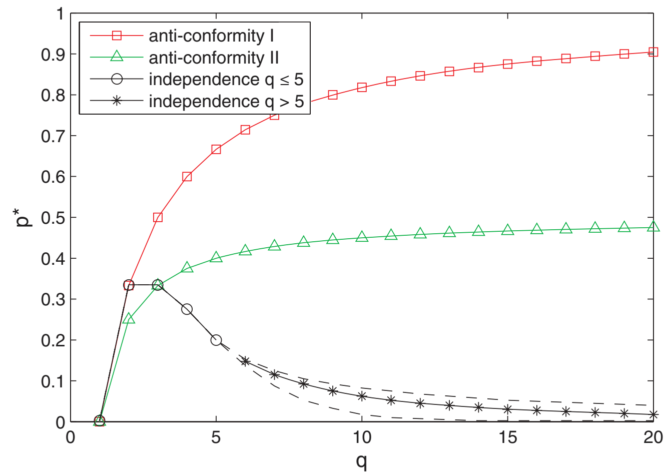

FIG. 10. (Color online) Transition points p^* as a function of q for all three models. Solid lines denote the line of the phase transition and dashed lines denote spinodal lines, i.e., determine the region with metastability. Several differences among models are visible. As seen, both models with anticonformity behave qualitatively the same: the critical value of p increases with q. However, for the model with independence the transition point decreases with p. Moreover, for $q \geqslant 6$ the phase transition changes its type from continuous to discontinuous.

for anticonformity II,

$$p^*(q) = \frac{q-1}{2q}, \tag{25}$$

and for independence with $q \leqslant 5$,

$$p^*(q) = \frac{q-1}{q-1+2^{q-1}}. \tag{26}$$

As we see, simple calculations allowed us to find the critical points for almost all cases, except for the model with nonconformity for $q \geqslant 6$. In all cases considered above, there is a continuous phase transition between phases with and without majority. However, for the model with independence and $q \geqslant 6$ the phase transition becomes discontinuous, which has been already discussed in Sec. III. This behavior can be also suspected from the form of the effective potential (17), which for $q \geqslant 6$ has the following properties (see also Fig. 9):

For $p \in \langle 0,p_1^*)$, $V(c,p,q)$ has two minima.
For $p = p_1^*$, in $V(c,p,q)$ a third minimum emerges.
For $p \in (p_1^*,p_2^*)$, $V(c,p,q)$ has three minima.
For $p = p_2^*$, $V(c,p,q)$ has three equal minima.
For $p \in (p_2^*,p_3^*)$, $V(c,p,q)$ has three minima.
For $p = p_3^*$, in $V(c,p,q)$ the third minimum disappears.
For $p \in (p_3^*,1\rangle$, $V(c,p,q)$ has one minimum.

As we see, there is an interval $p \in (p_1^*,p_3^*)$ in which potential $V(c,p,q)$ has three minima and therefore the stationary probability density function has three maxima (see Fig. 2). In this region we have a coexistence of two phases—with and without majority. For $p < p_2^*$ the state with majority is stable and the state without majority is metastable and for $p > p_2^*$ the state with majority is metastable and that without majority is stable. Consequently, the phase transition appears at $p = p_2^* = p^*$ and $p = p_1^*, p_3^*$ designate spinodal lines [27,28].

Transition points p^* as a functions of q for all three models are presented in Fig. 10. As seen, both models with anticonformity (I and II) behave qualitatively the same: the critical value of p increases with q. However, for the model with independence the transition point decreases with p. Moreover, for $q = 5$ in the case with independence, the phase transition changes its type from continuous to discontinuous. To clarify our results we decided to present the complete phase diagrams for the models with anticonformity and independence in Fig. 11. Because results for both models with anticonformity (I and II) are qualitatively the same we present the phase diagram only for the model with anticonformity II.

The first difference between models with anticonformity and independence—connected with the qualitative dependence between p^* and q—is easy to explain heuristically. It is quite obvious why in the model with independence the critical point p^* decreases when q increases. When q increases it becomes unlikely to choose randomly q parallel spins and therefore the noise term dominates because it is independent of the state of the q lobby. Similarly, it can be understood why in the model with anticonformity the critical point $p*$ increases with q. It should be recalled here that anticonformity takes place only when $q+1$ parallel spins are chosen randomly, which is more unlikely than choosing q parallel spins. Therefore the anticonformity term declines in importance even more than the conformity term as q increases. The second difference between models—the change of the transition type in the model with independence—is not so easy to understand intuitively. This result has been obtained numerically from the potential (17), but in the next section

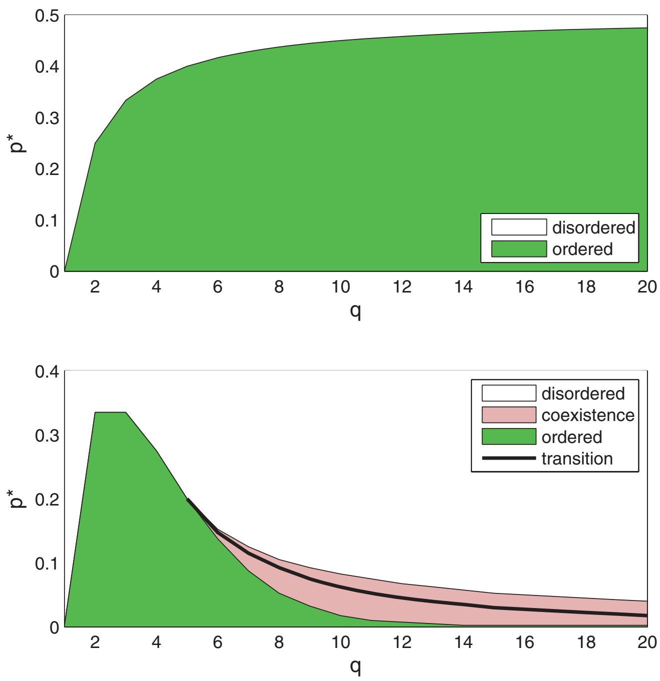

FIG. 11. (Color online) Phase diagrams for the models with anticonfomity II (top panel) and independence (bottom panel). As seen, for the model with anticonformity, the critical value p^* increases with q, and for the model with independence, it decreases with q. For anticonformity (top panel) there is a continuous phase transition (denoted by the solid line) between order (i.e., $c \neq 1/2$ or, equivalently, $m \neq 0$) and disorder (i.e., $c = 1/2$ or, equivalently, $m = 0$). In the model with independence (bottom panel) there is a continuous phase transition only for $q < 5$. At $q = 5$ the phase transition changes its type from continuous to discontinuous. For $q > 5$ an area in which one of two phases (ordered or disordered) is metastable is limited by so-called spinodal lines. This area is labeled as "coexistence," although the real coexistence occurs only on the transition line. However, in the region of metastability both phases can be observed depending on the initial conditions (hysteresis), which can be also seen from the flow diagram in Fig. 8.

we will show how this result could be also derived from an approximate Landau description.

VI. LANDAU DESCRIPTION

Although we were able to calculate critical points for the model with anticonformity and for the model with independence with $q \leqslant 5$ directly from the potentials (15)–(17), it can be instructive to use the classical description proposed by Landau for equilibrium phase transitions [26]. It has been shown that this kind of description can be also obtained as a mean-field approach for the Langevin equation of nonequilibrium systems with two (Z_2) symmetric absorbing states [21,22].

In our paper we have written the master equation as a function of concentration $c = N_\uparrow/N$ of up spins. We have decided to use this quantity for convenience since calculations are simple and equations have compact forms. However, to meet the symmetry requirement [21,22] one should use an order parameter (in this case magnetization) defined as

$$\phi = \frac{N_\uparrow + N_\downarrow}{N} \tag{27}$$

for which potentials (15)–(17) are symmetric under reversal $\phi \to -\phi$.

Following the approach presented in Refs. [21,22], which coincides with the classical approach proposed by Landau, we expand potentials (15)–(17), rewritten as a function of ϕ, into power series and keep only the first three terms of the expansion:

$$V(\phi) = A\phi^2 + B\phi^4 + C\phi^6, \tag{28}$$

where coefficients $A = A(p,q)$, $B = B(p,q)$, and $C = C(p,q)$ depend on the model.

For the model with independence,

$$\begin{aligned} A(p,q) &= -\frac{(1-p)}{2^q}\frac{(q-1)}{2} + \frac{p}{4},\\ B(p,q) &= -\frac{(1-p)}{2^q}\frac{q(q-1)(q-5)}{24},\\ C(p,q) &= -\frac{(1-p)}{2^q}\frac{q(q-1)(q-2)(q-3)(q-9)}{720}. \end{aligned} \tag{29}$$

From Landau theory it is known that for $B(p,q) > 0$ and $C(p,q) > 0$ there is a critical point at which $A(p,q)$ changes sign [26]. For $A < 0$ the potential $V(\phi)$ has two symmetric minima and thus the system is driven to one partially ordered state with $\phi \neq 0$. For $A > 0$ the potential $V(\phi)$ has a minimum at $\phi = 0$ and therefore the system remains in an active disordered state and a magnetization ϕ fluctuates around zero. From Eq. (29) it is easy to calculate that

$$\begin{aligned} A(p,q) = 0 \to p = p^* &= \frac{q-1}{q-1+2^{q-1}},\\ A < 0 \to p < p^*, &\quad \phi \neq 0,\\ A > 0 \to p > p^*, &\quad \phi = 0, \end{aligned} \tag{30}$$

which coincides with the result (26) obtained from the exact version of the potential (17).

As shown within classical Landau theory, for $B(p,q) < 0$ and $C(p,q) > 0$ a discontinuous jump in the order parameter is expected [26]. Again from Eq. (29) it is easy to see that $B(p,q) < 0$ for $q > 5$ (see also Fig. 12). Therefore we expect a discontinuous phase transition for $q > 5$, which also agrees with the results obtained from Eq. (17). It should be mentioned here that a transition for $B \leqslant 0$ could possibly be included in the class of generalized voter models (the so-called unique GV transition) [21]. It has been noticed for a general class of models with two (Z_2) symmetric absorbing states that for $B \leqslant 0$ the location of the potential minimum changes abruptly from $\phi = 0$ to $\phi \pm 1$; i.e., a discontinuous phase transition is observed [21]. In our model the situation is slightly different, because for $p > 0$ there are no absorbing states and below the transition point $|\phi| < 1$. However, still the system jumps from a totally disordered to a partially ordered state; i.e., a discontinuous phase transition is observed.

One should also notice that in the case of a q-voter model with independence for $q > 9$ also $C(p,q)$ becomes negative and than the approximation (28) is no longer valid.

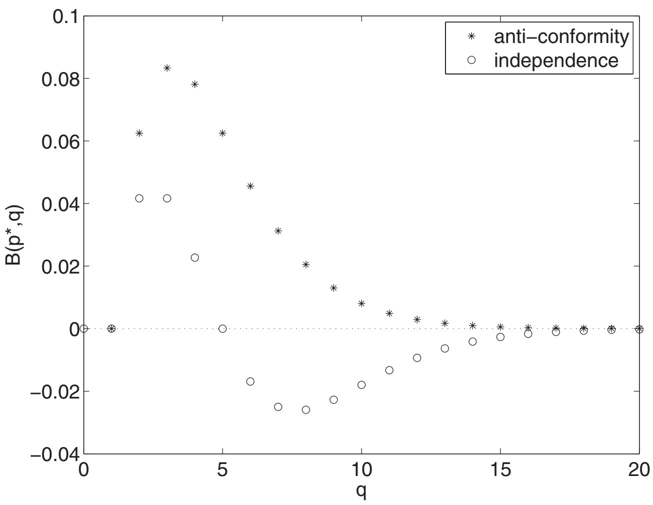

FIG. 12. Coefficient $B(p,q)$ [see the effective potential (28)] for a critical point $p = p^*$ at which $A(p,q)$ changes sign. For the model with independence (denoted by "o") coefficient $B < 0$ for $q > 5$, which suggests a discontinuous phase transition, whereas for the model with anticonformity (denoted by "*") coefficient $B \geqslant 0$ for any value of q and therefore the transition is continuous for arbitrary values of q.

Analogous calculations can be done for the models with anticonformity. Because both models with anticonformity are qualitatively the same we present here results for the model with anticonformity II. In this case,

$$\begin{aligned} A(p,q) &= \frac{2pq - q + 1}{2^q}, \\ B(p,q) &= -\frac{1}{4}\frac{1-p}{2^q}\left[\binom{q-1}{3} - \binom{q-1}{1}\right] + \frac{1}{4}\frac{p}{2^q}\binom{q+1}{3}, \\ C(p,q) &= -\frac{1}{6}\frac{1-p}{2^q}\left[\binom{q-1}{5} - \binom{q-1}{3}\right] + \frac{1}{6}\frac{p}{2^q}\binom{q+1}{5}. \end{aligned} \tag{31}$$

Therefore in the case with anticonformity,

$$\begin{aligned} A(p,q) = 0 \rightarrow p = p^* = \frac{q-1}{2q}, \\ A < 0 \rightarrow p < p^*, \quad \phi \neq 0, \\ A > 0 \rightarrow p > p^*, \quad \phi = 0, \end{aligned} \tag{32}$$

which coincides with the result (25). Moreover, for the model with anticonformity (see Fig. 12) coefficient $B(p = p^*,q) \geqslant 0$ for any value of q; i.e., the transition is continuous for arbitrary values of q.

VII. CONCLUSIONS

In this paper we have asked questions about the importance of the type of nonconformity (anticonformity and independence) that is often introduced in models of opinion dynamics (see, e.g., [13,14,19,20]). We realized that the differences between the different types of nonconformity are very important from social point of view [17] but we have expected that they may be irrelevant in terms of microscopic models of opinion dynamics. To check our expectations we have decided to investigate a nonlinear q-voter model on a complete graph, which has been recently introduced as a general model of opinion dynamics [12].

To our surprise, the results for the model with anticonformity are qualitatively different from those for the model with independence. In the first case there is a continuous order–disorder phase transition induced by the level of anticonformity p. The critical value of p grows with the size of the q lobby. On the other hand, for the model with independence the value of the transition point p^* decays with q. Moreover, the phase transition in this case is continuous only for $q \leqslant 5$. For larger values of q there is a discontinuous phase transition and coexistence of ordered (with majority) and disordered (without majority) phases is possible.

We have suggested in the title and the introduction of the paper that both types of nonconformity play the role of noise. However, only independence introduces real random noise, which plays a role similar to temperature. From this point of view the change of the type of transition resembles a similar phenomena in the Potts model (for a review see [29]). In the Potts model there is a first-order phase transition for $q > 4$ and a second-order phase transition for smaller values of q, where q denotes the number of spin states. Of course, in the case of our model q does not denote the number of states, which is always 2, but the size of the group. A similar observation has recently been made by Araujo *et al.* [30] within a model of tactical voting. They have considered q candidates on which citizens vote and proposed a balance function to quantify the degree of indecision in the society due to the coexistence of different opinions. It turned out that for some values of model parameters the model boiled down to the q-state Potts model, although similarly, like in our model, q denoted the number of candidates instead of the number of states. A similar change of the type of transition has been also observed in a general class of systems with two (Z_2) symmetric absorbing states within a Langevin description [21,22]. Moreover, it has been suggested that models with many intermediate states (i.e., the Potts model or a simple three-state model described in Ref. [22]) behave as equivalent two-state models with effective transitions that are nonlinear in the local densities [22], which is the case of a q-voter model or a two-state model of competition between two languages [31]. The theory presented in Refs. [21,22] suggests also that the continuous phase transition that is observed for $q < 5$ could possibly be included in the Ising class, whereas the discontinuous phase transition that is observed for $q > 5$ would fall into the class of generalized voter models.

Concluding the paper we would like to pay attention to one more phenomena that is visible in Fig. 10. For lobby $q = 2$ the results are the same for anticonformity and independence. Therefore it comes as no surprise that the difference between the two types of nonconformity has not been noticed while studying the Sznajd model (i.e., $q = 2$) [14,20].

ACKNOWLEDGMENT

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