State-space-split method for some generalized Fokker-Planck-Kolmogorov equations in high dimensions

Guo-Kang Er* and Vai Pan Iu
Faculty of Science and Technology, University of Macau, Macau Special Administrative Region, China
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The state-space-split method for solving the Fokker-Planck-Kolmogorov equations in high dimensions is extended to solving the generalized Fokker-Planck-Kolmogorov equations in high dimensions for stochastic dynamical systems with a polynomial type of nonlinearity and excited by Poissonian white noise. The probabilistic solution of the motion of the stretched Euler-Bernoulli beam with cubic nonlinearity and excited by uniformly distributed Poissonian white noise is analyzed with the presented solution procedure. The numerical analysis shows that the results obtained with the state-space-split method together with the exponential polynomial closure method are close to those obtained with the Monte Carlo simulation when the relative value of the basic system relaxation time and the mean arrival time of the Poissonian impulse is in some limited range.

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The investigation of the probabilistic solutions of nonlinear stochastic dynamical systems is attractive in various areas of science and engineering since many problems can be described with nonlinear stochastic dynamical systems. The statistical analysis of the nonlinear stochastic dynamical systems is based on the available probability density function (PDF) of the system responses [1–5]. When the noises are Gaussian white noise, the PDF of the system responses is governed by the Fokker-Planck-Kolmogorov (FPK) equations [6]. Various method were developed or employed to solve the FPK equation or the statistical quantities of the system responses [7–9]. However, obtaining the probabilistic solution of large-scale nonlinear stochastic dynamical systems or the FPK equations in high dimensions had been a challenge for almost a century since the probabilistic explanation of the Brownian motion of molecules by Einstein and the formulation of FPK equation by Fokker and Planck thereafter until the state-space-split (SSS) method was proposed recently for analyzing the probabilistic solutions of the large-scale nonlinear stochastic dynamical systems with a polynomial type of nonlinearity [10,11].

Gaussian white noise is a continuous process that can be considered as a special case of Poissonian white noise when the mean arrival rate of Poissonian impulse approaches infinity. Many noises in the real world can be modeled more reasonably with Poissonian white noise than Gaussian white noise because there is a time step between any two sequential impulses even if the time step can be small in many cases. Under the action of Poissonian white noise, the PDF of the responses of the nonlinear stochastic dynamical system is governed by the generalized Fokker-Planck-Kolmogorov (GFPK) equation or the Kolmogorov-Feller equation in the form of a truncated series [12]. The solutions of some two-dimensional reduced FPK and GFPK equations were analyzed with the exponential polynomial closure (EPC) method [8,13]. However, there are no reported solutions of the GFPK equations in higher dimensions because of the difficulties in solving the GFPK equation in high dimensions, although many problems in science and engineering are described as multiple-degree-of-freedom systems or high-dimensional nonlinear stochastic dynamic systems for which the PDF solution is governed by the GFPK equation. The equivalent linearization (EL) method was proposed and employed to analyze the multidimensional nonlinear stochastic dynamic systems excited by white noises to obtain the second moments of the system responses [1,14]. It is known that EL is suitable only for the weakly nonlinear systems when the Poissonian excitations are close to Gaussian or the ratio of the system relaxation time and the impulse mean arrival time of Poissonian noise is large because the system responses are close to Gaussian only in this case. The Monte Carlo simulation (MCS) is a numerical integration procedure [15]. With the MCS the computational effort is usually unacceptable for estimating the PDF of the responses of the large-scale nonlinear nonlinear stochastic dynamic systems, especially for small probability problems. The numerical convergence, stability, roundoff error, and requirement for great sample size in simulating small values of the PDF of system responses are inherent in the MCS in analyzing large-scale nonlinear stochastic dynamic systems.

In this paper the state-space-split method developed for analyzing the high-dimensional nonlinear stochastic dynamic systems with a polynomial type of nonlinearity and Gaussian white noise excitations or solving the FPK equation in high dimensions is further extended to analyze the high-dimensional nonlinear stochastic dynamic systems with Poissonian white noise excitations. The solution procedure is presented and the numerical results are given to show the effectiveness of the solution procedure. The limitations of the solution procedure are also stated.

In the following discussion the summation convention applies unless stated otherwise. The random state variable or vector is denoted by a capital letter and the corresponding deterministic state variable or vector is denoted by the same letter in lowercase. Consider the coupled Langevin equations or Ito differential equations

$$\frac{dX_i}{dt} = f_i(\mathbf{X}) + g_{ij} W_j(t), \tag{1}$$

where the state vector process $\mathbf{X} \in \mathbb{R}^{\mathbf{n_x}}$, X_i $(i = 1,2,\ldots,n_x)$ are components of the state vector process $\mathbf{X}$; $f_i(\mathbf{X}) : \mathbb{R}^{n_x} \to \mathbb{R}$;

*gker@umac.mo

and $W_j(t)$ $(j = 1,2,\ldots,m)$ are independent Poissonian white noises. The Poissonian white noise is formulated as

$$W_j(t) = \sum_{k=1}^{N_j(T)} \Omega_{jk}\delta(t-\tau_k), \tag{2}$$

where $N(T)$ is the total number of impulses that arrive in the time interval $(-\infty,T]$; Ω_{jk} is the impulse amplitude of the kth impulse arriving at time τ_k for $W_j(t)$; and $\delta(t)$ is the Dirac delta function. In this paper $N_j(T)$ is a counting process yielding the Poissonian law with a constant impulse arrival rate λ_j. The impulse amplitudes Ω_{jk} $[k = 1,2,\ldots,N_j(T)]$ are independent, identically distributed random variables with zero mean and also independent of the impulse arrival time τ_k. The ith moment of Ω_{jk} $[k = 1,2,\ldots,N_j(T)]$ is denoted by $E[\Omega_j^i]$ in view of Ω_{jk} being identically distributed random variables for any k.

The state vector $\mathbf{X}$ is Markovian and the PDF $p(\mathbf{x},t)$ of the Markovian vector is governed by the GFPK equation. The stationary PDF of the Markovian vector is governed by the reduced GFPK equation

$$\frac{\partial[f_i(\mathbf{x})p(\mathbf{x})]}{\partial x_i} - \frac{1}{2!}\frac{G_{ij}\partial^2 p(\mathbf{x})}{\partial x_i\partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^3 p(\mathbf{x})}{\partial x_i\partial x_j\partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^4 p(\mathbf{x})}{\partial x_i\partial x_j\partial x_m\partial x_n} + \cdots = 0\,(x_i,x_j,x_m,x_n \in \mathbb{R}^{n_x}), \tag{3}$$

where $\mathbf{x}$ is the deterministic state vector, $\mathbf{x} \in \mathbb{R}^{\mathbf{n_x}}$, $G_{ij} = \lambda_s E[\Omega_s^2]g_{is}g_{js}$, $G_{ijm} = \lambda_s E[\Omega_s^3]g_{is}g_{js}g_{ms}$, and $G_{ijmn} = \lambda_s E[\Omega_s^4]g_{is}g_{js}g_{ms}g_{ns}$. It is assumed that

$$\lim_{x_i\to\pm\infty}\left\{f_i(\mathbf{x})p(\mathbf{x}) - \frac{1}{2!}\frac{G_{ij}\partial p(\mathbf{x})}{\partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^2 p(\mathbf{x})}{\partial x_j\partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^3 p(\mathbf{x})}{\partial x_j\partial x_m\partial x_n} + \cdots\right\} = 0 \quad (x_i,x_j,x_m,x_n \in \mathbb{R}^{n_x}), \tag{4}$$

which can usually be fulfilled by the responses of the systems with a polynomial type of nonlinearity and white noise excitations.

With the SSS method [10,11], separate the state vector $\mathbf{X}$ into two parts $\mathbf{X}_1 \in \mathbb{R}^{n_{x_1}}$ and $\mathbf{X}_2 \in \mathbb{R}^{n_{x_2}}$, i.e., $\mathbf{X} = \{\mathbf{X}_1,\mathbf{X}_2\} \in \mathbb{R}^{n_{x_1}} \times \mathbb{R}^{n_{x_2}}$. Denote the PDF of X_1 by $p_1(\mathbf{x}_1)$. In order to obtain the governing equation for $p_1(\mathbf{x}_1)$, integrating both sides of Eq. (3) over $\mathbb{R}^{n_{x_2}}$ by part and noting Eq. (4) gives

$$\int_{\mathbb{R}^{n_{x_2}}} \frac{\partial f_i(\mathbf{x})p(\mathbf{x})}{\partial x_i}d\mathbf{x}_2 - \frac{1}{2!}\frac{G_{ij}\partial^2 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^3 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^4 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m\partial x_n} + \cdots = 0 \quad (x_i,x_j,x_m,x_n \in \mathbb{R}^{n_{x_1}}), \tag{5}$$

which can also be written equivalently, by exchanging the order of integral and derivative, as

$$\frac{\partial \int_{\mathbb{R}^{n_{x_2}}} f_i(\mathbf{x})p(\mathbf{x})d\mathbf{x}_2}{\partial x_i} - \frac{1}{2!}\frac{G_{ij}\partial^2 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^3 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^4 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m\partial x_n} + \cdots = 0 \quad (x_i,x_j,x_m,x_n \in \mathbb{R}^{n_{x_1}}). \tag{6}$$

Separate $f_i(\mathbf{x})$ into two parts as

$$f_i(\mathbf{x}) = f_i^{\mathrm{I}}(\mathbf{x}_1) + f_i^{\mathrm{II}}(\mathbf{x}). \tag{7}$$

Then Eq. (6) can be written as

$$\frac{\partial\left[f_i^{\mathrm{I}}(\mathbf{x}_1)p_1(\mathbf{x}_1) + \int_{\mathbb{R}^{n_{x_2}}} f_i^{\mathrm{II}}(\mathbf{x})p(\mathbf{x})d\mathbf{x}_2\right]}{\partial x_i} - \frac{1}{2!}\frac{G_{ij}\partial^2 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^3 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^4 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m\partial x_n} + \cdots = 0 \quad (x_i,x_j,x_m,x_n \in \mathbb{R}^{n_{x_1}}). \tag{8}$$

Set $f_i^{\mathrm{II}}(\mathbf{x}) = \sum_k f_i^{\mathrm{II}}(\mathbf{x}_1,\mathbf{z}_k)$ in which $\mathbf{z}_k \in \mathbb{R}^{n_{z_k}} \subset \mathbb{R}^{n_{x_2}}$ and n_{z_k} denotes the number of the state variables in $\mathbf{z}_k$. Then Eq. (8) can be expressed as

$$\frac{\partial\left[f_i^{\mathrm{I}}(\mathbf{x}_1)p_1(\mathbf{x}_1) + \sum_k \int_{\mathbb{R}^{n_{z_k}}} f_i^{\mathrm{II}}(\mathbf{x}_1,\mathbf{z}_k)p(\mathbf{x}_1,\mathbf{z}_k)d\mathbf{z}_k\right]}{\partial x_i} - \frac{1}{2!}\frac{G_{ij}\partial^2 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^3 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^4 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m\partial x_n} + \cdots = 0 \quad (x_i,x_j,x_m,x_n \in \mathbb{R}^{n_{x_1}}), \tag{9}$$

in which $p(\mathbf{x}_1,\mathbf{z}_k)$ denotes the joint PDF of $\{\mathbf{X}_1,\mathbf{Z}_k\}$. Because $p(\mathbf{x}_1,\mathbf{z}_k) = p_1(\mathbf{x}_1)q(\mathbf{z}_k;\mathbf{x}_1)$ in which $q(\mathbf{z}_k;\mathbf{x}_1)$ is the conditional PDF of $\mathbf{Z}_k$ for given $\mathbf{X}_1 = \mathbf{x}_1$, Eq. (9) can then be written as

$$\frac{\partial\left[f_i^{\mathrm{I}}(\mathbf{x}_1)p_1(\mathbf{x}_1) + p_1(\mathbf{x}_1)\sum_k \int_{\mathbb{R}^{n_{z_k}}} f_i^{\mathrm{II}}(\mathbf{x}_1,\mathbf{z}_k)q(\mathbf{z}_k;\mathbf{x}_1)d\mathbf{z}_k\right]}{\partial x_i} - \frac{1}{2!}\frac{G_{ij}\partial^2 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^3 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^4 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m\partial x_n} + \cdots = 0 \quad (x_i,x_j,x_m,x_n \in \mathbb{R}^{n_{x_1}}). \tag{10}$$

By approximately replacing $q(\mathbf{z}_k;\mathbf{x}_1)$ by the conditional PDF $\overline{q}(\mathbf{z}_k;\mathbf{x}_1)$ of $\mathbf{Z}_k$ for given $\mathbf{X}_1 = \mathbf{x}_1$ obtained from the EL method Eq. (10) is approximately written as

$$\frac{\partial\left[f_i^{\mathrm{I}}(\mathbf{x}_1)p_1(\mathbf{x}_1) + p_1(\mathbf{x}_1)\sum_k \int_{\mathbb{R}^{n_{z_k}}} f_i^{\mathrm{II}}(\mathbf{x}_1,\mathbf{z}_k)\overline{q}(\mathbf{z}_k;\mathbf{x}_1)d\mathbf{z}_k\right]}{\partial x_i} - \frac{1}{2!}\frac{G_{ij}\partial^2 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^3 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^4 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m\partial x_n} + \cdots = 0 \quad (x_i,x_j,x_m,x_n \in \mathbb{R}^{n_{x_1}}). \tag{11}$$

By setting

$$\overline{f}_i(\mathbf{x}_1) = f_i^{\mathrm{I}}(\mathbf{x}_1) + \sum_k \int_{\mathbb{R}^{n_{z_k}}} f_i^{\mathrm{II}}(\mathbf{x}_1,\mathbf{z}_k)\overline{q}(\mathbf{z}_k;\mathbf{x}_1)d\mathbf{z}_k, \tag{12}$$

Eq. (11) can be finally written as

$$\frac{\partial[\overline{f}_i(\mathbf{x}_1)p_1(\mathbf{x}_1)]}{\partial x_i} - \frac{1}{2!}\frac{G_{ij}\partial^2 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^3 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^4 p_1(\mathbf{x}_1)}{\partial x_i\partial x_j\partial x_m\partial x_n} + \cdots = 0 \quad (x_i,x_j,x_m,x_n \in \mathbb{R}^{n_{x_1}}), \tag{13}$$

which is the approximate GFPK equation for the joint PDF of the state variables in the substate space $\mathbb{R}^{n_{x_1}}$.

When the derivative terms that are higher than fourth order are neglected because the contribution of the higher-order terms is small when the mean arrival time of the Poissonian impulse is usually small enough and within some limited range, Eq. (13) can be reduced to

$$\frac{\partial[\overline{f}_i(\mathbf{x}_1)p_1(\mathbf{x}_1)]}{\partial x_i} - \frac{1}{2!}\frac{G_{ij}\partial^2 p_1(\mathbf{x}_1)}{\partial x_i \partial x_j} + \frac{1}{3!}\frac{G_{ijm}\partial^3 p_1(\mathbf{x}_1)}{\partial x_i \partial x_j \partial x_m} - \frac{1}{4!}\frac{G_{ijmn}\partial^4 p_1(\mathbf{x}_1)}{\partial x_i \partial x_j \partial x_m \partial x_n} = 0 \quad (x_i, x_j, x_m, x_n \in \mathbb{R}^{n_{x_1}}). \tag{14}$$

If $\mathbf{x}_1$ is chosen to have few-state variables, for instance, two-state variables, the resulting GFPK equation is of low dimension and the EPC method can be employed to solve Eq. (14) [8,13]. The numerical analysis is based on Eq. (14) in the following discussion.

Consider the stretched Euler-Bernoulli beam with a constant cross section. The governing equation about the motion of the beam is

$$\rho\ddot{y}(x,t) + c\dot{y}(x,t) + EI\frac{\partial^4 y(x,t)}{\partial x^4} - \frac{EA}{2L}\frac{\partial^2 y(x,t)}{\partial x^2}\int_0^L \left[\frac{\partial y(x,t)}{\partial x}\right]^2 dx = q_0 W(t), \tag{15}$$

where $y(x,t)$ is the deflection of the beam, L is the length of the beam, E is Young's modulus, I is the moment inertia of the cross section of the beam, A is the area of the cross section of the beam, ρ (kg/m) is the mass density of the beam, c is the damping constant of the beam, $q_0 W(t)$ is the distributed loading laterally applied on the beam, $W(t)$ is Poissonian white noise whose impulse amplitude is zero-mean Gaussian with mean square $E[\Omega^2]$, and q_0 is a constant. For the beam with two ends supported with hinges, the boundary conditions of the beam are

$$y(0,t) = y(L,t) = \frac{\partial^2 y(0,t)}{\partial x^2} = \frac{\partial^2 y(L,t)}{\partial x^2} = 0. \tag{16}$$

In order to solve this problem using the Galerkin method, $y(x,t)$ is expressed as

$$y(x,t) = \sum_{i=1,3,\ldots}^{2m-1} a_i(t)\sin\frac{i\pi x}{L}. \tag{17}$$

With the Galerkin method, the following equations are obtained:

$$\ddot{a}_i + \frac{c}{\rho}\dot{a}_i + \frac{EI\pi^4 i^4}{\rho L^4}a_i + \frac{EA\pi^4 i^2}{4L^4\rho}\sum_{j=1,3,\ldots}^{2m-1} j^2 a_i a_j^2 = \beta_i W(t), \quad i = 1,3,\ldots,2m-1, \tag{18}$$

in which $\beta_i = \frac{4q_0}{i\pi\rho}$.

Equation (18) represents a coupled nonlinear stochastic dynamic system with m degrees of freedom, a polynomial type of nonlinearity, and excited by Poissonian white noise. Many systems similar to Eq. (18) can be obtained from practical problems in science and engineering. In order to conduct a numerical analysis, a four-degree-of-freedom nonlinear stochastic dynamic system is formulated by setting $m = 4$ in Eq. (18). Taking the deflection and the velocity in the middle of the beam with $x = 0.5L$ as the state variables formulating the substate vector $\mathbf{X}_1$, the PDF solutions of the deflection and the velocity in the middle of the beam are analyzed using the SSS-EPC method. The Monte Carlo simulation is also conducted to verify the effectiveness of the SSS-EPC method in solving the GFPK equations in high dimensions or analyzing the PDF solution of the multiple-degree-of-freedom systems with a polynomial type of nonlinearity and excited by Poissonian white noise. In the numerical analysis, the parameter values are given as $L = 5$ m, $E = 2.1 \times 10^{11}$ N/m^2, $I = 2.17 \times 10^{-4}$ m^4, $A = 8.6112 \times 10^{-3}$ m^2, $\rho = 67.598$ kg/m, $c = 10^3$ N s/m^2, and $q_0 = 10^4$ N/m. The mean arrival rate λ of the Poissonian impulse equals 0.1 and $\lambda E[\Omega^2] = 2$.

Figures 1 and 2 present the PDFs and logarithmic PDFs of the deflection in the middle of the beam, respectively. The MCS is conducted to verify the effectiveness of SSS-EPC method in this case. The MCS was conducted on the original multiple-degree-of-freedom system rather than the low- or two-dimensional systems. The sample size used in the simulation is 10^8. From Fig. 1 it can be seen that the PDF of the deflection obtained by the SSS-EPC method with the polynomial degree n being four in the EPC procedure is close to that obtained by the MCS. The tails of the PDF

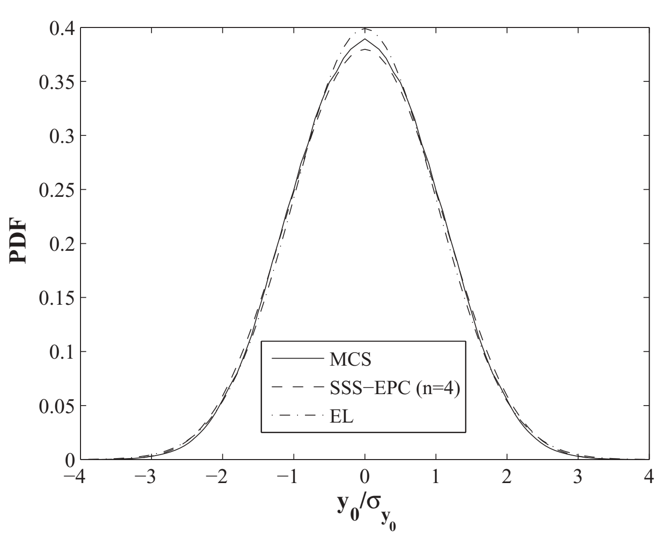

FIG. 1. The PDF of the beam deflection at $x = 0.5L$.

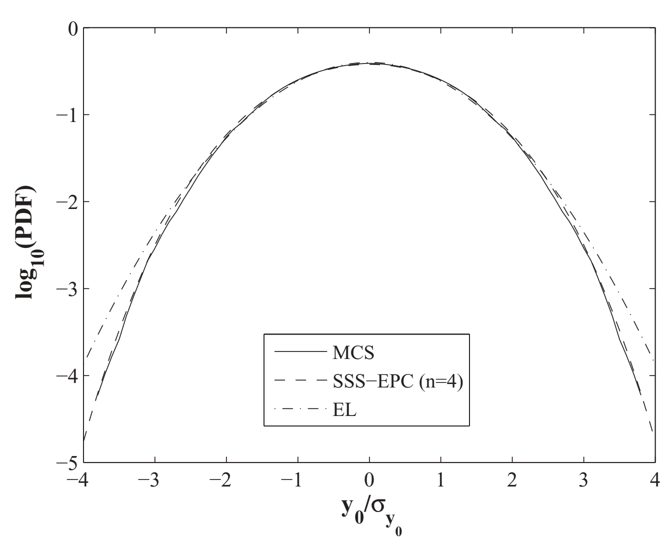

FIG. 2. The logarithm of the PDF of the beam deflection at $x = 0.5L$.

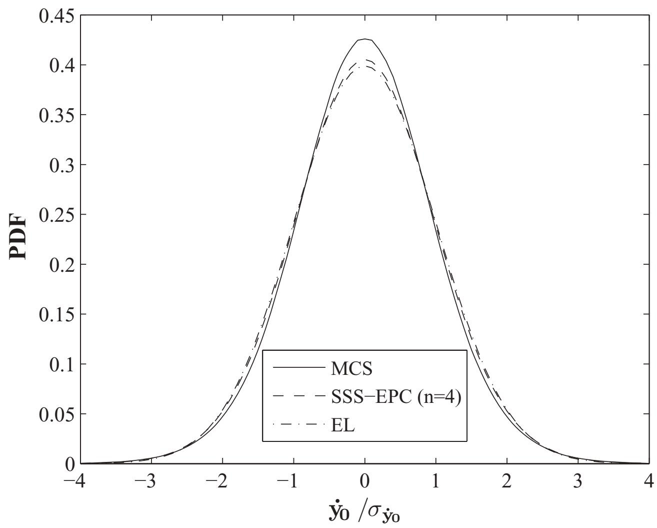

FIG. 3. The PDF of the beam velocity at $x = 0.5L$.

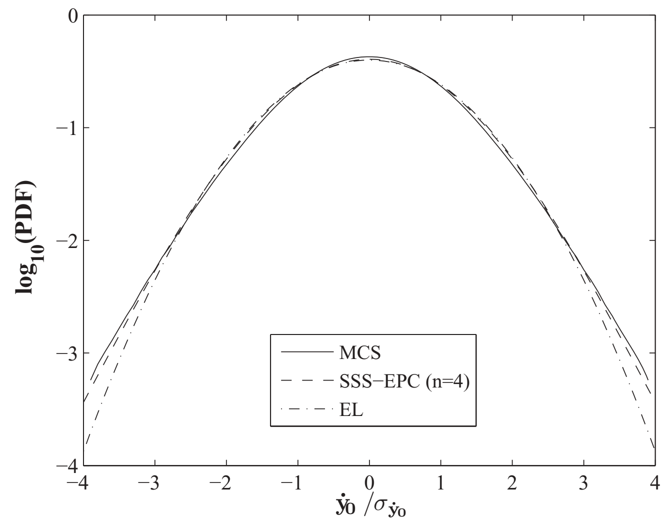

FIG. 4. The logarithm of the PDF of the beam velocity at $x = 0.5L$.

of the deflection and their comparison are shown clearly in Fig. 2, from which it can be seen that the PDF value of the deflection obtained by the SSS-EPC method is also close to that obtained by the MCS even in the tails of the PDF. The PDFs of the velocity in the middle of the beam obtained by the SSS-EPC method, EL, and MCS are compared in Figs. 3 and 4, respectively. It can also be observed that the PDF and the tails of the PDF of the velocity obtained by the SSS-EPC method are close to those obtained by the MCS. In Figs. 1–4 it is shown that the results obtained by EL deviate greatly from those from the MCS, but it should be mentioned that the result obtained by EL is needed in the solution procedure with the SSS-EPC method. The numerical results obtained by the SSS-EPC method are better than those obtained by EL because the results from EL are only used to approximately transform the state variables in X_2 into the state variables in X_1 as shown in Eq. (11) and the obtained approximate FPK equation is solved by the EPC method with which the obtained solution of the nonlinear system is of higher accuracy than that obtained by EL.

The above numerical analysis shows that the SSS-EPC method is also effective in analyzing the PDF of the responses of the multiple-degree-of-freedom nonlinear stochastic dynamic systems with some polynomial type of nonlinearity and excited by Poissonian white noise. When the ratio of the relaxation time corresponding to the dominate frequency of the system responses and the impulse mean arrival time is greater than some value, e.g., 10, the terms with fifth-order and higher derivatives in the GFPK equations can be neglected because the coefficients of the higher derivatives become small compared to those of other terms. Otherwise, neglecting these terms can lead to large error in the solution.

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[1] R. C. Booton, IRE Trans. Circuit Theory **1**, 9 (1954).
[2] S. M. Weng, Z. M. Sheng, M. Q. He, P. A. Norreys, M. Sherlock, and A. P. L. Robinson, Phys. Rev. Lett. **100**, 185001 (2008).
[3] L. Sommerlade, M. Eichler, M. Jachan, K. Henschel, J. Timmer, and B. Schelter, Phys. Rev. E **80**, 051128 (2009).
[4] B. Shotorban, Phys. Rev. E **81**, 046706 (2010).
[5] P. N. Patrone, T. L. Einstein, and D. Margetis, Phys. Rev. E **82**, 061601 (2010).
[6] H. Risken, *The Fokker-Planck Equation, Methods of Solution and Applications* (Springer, Berlin, 1989).
[7] D. S. Zhang, G. W. Wei, D. J. Kouri, and D. K. Hoffman, J. Chem. Phys. **106**, 5216 (1997).
[8] G. K. Er, Physica A **262**, 118 (1999).
[9] R. V. Bobryk, Phys. Rev. E **83**, 057701 (2011).
[10] G. K. Er, Ann. Phys. (Berlin) **523**, 247 (2011).
[11] G. K. Er and V. P. Iu, in *A New Method for the Probabilistic Solutions of Large-Scale Nonlinear Stochastic Dynamic Systems*, edited by W. Q. Zhu, Y. K. Lin, and G. Q Cai, *Proceedings of IUTAM Symposium on Nonlinear Stochastic Dynamics and Control*, IUTAM Book Series, Vol. 29 (Springer, Berlin, 2011), pp. 25–34.
[12] A. V. Plyukhin, Phys. Rev. E **77**, 061136 (2008).
[13] G. K. Er, H. T. Zhu, V. P. Iu, and K. P. Kou, AIAA J. **46**, 2839 (2008).
[14] L. Socha, *Linearization Methods for Stochastic Dynamic Systems* (Springer, Berlin, 2008).
[15] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations* (Springer, Berlin, 1995).