Acoustic vector solitons

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A theory of an acoustic vector soliton of self-induced transparency is constructed. By using the perturbative reduction method the magnetic Bloch equations and the equation of motion for the displacement field for the small area pulse are reduced to a system of two coupled nonlinear Schrödinger equations. The shape of an acoustic vector soliton with the sum and difference of the frequencies is presented. Explicit analytical expressions for the parameters of an acoustic vector soliton are obtained as well as simulations of an acoustic vector soliton presented with realistic parameters which can be reached in experiments. It is shown that the vector soliton in the special case can be reduced to the breather solution, and these nonlinear waves have different profiles.

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The existence of acoustical nonlinear solitary waves is one of the most interesting and important manifestations of nonlinearity in various solids (crystals and nanostructures). The determination of the mechanisms causing the formation of acoustical nonlinear waves and the investigation of their properties are among the principal problems of the physics of acoustical nonlinear waves. Depending on the nature of the nonlinearity, the resonant and nonresonant nonlinear waves can be considered. The resonant acoustical nonlinear waves can be formed with the help of the resonance (McCall-Hahn) mechanism of the formation of nonlinear waves, i.e., from a nonlinear coherent interaction of an acoustic pulse with resonance impurity atoms, when the conditions of acoustic self-induced transparency (SIT), $\omega T_p \gg 1$ and $T_p \ll T_{1,2}$, are fulfilled [1–3]. If the area of the pulse envelope as a measure of the pulse-matter interaction strength $\Theta > \pi$, a soliton is formed, and if $\Theta \ll 1$, a breather (pulsing soliton) is generated. Here Θ, ω, and T_p are the acoustic pulse area, frequency, and width, respectively, while T_1 and T_2 are the longitudinal and transverse relaxation times of the resonant impurity atoms.

Originally acoustic SIT was investigated in atomic systems but later the search for acoustical nonlinear waves was extended to nanostructures. The acoustic nonlinear waves have been experimentally observed in a variety of paramagnetic crystals, including CaF_2:U^{4+}, MgO:Fe^{2+}, MgO:Ni^{2+}, and $LiNbO_3$:Fe^{2+} [1,4], as well as in superfluid ^{3}He-A [5], in a glass [6], in a liquid [7], and in nanostructures [8].

The theoretical development of the effect of acoustic SIT is based on the magnetic Bloch equations and the equation of motion for the displacement one-component field (see, for instance, Ref. [9] and references therein). Such one-component nonlinear waves form when a single acoustic pulse propagates inside a medium containing resonance impurity atoms in such a way that it maintains its state. When these conditions are not satisfied, one has to consider interaction of two field components at different frequencies or polarizations as a bound state, and the magnetic Bloch equations and the equation of motion for the displacement field are reduced to the coupled nonlinear Schrödinger (NLS) equations. A shape-preserving solution of these equations is an acoustic vector pulse because of its two-component nature.

It is very important to find two-component double periodic solutions of nonlinear equations to provide more information for understanding many physical phenomena arising in numerous scientific fields and applications. In the last few decades, direct search for double periodic solutions of different nonlinear equations has become increasingly attractive [10–12]. The vector solitons arise in many physical phenomena when two wave packets interact with each other, but double periodic acoustic waves have not been considered up to now.

The purpose of the present work is to consider the conditions of realization of the resonant acoustic vector soliton of SIT with the sum and difference of the frequencies and the determination of the explicit analytic expressions for the parameters of the acoustic vector pulse.

We consider the formation of acoustical vector solitons of SIT in paramagnetic crystals with cubic symmetry which contain a small concentration of paramagnetic impurity atoms n_0, whose effective spin $S = \frac{1}{2}$. Suppose that an external constant magnetic field H_0 is applied parallel to one of the crystal axes of the fourth order, which we will take to be along the z axis (both the x and y axes are taken to be parallel to the other crystal axes of the fourth order). We shall consider a transverse circular-polarized acoustic pulse with width $T_p \ll T_{1,2}$, frequency $\omega \gg T_p^{-1}$, and wave vector $\vec{k}$, propagating along the positive z axis and parallel to the external constant magnetic field.

We model the paramagnetic impurity atoms with a two-level system which can be described by states $|1\rangle$ and $|2\rangle$, with energies $\mathcal{E}_1 = 0$ and $\mathcal{E}_2 = \hbar\omega_0$, respectively, where $|1\rangle$ is the ground state and $\hbar$ is Planck's constant. We take $\omega_0 = \gamma_M H_0$ to be tuned to the Zeeman frequency, where γ_M is the gyromagnetic ratio.

The Hamiltonian and wave function of this system are [9,13]:

$$\hat{H} = \hat{H}_Z + \hat{V}, \quad |\Psi\rangle = \sum_{n=1,2} c_n(t) e^{-\frac{i}{\hbar}\mathcal{E}_n t} |n\rangle,$$

where $\hat{H}_Z = \hbar\omega_0 \hat{s}^z$ is the Zeeman Hamiltonian of the two-level spin-system, and $\hat{V} = \frac{L}{2}(\varepsilon^+ \hat{s}^- + \varepsilon^- \hat{s}^+)$ is the Hamiltonian of the spin-phonon interaction. The quantities $\mathcal{E}_1$ and $\mathcal{E}_2$ are eigenvalues of the Hamiltonian $\hat{H}_Z$, $c_n(t)$ is the amplitudes of probabilities of the corresponding states, $\hat{s}^\pm = \hat{s}^x \pm i\hat{s}^y$, $\varepsilon^\pm = \varepsilon_{xz} \pm i\varepsilon_{yz}$, $\hat{s}^{x,y,z}$ are the x, y, and z components of the

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electron spin-operators for the paramagnetic impurity atoms, and ε_{xz} and ε_{yz} are components of the deformation tensor [14]. $L = \beta_0 H_0 F_{xzxz}$, β_0 is the Bohr magneton, and $F_{xzxz} = F_{yzyz}$ are components of the spin-phonon coupling tensor. We will consider only a transversely circularly polarized acoustic wave, and we also assume translational invariance in the x and y directions so that all field quantities do not depend on the coordinates x and y, i.e., $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$.

The average values s^i of the spin operators $\hat{s}^i$ for the state $|\Psi>$, are $s^i = \mathrm{Tr}<\Psi|\hat{s}^i|\Psi>$, where $i = x,y,z$ [13].

Using the method of slowly changing profile, we represent the functions ε_{xz} and s^x in the form

$$\varepsilon_{xz} = \frac{1}{2}\sum_{l=\pm1} \hat{E}_l Z_l, \quad s^x = -\frac{i}{2}\sum_{l=\pm1} l\,\rho_{-l} Z_l, \tag{1}$$

where $\hat{E}_l$ and ρ_l are the complex envelope of the acoustic field and spin variable, and $Z_l = e^{il(kz-\omega t)}$ contains the rapidly varying phase of the carrier wave. To guarantee the reality of the quantity ε_{xz}, we set $\hat{E}_l = \hat{E}^*_{-l}$.

The magnetic Bloch equations have the following form:

$$\frac{\partial \rho_l}{\partial t} = il\Delta\rho_l - \frac{L}{\hbar}\hat{E}_{-l}s_z, \quad \frac{\partial s_z}{\partial t} = \frac{L}{2\hbar}\sum_{l=\pm1}\hat{E}_l\rho_l, \tag{2}$$

where $\Delta = \omega_0 - \omega$. Equations (2) are exact only in the limit of infinite relaxation times.

To study acoustic solitons as self-consisted solutions of the magnetic Bloch's equations and the equation of the acoustic field, we need, in addition to Eqs. (2) for the material, a description of the pulse propagation in the medium by the equations of the theory of elasticity for the transverse-polarized acoustic pulse [14]:

$$\frac{\partial^2 \varepsilon_{xz}}{\partial t^2} = c_t^2 \frac{\partial^2 \varepsilon_{xz}}{\partial z^2} + \frac{Ln_0}{2\rho}\frac{\partial^2 s_x}{\partial z^2}, \tag{3}$$

where ρ is the density of the medium, and c_t is the velocity of the transverse-polarized linear acoustic wave in medium.

Upon taking into account the dispersion law for the transverse-polarized acoustic wave in a medium,

$$\omega^2 = c_t^2 k^2, \tag{4}$$

for a complex envelope of the acoustic wave $\hat{E}_l$ we obtain

$$\sum_{l=\pm1} Z_l\Big[-2il\omega\frac{\partial \hat{E}_l}{\partial t} - 2ilkc_t^2\frac{\partial \hat{E}_l}{\partial z} + \frac{\partial^2 \hat{E}_l}{\partial t^2} - c_t^2\frac{\partial^2 \hat{E}_l}{\partial z^2} - il\frac{Ln_0k^2}{2\rho}\int \frac{g(\Delta)d\Delta}{1+T_p^2\Delta^2}\rho_{-l}|_{\Delta=0}\Big] = 0, \tag{5}$$

where $g(\Delta)$ is the inhomogeneous broadening function of the spectral line of the impurities.

Equations (2), (4), and (5) are the general equations for the slowly varying complex amplitudes $\hat{E}_l$ and ρ_l by means of which we can consider a quite wide class of coherent acoustic phenomena, for instance, Rabi oscillations, spin echo, acoustic SIT, and others.

To further analyze these equations we make use of the multiple scale perturbative reduction method [15], in the limit that $\psi_l(z,t) = \int_{-\infty}^{t}\hat{E}_l(z,t')\,dt'$ is $O(\epsilon)$, with its scale-length being of order $O(\epsilon^{-1})$. This is the typical scaling for the coupled NLS equations and would then also be the scaling for two-component soliton. In this case $\psi_l(z,t)$ can be represented as

$$\psi_l(z,t) = \sum_{\alpha=1}^{\infty}\varepsilon^{\alpha}\psi_l^{(\alpha)} = \sum_{\alpha=1}^{\infty}\sum_{n=-\infty}^{+\infty}\varepsilon^{\alpha}Y_{l,n}\varphi_{l,n}^{(\alpha)}(\zeta_{l,n},\tau), \tag{6}$$

where $\zeta_{l,n} = \varepsilon Q_{l,n}(z - v_{g(l,n)}t)$, $\tau = \varepsilon^2 t$,

$$Y_{l,n} = e^{in(Q_{l,n}z - \Omega_{l,n}t)}, \quad v_{g(l,n)} = \frac{\partial \Omega_{l,n}}{\partial Q_{l,n}},$$

and ε is a small parameter. Such a representation allows us to separate from $\psi_l(z,t)$ the still more slowly changing quantities $\varphi_{l,n}^{(\alpha)}$. Consequently it is assumed that the quantities $\Omega_{l,n}$, $Q_{l,n}$, and $\varphi_{l,n}^{(\alpha)}$ satisfy the inequalities for any l and n: $\omega \gg \Omega$, $k \gg Q$,

$$\left|\frac{\partial \varphi_{l,n}^{(\alpha)}}{\partial t}\right| \ll \Omega|\varphi_{l,n}^{(\alpha)}|, \quad \left|\frac{\partial \varphi_{l,n}^{(\alpha)}}{\partial z}\right| \ll Q|\varphi_{l,n}^{(\alpha)}|.$$

We have to note that the quantities Q and Ω depend on l and n, but for simplicity, we omit these indexes in equations where this will not bring about any confusion. From the condition $\hat{E}_l = \hat{E}^*_{-l}$ it follows that $\varphi^{*(\alpha)}_{l,n} = \varphi^{(\alpha)}_{-l,-n}$.

Substituting Eq. (6) into Eqs. (2) and (5), we obtain the nonlinear wave equation

$$\sum_{l=\pm1}\sum_{\alpha=1}\sum_{n=-\infty}^{+\infty}\varepsilon^{\alpha}Z_lY_{l,n}\left\{\tilde{W}_{l,n} + \varepsilon J_{l,n}\frac{\partial}{\partial \zeta} + \varepsilon^2 h_{l,n}\frac{\partial}{\partial \tau} + \varepsilon^2 iH_{l,n}\frac{\partial^2}{\partial \zeta^2}\right\}\varphi_{l,n}^{(\alpha)} = -\varepsilon^3 i\frac{L^2\alpha_0^2}{4\hbar^2}\sum_{l=\pm1} lZ_l\int \frac{\partial \psi_l^{(1)}}{\partial t}\psi_{-l}^{(1)}\psi_l^{(1)}dt' + O(\varepsilon^4), \tag{7}$$

where

$$W_{l,n} = in\Omega\left(A_l n\Omega - B_l nQ + n^2\Omega^2 - c_t^2 n^2 Q^2 - \frac{l}{n}\frac{\alpha_0^2}{\Omega}\right),$$

$$J_{l,n} = nQ[2A_l\Omega v_g - B_l(Qv_g + \Omega) + 3n\Omega^2 v_g - c_t^2 nQ(Qv_g + 2\Omega)], \quad h_{l,n} = -2nA_l\Omega + B_l nQ - 3n^2\Omega^2 + c_t^2 n^2 Q^2,$$

$$H_{l,n} = Q^2[-A_l v_g^2 + B_l v_g - 3n\Omega v_g^2 + c_t^2 n(2Qv_g + \Omega)], \quad A_l = 2l\omega, \quad B_l = 2lkc_t^2, \quad \alpha_0^2 = \frac{L^2 n_0 k^2}{4\rho\hbar}\int \frac{g(\Delta)\,d\Delta}{1+\Delta^2 T_p^2}. \tag{8}$$

To determine the values of $\varphi_{l,n}^{(\alpha)}$, we equate to zero the various terms corresponding to the same powers of ε. As a result, we obtain a chain of equations. Starting with first order in ε, we have

$$\sum_{l=\pm 1}\sum_{n=\pm l} Z_l Y_{l,n}\tilde{W}_{l,n}\varphi_{l,n}^{(1)} = 0. \tag{9}$$

We shall be interested in localized solitary waves, which vanish as $t \to \pm\infty$. Consequently, according to Eq. (9), only the following components of $\varphi_{l,n}^{(1)}$ can differ from zero: $\varphi_{\pm 1,\pm 1}^{(1)}$ or $\varphi_{\pm 1,\mp 1}^{(1)}$. The relation between the parameters Ω and Q is determined from Eq. (9) and has the form

$$A_l n\Omega^2 - B_l n Q\Omega + \Omega^3 - c_t^2 Q^2\Omega - \mathrm{ln}\alpha_0^2 = 0. \tag{10}$$

Substituting Eq. (10) into Eq. (8), we easily see that the following relation holds: $J_{\pm 1,\pm 1} = J_{\pm 1,\mp 1} = 0$. To second order in ε we obtain the relation for $\varphi_{l,n}^{(2)}$. From Eq. (7), to third order in ε, we finally obtain two coupled NLS equations for functions $u_\pm = \varepsilon\varphi_{+1,\pm 1}^{(1)}$ that describe the coupling between two components of the pulse

$$i\left(\frac{\partial u_\pm}{\partial t} + v_\pm\frac{\partial u_\pm}{\partial z}\right) + p_\pm\frac{\partial^2 u_\pm}{\partial z^2} + g_\pm|u_\pm|^2 u_\pm + r_\pm|u_\mp|^2 u_\pm = 0, \tag{11}$$

where

$$v_\pm = v_{g(+1,\pm 1)}, \quad p_\pm = \frac{-H_{+1,\pm 1}}{h_{+1,\pm 1}Q_{\pm 1}^2}, \qquad g_\pm = \frac{-L^2\alpha_0^2}{4\hbar^2\, h_{+1,\pm 1}}, \quad r_\pm = g_\pm\left(1 - \frac{\Omega_{\mp 1}}{\Omega_{\pm 1}}\right), \tag{12}$$

$\Omega_{\pm 1} = \Omega_{ln\gtrless 0}$, $Q_{\pm 1} = Q_{ln\gtrless 0}$. The nonlinear equations (11) describe the slowly varying envelope functions $u_\pm$, where u_+ describes the envelope wave of the frequency $\omega + \Omega_{+1}$ and u_- describes the wave with frequency $\omega - \Omega_{-1}$. The nonlinear coupling between the two waves is governed by the terms $r_\pm|u_\mp|^2 u_\pm$. We must consider interaction of these field components at different frequencies and solve simultaneously a set of coupled NLS equations (11). A shape-preserving solution of Eqs. (11) is a vector pulse because of its two-component structure.

The steady-state solutions for complex amplitudes have the following form:

$$u_\pm(z,t) = \frac{A_\pm}{bT_p}\mathrm{sech}\left(\frac{t - \frac{z}{V_0}}{T_p}\right)e^{i\phi_\pm} \tag{13}$$

which is a well-known steady-state 2π pulse (soliton) of SIT [1]. Here V_0 is the constant pulse velocity:

$$b^2 = V_0^2\frac{A_+^2 q_+ + A_-^2 r_+}{2p_+}, \quad T_p^{-2} = V_0^2\frac{v_+k_+ + k_+^2 p_+ - \omega_+}{p_+}. \tag{14}$$

Here $\phi_\pm = k_\pm z - \omega_\pm t$ are the phase functions, and $A_\pm$, $k_\pm$, and $\omega_\pm$ are all real constants. Derivatives of the phases $\phi_\pm$ are assumed to be small; i.e., the functions $e^{i\phi_\pm}$ are slow in comparison with oscillations of the pulse, and consequently, the inequalities $k_\pm \ll Q_{\pm 1}$, $\omega_\pm \ll \Omega_{\pm 1}$ are satisfied.

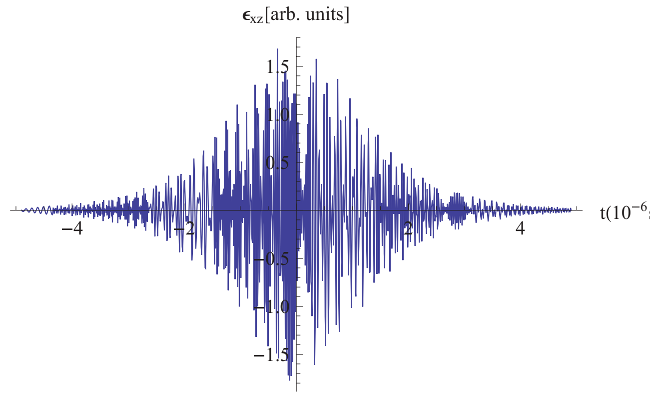

FIG. 1. (Color online) ε_{xz} component of the deformation tensor the two-component vector soliton is shown for a fixed value of z. The nonlinear pulse oscillates with the sum $\omega + \Omega_{+1} + \omega_+$ and difference $\omega - \Omega_{-1} + \omega_-$ of the frequencies along the t axis.

Substituting the solutions for the functions $u_\pm$ Eq. (13) of the coupled NLS equations (11) into Eqs. (1) and (6), we obtain for the ε_{xz} component of the deformation tensor the two-component vector soliton solution:

$$\varepsilon_{xz} = \frac{1}{bT_p}\mathrm{sech}\left(\frac{t - \frac{z}{V_0}}{T_p}\right)\{(\Omega_{+1} + \omega_+)A_+ \times \sin[(k + Q_{+1} + k_+)z - (\omega + \Omega_{+1} + \omega_+)t] - (\Omega_{-1} - \omega_-)A_- \sin[(k - Q_{-1} + k_-)z - (\omega - \Omega_{-1} + \omega_-)t]\}, \tag{15}$$

where the relations between the parameters $A_\pm$, $\omega_\pm$, and $k_\pm$ have the form

$$A_+^2 = \frac{p_+q_- - p_-r_+}{p_-q_+ - p_+r_-}A_-^2, \quad k_\pm = \frac{V_0 - v_\pm}{2p_\pm}, \qquad \omega_+ = \frac{p_+}{p_-}\omega_- + \frac{V_0^2(p_-^2 - p_+^2) + v_-^2p_+^2 - v_+^2p_-^2}{4p_+p_-^2}. \tag{16}$$

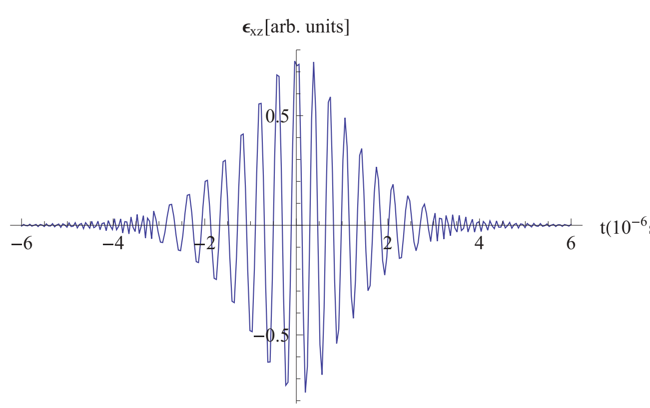

FIG. 2. (Color online) ε_{xz} component of the deformation tensor the breather for the parameters [16] is shown for a fixed value of z.

The appearance in expression (15) of the functions $\sin[(k + Q_{+1} + k_+)z - (\omega + \Omega_{+1} + \omega_+)t]$ and $\sin[(k - Q_{-1} + k_-)z - (\omega - \Omega_{-1} + \omega_-)t]$ indicates the formation of double periodic beats with coordinate and time relative to the frequency and wave number of the carrier wave (ω, k), with characteristic parameters $(\omega + \Omega_{+1} + \omega_+, k + Q_{+1} + k_-)$ and $(\omega - \Omega_{-1} + \omega_-, k - Q_{-1} + k_-)$, respectively, as a result of which the one-component soliton solutions (13) for u_+ and u_- is transformed into two-component vector soliton solution (15) for the the ε_{xz} component of the deformation tensor of the acoustic pulse. Equation (15) is an exact regular time and space double periodic solution of the nonlinear wave equation (7), which, like the one-component soliton and breather, loses no energy in the process of propagation through a medium over considerable distances.

We have shown that in the propagation of an acoustic pulse through a resonance medium containing an impurity atoms under the condition of SIT acoustic vector soliton can arise. The explicit form and parameters of the acoustic two-component vector soliton are given by Eqs. (8), (12), (14), and (16). The dispersion equation and the relations between quantities $\Omega_{\pm1}$ and $Q_{\pm1}$ are given by Eqs. (4) and (10), respectively.

Using typical parameters for the pulse, the materials, and the paramagnetic impurities [16], we can construct a plot of the ε_{xz} component of the deformation tensor for a two-component vector soliton Eq. (15) (shown in Fig. 1 for a fixed value of the z coordinate). This case corresponds to a bright-bright soliton pair, because the conditions $p_+g_+ > 0$ and $p_-g_- > 0$, are fulfilled, and both components of a vector soliton are bright solitons.

The one-component breather is the special case of the vector soliton. The profile of the breather with the same value of the parameters as for a vector soliton [Eq. (15)] (for a fixed value of $z = 0$) is shown in Fig. 2. It is obvious that the shape of the two-component vector soliton (Fig. 1) is different in the comparison with the shape of the one-component breather (Fig. 2).

We consider paramagnetic crystals, but these results also can be transformed for the other physical systems, for instance, in nanostructures.

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