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Bounds for the number of degrees of freedom of incompressible magnetohydrodynamic turbulence in two and three dimensions

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We study incompressible magnetohydrodynamic turbulence in both two and three dimensions, with an emphasis on the number of degrees of freedom N. This number is estimated in terms of the magnetic Prandtl number Pr, kinetic Reynolds number Re, and magnetic Reynolds number Rm. Here Re and Rm are dynamic in nature, defined in terms of the kinetic and magnetic energy dissipation rates (or averages of the velocity and magnetic field gradients), viscosity and magnetic diffusivity, and the system size. It is found that for the two-dimensional case, N satisfies $N \leqslant \mathrm{Pr}\,\mathrm{Re}^{3/2} + \mathrm{Rm}^{3/2}$ for $\mathrm{Pr} > 1$ and $N \leqslant \mathrm{Re}^{3/2} + \mathrm{Pr}^{-1}\,\mathrm{Rm}^{3/2}$ for $\mathrm{Pr} \leqslant 1$. In three dimensions, on the other hand, N satisfies $N \leqslant (\mathrm{Pr}\,\mathrm{Re}^{3/2} + \mathrm{Rm}^{3/2})^{3/2}$ for $\mathrm{Pr} > 1$ and $N \leqslant (\mathrm{Re}^{3/2} + \mathrm{Pr}^{-1}\,\mathrm{Rm}^{3/2})^{3/2}$ for $\mathrm{Pr} \leqslant 1$. In the limit $\mathrm{Pr} \to 0$, $\mathrm{Re}^{3/2}$ dominates $\mathrm{Pr}^{-1}\,\mathrm{Rm}^{3/2}$, and the present estimate for N appropriately reduces to $\mathrm{Re}^{9/4}$ as in the case of usual Navier-Stokes turbulence. For $\mathrm{Pr} \approx 1$, our results imply the classical spectral scaling of the energy inertial range and dissipation wave number (in the form of upper bounds). These bounds are consistent with the existing predictions in the literature for turbulence with or without Alfvén wave effects. We discuss the possibility of solution regularity, with an emphasis on the two-dimensional case in the absence of either one or both of the dissipation terms.

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I. INTRODUCTION

In incompressible magnetohydrodynamic (MHD) turbulence, the total (kinetic plus magnetic) energy is conserved and transferred from large to small scales (direct transfer). This dynamical behavior is common to both two and three dimensions (2D and 3D), making MHD turbulence markedly different from its Navier-Stokes counterpart, for which the energy transfer reverses direction upon the reduction of dimensions from three to two. The reason for this difference is that the dimension reduction removes the vortex stretching mechanism in both cases, but retains the magnetic stretching mechanism in the MHD case. Furthermore, the Lorentz force, which is quadratic in the magnetic field, can act as a strong vorticity source. In the absence of vortex stretching, this force, together with the magnetic stretching mechanism, can maintain certain nonlinear aspects in the dynamics of 2D MHD turbulence. As a consequence, upon dimension reduction, 2D MHD turbulence can remain nonlinear, whereas 2D Navier-Stokes turbulence effectively becomes linearized [1,2]. Hence, it is not surprising that the energy of 2D MHD turbulence is transferred to small scales as is the case in its 3D parent system. This direct transfer of the total energy has been well documented by numerical simulations and observational data [3–6], although the energy spectrum in the inertial range by which this transfer takes place may [4,6] or may not [6–8] be the classical $k^{-5/3}$ of Kolmogorov. In addition to the direct energy transfer, the variance of the 2D magnetic potential is transferred from small to large scales [9–11].

Given the direct energy transfer by quadratic nonlinearity and small enough viscosity and magnetic diffusivity, classical arguments on the basis of Kolmogorov's phenomenology are applicable to both 2D and 3D MHD turbulence (see, however, Refs. [7,8,12,13] for modifications to Kolmogorov's theory). Recently, several studies [1,14–17] have used a dynamical systems approach as an alternative to the classical method, recovering consistent results in the form of estimates. These include the system's number of degrees of freedom, the slope of the energy spectrum in the inertial range, and the dissipation wave number. An advantage of the new approach is that the derived estimates are rigorous and valid whenever the dynamical quantities involved can be fully controlled. Another advantage is that the new approach can handle systems to which Kolmogorov's phenomenological method fails to apply. These include, for example, the Burgers equation [15] and systems having a credible possibility that the energy dissipation rate depends on viscosity and vanishes in the inviscid limit [16]. Such a possibility is consistent with the existing numerical results for surface quasigeostrophic turbulence [18,19], suggesting no finite-time singularities in the inviscid dynamics.

In this study, we extend the results of Refs. [1,14–17] to incompressible MHD turbulence in both 2D and 3D. It is found that for 3D, the number of degrees of freedom N satisfies $N \leqslant (\mathrm{Pr}\,\mathrm{Re}^{3/2} + \mathrm{Rm}^{3/2})^{3/2}$ for $\mathrm{Pr} > 1$ and $N \leqslant (\mathrm{Re}^{3/2} + \mathrm{Pr}^{-1}\,\mathrm{Rm}^{3/2})^{3/2}$ for $\mathrm{Pr} \leqslant 1$. Here Pr is the magnetic Prandtl number and Re and Rm are the kinetic and magnetic Reynolds numbers, respectively. These Reynolds numbers are defined in terms of averages of the velocity and magnetic field gradients (or energy dissipation-like rates), viscosity and magnetic diffusivity, and the system's size. For 2D, N is found to scale as $N \leqslant \mathrm{Pr}\,\mathrm{Re}^{3/2} + \mathrm{Rm}^{3/2}$ for $\mathrm{Pr} > 1$ and $N \leqslant \mathrm{Re}^{3/2} + \mathrm{Pr}^{-1}\,\mathrm{Rm}^{3/2}$ for $\mathrm{Pr} \leqslant 1$, where the Reynolds numbers are defined in a more conventional way using the actual energy dissipation rates. For $\mathrm{Pr} \approx 1$, these results imply the upper bound $\alpha \leqslant 5/3$ for the exponent α of the power-law energy spectrum $k^{-\alpha}$ in the inertial range. The bound $\alpha \leqslant 5/3$ is sharp, provided that the estimates for N are optimal. For $\alpha = 5/3$, we recover the well-known Kolmogorov spectrum and dissipation wave number. When these estimates are excessive, the upper bound $\alpha \leqslant 5/3$ still holds but is no longer sharp. This situation may occur in the following circumstances. One is the case considered by Iroshnikov [7], Kraichnan [8], and others [12,13], where strong Alfvén waves can reduce the energy transfer (suppression of turbulence), thereby rendering overestimates for N and α. The other is concerned with the extreme limits of the magnetic Prandtl number, where the dynamics can change in fundamental ways. The limit $\mathrm{Pr} \to 0$ corresponds to the Navier-Stokes regime, where $\mathrm{Re}^{3/2}$ is expected to predominate $\mathrm{Pr}^{-1}\,\mathrm{Rm}^{3/2}$. The above estimate for N appropriately reduces to $N \leqslant \mathrm{Re}^{9/4}$ for 3D, but excessively reduces to $N \leqslant \mathrm{Re}^{3/2}$ for 2D. The limit $\mathrm{Pr} \to \infty$ corresponds to the passive advection regime, where $\mathrm{Rm}^{3/2}$ is expected to predominate $\mathrm{Pr}\,\mathrm{Re}^{3/2}$. In this regime, the estimate for N becomes excessive for each of the 2D and 3D cases. We discuss the related issue of solution regularity, with an emphasis on the 2D case in the absence of either one or both of the dissipation mechanisms.

II. THEORETICAL BACKGROUND

A. Governing equations and conservation laws

Consider an incompressible electrically conducting fluid which is under no influence of external field or stirring. The fluid motion and the evolution of the internally generated magnetic field are governed by

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \nu\Delta\boldsymbol{u}, \tag{1}$$

$$\boldsymbol{b}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} + \mu\Delta\boldsymbol{b}, \tag{2}$$

$$\nabla\cdot\boldsymbol{u} = 0 = \nabla\cdot\boldsymbol{b}, \tag{3}$$

where $\boldsymbol{u}(\boldsymbol{x},t)$ is the fluid velocity, $\boldsymbol{b}(\boldsymbol{x},t)$ is the magnetic field vector, $p(\boldsymbol{x},t)$ is the total pressure, ν is the viscosity, and μ is the magnetic diffusivity. For convenience, we consider doubly and triply periodic domains of size L, and all fields involved are assumed to have zero average. In Eq. (1), the term $(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}$ represents the Lorentz force, which can act as a strong source of vorticity. In Eq. (2), the stretching term $(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}$ allows $\boldsymbol{b}$ to be amplified (in the expense of $\boldsymbol{u}$) by the fluid velocity gradients. This term survives the reduction of dimensions from three to two.

The inviscid and diffusionless version of the MHD system possesses a number of conservation laws. Most important is the conservation of the total energy density $\|\boldsymbol{u}\|^2/2 + \|\boldsymbol{b}\|^2/2 = \langle|\boldsymbol{u}|^2\rangle/2 + \langle|\boldsymbol{b}|^2\rangle/2$, where $\langle\cdot\rangle$ denotes a spatial average. This conservation law is a straightforward consequence of the exact cancellation of the triple-product terms on the right-hand sides of the following evolution equations for the kinetic and magnetic energy:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2 = \langle\boldsymbol{u}\cdot(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\rangle - \nu\|\nabla\boldsymbol{u}\|^2, \tag{4}$$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{b}\|^2 = \langle\boldsymbol{b}\cdot(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}\rangle - \mu\|\nabla\boldsymbol{b}\|^2. \tag{5}$$

The triple-product terms in Eqs. (4) and (5) represent the conversion from kinetic into magnetic energy (dynamo action) and vice versa (referred to as antidynamo action in this study). These processes have become major research subjects [20–22] for their importance in flows in the liquid core of the earth, in the sun and stars, and in the interplanetary medium. In general, dynamo and antidynamo play crucial roles in the energy transfer. In particular, as vorticity amplification by vortex stretching is absent in 2D, dynamo action is *a priori* responsible for ridding the large scales of kinetic energy, which would otherwise undergo an inverse transfer. This makes the direct energy transfer in 2D different from that in 3D in fundamental ways. Hence, to fully understand 2D MHD turbulence, a detailed knowledge of dynamo and antidynamo action is absolutely necessary. Other well-known invariants include the magnetic helicity $\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle$, where $\boldsymbol{a}$ is the magnetic vector potential, and cross helicity $\langle\boldsymbol{u}\cdot\boldsymbol{b}\rangle$. The latter, together with the conservation of total energy, further implies the conservation of $\langle|\boldsymbol{Z}^{\pm}|^2\rangle$, where $\boldsymbol{Z}^{\pm} = \boldsymbol{u}\pm\boldsymbol{b}$ are known as the Elsässer variables, which can be more conveniently used in the study of Alfvén waves.

For a smooth solution with initial condition $(\boldsymbol{u},\boldsymbol{b}) = (\boldsymbol{u}_0,\boldsymbol{b}_0)$, the decay of energy is governed by

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\boldsymbol{u}\|^2 + \|\boldsymbol{b}\|^2) &= -\nu\|\nabla\boldsymbol{u}\|^2 - \mu\|\nabla\boldsymbol{b}\|^2\\ &= -\epsilon_{\boldsymbol{u}} - \epsilon_{\boldsymbol{b}} = -\epsilon.\end{aligned} \tag{6}$$

It follows that

$$\int_0^\infty \epsilon(t)\,dt = \frac{1}{2}(\|\boldsymbol{u}_0\|^2 + \|\boldsymbol{b}_0\|^2). \tag{7}$$

The energy dissipation rates $\epsilon_{\boldsymbol{u}}(t)$ (kinetic), $\epsilon_{\boldsymbol{b}}(t)$ (magnetic), and $\epsilon(t)$ (total) are key dynamical parameters, and their detailed behavior is an issue of fundamental importance. Given $(\boldsymbol{u}_0,\boldsymbol{b}_0)$, these rates depend on ν and μ, and in the spirit of Kolmogorov, remain bounded for all ν and μ, including the limits $\nu \to 0$ and $\mu \to 0$ taken simultaneously or individually. Here, we are primarily concerned with their instantaneous values for $\nu > 0$ and $\mu > 0$, and any assumptions on their asymptotic behavior, if necessary, will be stated in due course.

B. Preliminary estimates

Let $S_n = \{\boldsymbol{e}_1,\boldsymbol{e}_2,\ldots,\boldsymbol{e}_n\}$ be an orthonormal set of periodic and zero average vector-valued functions of space variables $\boldsymbol{x} = (x,y)$ or $\boldsymbol{x} = (x,y,z)$. Here $\boldsymbol{e}_i = (\boldsymbol{e}_i^1,\boldsymbol{e}_i^2)$, where both $\boldsymbol{e}_i^1$ and $\boldsymbol{e}_i^2$ are vector fields in 2D or 3D. In other words, the $\boldsymbol{e}_i$'s are four-vectors and six-vectors for 2D and 3D, respectively. Orthonormality means that $\langle\boldsymbol{e}_i\cdot\boldsymbol{e}_j\rangle = \langle\boldsymbol{e}_i^1\cdot\boldsymbol{e}_j^1 + \boldsymbol{e}_i^2\cdot\boldsymbol{e}_j^2\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta symbol. For the present application, we also assume $\nabla\cdot\boldsymbol{e}_i^1 = 0 = \nabla\cdot\boldsymbol{e}_i^2$. The norms of $\boldsymbol{e}_i$ and of its gradients are defined by $\|\boldsymbol{e}_i\| = (\|\boldsymbol{e}_i^1\|^2 + \|\boldsymbol{e}_i^2\|^2)^{1/2}$ and $\|\nabla\boldsymbol{e}_i\| = (\|\nabla\boldsymbol{e}_i^1\|^2 + \|\nabla\boldsymbol{e}_i^2\|^2)^{1/2}$. For the periodic domains of size L under consideration, the eigenvalues of $-\Delta$ of the first n basic Fourier modes sum up to approximately n^2/L^2 and $n^{5/3}/L^2$ for 2D and 3D, respectively. The Rayleigh-Ritz principle implies that

$$\sum_{i=1}^{n}\|\nabla\boldsymbol{e}_i\|^2 \geqslant c_1\frac{n^2}{L^2} \quad \text{for 2D}, \tag{8}$$

$$\sum_{i=1}^{n}\|\nabla\boldsymbol{e}_i\|^2 \geqslant c_2\frac{n^{5/3}}{L^2} \quad \text{for 3D}, \tag{9}$$

where c_1 and c_2 are nondimensional constants independent of the given sets and of their size n. On the other hand, we have the Lieb-Thirring inequalities [16,17,23–25]

$$\left\| \sum_{i=1}^{n} |\boldsymbol{e}_i|^2 \right\| \leqslant c_3 L \left(\sum_{i=1}^{n} \|\nabla \boldsymbol{e}_i\|^2 \right)^{1/2} \quad \text{for 2D}, \tag{10}$$

$$\left\| \sum_{i=1}^{n} |\boldsymbol{e}_i|^2 \right\| \leqslant c_4 L^{3/2} \left(\sum_{i=1}^{n} \|\nabla \boldsymbol{e}_i\|^2 \right)^{3/4} \quad \text{for 3D}, \tag{11}$$

where c_3 and c_4 are nondimensional constants, which are, as c_1 and c_2, independent of the given sets and of their size n. Except possibly for Eq. (11), which has been given for completeness and for comparison, the analytic estimates (8), (9), and (10) have been known to provide sharp estimates in their application to fluid turbulence. These are used in the subsequent sections to derive bounds for the number of degrees of freedom N.

III. LYAPUNOV STABILITY ANALYSIS AND NUMBER OF DEGREES OF FREEDOM

This section is concerned with the problem of Lyapunov stability, from which the notion of number of degrees of freedom arises and its estimates can be deduced. The mathematical framework is similar to that in recent studies [1,14–16] on the same problem for other dynamical systems, differing only in minor details.

In the early 1980s, a rigorous notion of the number of degrees of freedom of turbulence emerged from dynamical systems theory. For a regular forced-dissipative dynamical system, solutions starting from bounded initial conditions can asymptotically approach a universal set in phase space (solution space). Such a set is known as a global attractor, which is invariant and represents the system's dynamical behavior in the long term. In general, an attractor is fractal and has finite generalized dimensions, such as the Kaplan-Yorke, Hausdorff, and box-counting dimensions. In the context of an infinite-dimensional dynamical system, these dimensions are virtually indistinguishable, and any one of them can represent the number of degrees of freedom of the system in question. Several authors have calculated these dimensions for fluid systems such as the 2D Navier-Stokes and MHD equations (cf. Ref. [25] and references therein), for which the existence of global attractors has been well established. Similar calculations have also been carried out for hypothetical attractors of other fluid equations, most notably the 3D Navier-Stokes equations [26,27], whose solution regularity is unknown.

For unforced-dissipative systems, the concept of an attractor is irrelevant since the asymptotic dynamics are trivial. Nevertheless, the notion of the number of degrees of freedom remains meaningful for the transient dynamics. For a given solution at a given time, Tran and Blackbourn [14] identified this number with the dimension of the subspace spanned by a sufficiently large number of least Lyapunov stable modes of the system linearized about the solution in question. Here, "sufficiently large" means that the sum of the Lyapunov exponents of these modes becomes negative. This formulism applies to (regular) solutions on the global attractors of forced systems, as well as to solutions of a general system whose global regularity is unknown. For the present case, the problem of Lyapunov stability and number of degrees of freedom is formulated and further explained.

A. Lyapunov versus Fourier

Given the solution $(\boldsymbol{u},\boldsymbol{b})$ starting from some smooth initial condition $(\boldsymbol{u}_0,\boldsymbol{b}_0)$, consider an admissible disturbance $(\boldsymbol{u}',\boldsymbol{b}')$ [i.e., one satisfying the same conditions as $(\boldsymbol{u},\boldsymbol{b})$]. The linear evolution of this disturbance is governed by

$$\begin{aligned} \boldsymbol{u}'_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}' + (\boldsymbol{u}'\cdot\nabla)\boldsymbol{u} + \nabla p' \\ = (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}' + (\boldsymbol{b}'\cdot\nabla)\boldsymbol{b} + \nu\Delta\boldsymbol{u}', \end{aligned} \tag{12}$$

$$\begin{aligned} \boldsymbol{b}'_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b}' + (\boldsymbol{u}'\cdot\nabla)\boldsymbol{b} \\ = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}' + (\boldsymbol{b}'\cdot\nabla)\boldsymbol{u} + \mu\Delta\boldsymbol{b}', \end{aligned} \tag{13}$$

where p' is the perturbed pressure. By multiplying Eq. (12) by $\boldsymbol{u}'$ and Eq. (13) by $\boldsymbol{b}'$ and taking the spatial average of the resulting equations we obtain the equations governing the evolution of $\|\boldsymbol{u}'\|^2$ and $\|\boldsymbol{b}'\|^2$. These equations are given by

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}'\|^2 = -\langle \boldsymbol{u}'\cdot(\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}\rangle + \langle \boldsymbol{u}'\cdot(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}'\rangle \\ + \langle \boldsymbol{u}'\cdot(\boldsymbol{b}'\cdot\nabla)\boldsymbol{b}\rangle - \nu\|\nabla\boldsymbol{u}'\|^2, \end{aligned} \tag{14}$$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{b}'\|^2 = -\langle \boldsymbol{b}'\cdot(\boldsymbol{u}'\cdot\nabla)\boldsymbol{b}\rangle + \langle \boldsymbol{b}'\cdot(\boldsymbol{b}\cdot\nabla)\boldsymbol{u}'\rangle \\ + \langle \boldsymbol{b}'\cdot(\boldsymbol{b}'\cdot\nabla)\boldsymbol{u}\rangle - \mu\|\nabla\boldsymbol{b}'\|^2. \end{aligned} \tag{15}$$

Adding up Eqs. (14) and (15), noting the cancellation of two of the triple-product terms and the notation $\|(\boldsymbol{u}',\boldsymbol{b}')\| = (\|\boldsymbol{u}'\|^2 + \|\boldsymbol{b}'\|^2)^{1/2}$ yields

$$\begin{aligned} &\|(\boldsymbol{u}',\boldsymbol{b}')\|\frac{d}{dt}\|(\boldsymbol{u}',\boldsymbol{b}')\| \\ &= \langle \boldsymbol{b}'\cdot(\boldsymbol{b}'\cdot\nabla)\boldsymbol{u}\rangle - \langle \boldsymbol{b}'\cdot(\boldsymbol{u}'\cdot\nabla)\boldsymbol{b}\rangle + \langle \boldsymbol{u}'\cdot(\boldsymbol{b}'\cdot\nabla)\boldsymbol{b}\rangle \\ &\quad - \langle \boldsymbol{u}'\cdot(\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}\rangle - \nu\|\nabla\boldsymbol{u}'\|^2 - \mu\|\nabla\boldsymbol{b}'\|^2. \end{aligned} \tag{16}$$

The exponential growth or decay rate λ of $\|(\boldsymbol{u}',\boldsymbol{b}')\|$ is given by

$$\begin{aligned} \lambda &= \frac{1}{\|(\boldsymbol{u}',\boldsymbol{b}')\|}\frac{d}{dt}\|(\boldsymbol{u}',\boldsymbol{b}')\| \\ &= \frac{\langle \boldsymbol{b}'\cdot(\boldsymbol{b}'\cdot\nabla)\boldsymbol{u}\rangle - \langle \boldsymbol{u}'\cdot(\boldsymbol{u}'\cdot\nabla)\boldsymbol{u}\rangle + \langle \boldsymbol{u}'\cdot(\boldsymbol{b}'\cdot\nabla)\boldsymbol{b}\rangle - \langle \boldsymbol{b}'\cdot(\boldsymbol{u}'\cdot\nabla)\boldsymbol{b}\rangle - \nu\|\nabla\boldsymbol{u}'\|^2 - \mu\|\nabla\boldsymbol{b}'\|^2}{\|(\boldsymbol{u}',\boldsymbol{b}')\|^2}. \end{aligned} \tag{17}$$

An orthonormal set of n least stable disturbances $\{(\boldsymbol{u}_1,\boldsymbol{b}_1),(\boldsymbol{u}_2,\boldsymbol{b}_2),\ldots,(\boldsymbol{u}_n,\boldsymbol{b}_n)\}$ and the corresponding greatest growth rates $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ (local Lyapunov exponents) can be derived by successively maximizing λ with respect to all admissible disturbances $(\boldsymbol{u}',\boldsymbol{b}')$ subject to the following orthonormality constraint. At each step i in the process, $(\boldsymbol{u}',\boldsymbol{b}')$ is required to satisfy both $\|(\boldsymbol{u}',\boldsymbol{b}')\| = 1$ and $\langle(\boldsymbol{u}',\boldsymbol{b}')\cdot(\boldsymbol{u}_j,\boldsymbol{b}_j)\rangle = 0$, for $j = 1,2,\ldots,i-1$, where $(\boldsymbol{u}_j,\boldsymbol{b}_j)$ is the maximizer obtained at the jth step. This process eventually exhausts all unstable mutually orthogonal disturbances and reaches the stable regime where $\lambda_i < 0$. It follows that there exists an integer N satisfying

$$\sum_{i=1}^{N}\lambda_i \leqslant 0 < \sum_{i=1}^{N-1}\lambda_i. \tag{18}$$

The orthonormal set $\{(\boldsymbol{u}_1,\boldsymbol{b}_1),(\boldsymbol{u}_2,\boldsymbol{b}_2),\ldots,(\boldsymbol{u}_N,\boldsymbol{b}_N)\}$ then consists of all unstable modes and a number of stable modes, which can adequately describe the solution $(\boldsymbol{u},\boldsymbol{b})$ at least locally in time. In other words, the N-dimensional linear subspace spanned by the N-dimensional "Lyapunov basis" $\{(\boldsymbol{u}_1,\boldsymbol{b}_1),(\boldsymbol{u}_2,\boldsymbol{b}_2),\ldots,(\boldsymbol{u}_N,\boldsymbol{b}_N)\}$ adequately "accommodates" the solution. Indeed, in principle, one can formally express $(\boldsymbol{u},\boldsymbol{b})$ in terms of this basis (instead of the Fourier modes) as

$$(\boldsymbol{u},\boldsymbol{b}) \approx \sum_{i=1}^{N}\rho_i(t)(\boldsymbol{u}_i,\boldsymbol{b}_i), \tag{19}$$

with the assurance that the "truncation" mode $(\boldsymbol{u}_N,\boldsymbol{b}_N)$ lies well within the dissipation range (negative and large Lyapunov exponent). For this obvious reason, N can represent the system's number of degrees of freedom.

Estimates for individual λ_i and for N for 3D Navier-Stokes turbulence were first derived by Ruelle [28] and then improved by Lieb [29]. These pioneering studies, together with that of Babin and Vishik [30], triggered a series of investigations lasting for nearly two decades into the attractor dimension of the Navier-Stokes equations. Ruelle's formulation made use of the notion of invariant measure and its properties. Here, this technical detail is unnecessary and can be circumvented, by allowing the Lyapunov basis and the associated set of exponents to depend on the solution and time. The time dependence is largely immaterial for our purposes and can be removed. Indeed, by considering a general solution at an arbitrary time, we derive in the next section upper bounds for N in terms of physical parameters and time-dependent dynamical quantities of the given solution. These bounds are globally valid as far as regular solutions are concerned. Furthermore, they can be readily made valid uniformly in time when the time-dependent dynamical quantities involved are replaced by their uniform-in-time upper bounds whose existence is guaranteed for 2D and is assumed for 3D.

B. Estimates for number of degrees of freedom

Consider n normalized solutions $[\lambda_i,(\boldsymbol{u}_i,\boldsymbol{b}_i)]$ of the linearized problem discussed in the preceding section. Since each solution satisfies Eq. (17), we have

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i &= \sum_{i=1}^{n}[\langle\boldsymbol{b}_i\cdot(\boldsymbol{b}_i\cdot\nabla)\boldsymbol{u}\rangle - \langle\boldsymbol{b}_i\cdot(\boldsymbol{u}_i\cdot\nabla)\boldsymbol{b}\rangle + \langle\boldsymbol{u}_i\cdot(\boldsymbol{b}_i\cdot\nabla)\boldsymbol{b}\rangle - \langle\boldsymbol{u}_i\cdot(\boldsymbol{u}_i\cdot\nabla)\boldsymbol{u}\rangle - \nu\|\nabla\boldsymbol{u}_i\|^2 - \mu\|\nabla\boldsymbol{b}_i\|^2]\\ &\leqslant \sum_{i=1}^{n}\left[\langle(|\boldsymbol{u}_i|^2+|\boldsymbol{b}_i|^2)|\nabla\boldsymbol{u}|\rangle + 2\langle|\boldsymbol{u}_i||\boldsymbol{b}_i||\nabla\boldsymbol{b}|\rangle - \nu\|\nabla\boldsymbol{u}_i\|^2 - \mu\|\nabla\boldsymbol{b}_i\|^2\right]\\ &\leqslant \sum_{i=1}^{n}\left[\langle(|\boldsymbol{u}_i|^2+|\boldsymbol{b}_i|^2)(|\nabla\boldsymbol{u}|+|\nabla\boldsymbol{b}|)\rangle - \nu\|\nabla\boldsymbol{u}_i\|^2 - \mu\|\nabla\boldsymbol{b}_i\|^2\right],\end{aligned} \tag{20}$$

where the inequalities are elementary. The above estimate is applicable to both 2D and 3D. We proceed to treat these cases separately.

For 3D, we follow the treatment of 3D Navier-Stokes turbulence in Ref. [1] by defining two spatial average quantities Ω and I by

$$\begin{aligned}\Omega &= \frac{1}{n}\sum_{i=1}^{n}\langle\boldsymbol{b}_i\cdot(\boldsymbol{b}_i\cdot\nabla)\boldsymbol{u} - \boldsymbol{u}_i\cdot(\boldsymbol{u}_i\cdot\nabla)\boldsymbol{u}\rangle\\ &\leqslant \frac{1}{n}\sum_{i=1}^{n}\langle(|\boldsymbol{u}_i|^2+|\boldsymbol{b}_i|^2)|\nabla\boldsymbol{u}|\rangle,\\ I &= \frac{1}{n}\sum_{i=1}^{n}\langle\boldsymbol{u}_i\cdot(\boldsymbol{b}_i\cdot\nabla)\boldsymbol{b} - \boldsymbol{b}_i\cdot(\boldsymbol{u}_i\cdot\nabla)\boldsymbol{b}\rangle\\ &\leqslant \frac{1}{n}\sum_{i=1}^{n}\langle(|\boldsymbol{u}_i|^2+|\boldsymbol{b}_i|^2)|\nabla\boldsymbol{b}|\rangle.\end{aligned} \tag{21}$$

By their very definition, Ω and I satisfy $\Omega \leqslant \|\nabla\boldsymbol{u}\|_\infty$ and $I \leqslant \|\nabla\boldsymbol{b}\|_\infty$ since $\|(\boldsymbol{u}_i,\boldsymbol{b}_i)\| = 1$. It is probably the case that $\Omega \approx \|\nabla\boldsymbol{u}\|$ and $I \approx \|\nabla\boldsymbol{b}\|$, even for fully developed turbulence at small ν and μ, for which one can expect $\|\nabla\boldsymbol{u}\| \ll \|\nabla\boldsymbol{u}\|_\infty$ and $\|\nabla\boldsymbol{b}\| \ll \|\nabla\boldsymbol{b}\|_\infty$ (see the discussion below). Substituting Ω and I into Eq. (20) yields

$$\begin{aligned}\sum_{i=1}^{n}\lambda_i &= n(\Omega+I) - \sum_{i=1}^{n}(\nu\|\nabla\boldsymbol{u}_i\|^2 + \mu\|\nabla\boldsymbol{b}_i\|^2)\\ &\leqslant n(\Omega+I) - \min\{\nu,\mu\}\sum_{i=1}^{n}\|(\nabla\boldsymbol{u}_i,\nabla\boldsymbol{b}_i)\|^2\\ &\leqslant n(\Omega+I) - \min\{\nu,\mu\}\,\frac{c_2 n^{5/3}}{L^2},\end{aligned} \tag{22}$$

where, in the final step, Eq. (9) with $\boldsymbol{e}_i$ replaced by $(\boldsymbol{u}_i,\boldsymbol{b}_i)$ has been used. The condition $\sum_{i=1}^{n}\lambda_i \leqslant 0$ is satisfied when

$$\Omega + I \leqslant \min\{\nu,\mu\}\,\frac{c_2 n^{2/3}}{L^2}. \tag{23}$$

It follows that

$$N^{2/3} \leqslant \frac{L^2}{\min\{\nu,\mu\}}\,(\Omega + I), \tag{24}$$

where the constant c_2 has been omitted. We define the kinetic and magnetic Reynolds numbers Re and Rm by

$$\mathrm{Re} = \frac{L^{4/3}\epsilon_{\boldsymbol{u}}'^{\,1/3}}{\nu} = \frac{L^{4/3}(\nu\Omega^2)^{1/3}}{\nu}, \tag{25}$$

$$\mathrm{Rm} = \frac{L^{4/3}\epsilon_{\boldsymbol{b}}'^{\,1/3}}{\mu} = \frac{L^{4/3}(\mu I^2)^{1/3}}{\mu}, \tag{26}$$

where $\epsilon_{\boldsymbol{u}}' = \nu\Omega^2$ and $\epsilon_{\boldsymbol{b}}' = \mu I^2$ are kinetic and magnetic energy dissipation-like rates, which, as mentioned earlier, are expected to approximate their corresponding actual dissipation rates. In terms of Re and Rm Eq. (24) becomes

$$\begin{aligned} N &\leqslant (\mathrm{Pr}\,\mathrm{Re}^{3/2} + \mathrm{Rm}^{3/2})^{3/2} \quad \text{for} \quad \mathrm{Pr} > 1, \\ N &\leqslant (\mathrm{Re}^{3/2} + \mathrm{Pr}^{-1}\mathrm{Rm}^{3/2})^{3/2} \quad \text{for} \quad \mathrm{Pr} \leqslant 1. \end{aligned} \tag{27}$$

Here again $\mathrm{Pr} = \nu/\mu$ is the magnetic Prandtl number.

For 2D, one could proceed as above until the final step of Eq. (22), upon which Eq. (8) would be invoked to yield $\sum_{i=1}^n \lambda_i \leqslant (\Omega + I) - c_1 \min\{\nu,\mu\} n/L^2$. From this estimate, it is easy to see that N satisfies Eq. (27) with the overall exponent $3/2$ removed. However, it turns out that equally optimal bounds in terms of $\|\nabla \boldsymbol{u}\|$ and $\|\nabla \boldsymbol{b}\|$ instead of Ω and I can be obtained by utilizing the Lieb-Thirring inequality. The detailed steps leading to these bounds are described in what follows.

Recalling the final equation of Eq. (20), we have

$$\begin{aligned} \sum_{i=1}^n \lambda_i &\leqslant \sum_{i=1}^n \left[\langle (|\boldsymbol{u}_i|^2 + |\boldsymbol{b}_i|^2)(|\nabla \boldsymbol{u}| + |\nabla \boldsymbol{b}|)\rangle - \nu\|\nabla \boldsymbol{u}_i\|^2 - \mu\|\nabla \boldsymbol{b}_i\|^2\right] \\ &\leqslant \left\| \sum_{i=1}^n |\boldsymbol{u}_i|^2 + |\boldsymbol{b}_i|^2 \right\| (\|\nabla \boldsymbol{u}\| + \|\nabla \boldsymbol{b}\|) - \sum_{i=1}^n (\nu\|\nabla \boldsymbol{u}_i\|^2 + \mu\|\nabla \boldsymbol{b}_i\|^2) \\ &\leqslant \left\| \sum_{i=1}^n |(\boldsymbol{u}_i,\boldsymbol{b}_i)|^2 \right\| (\|\nabla \boldsymbol{u}\| + \|\nabla \boldsymbol{b}\|) - \min\{\nu,\mu\} \sum_{i=1}^n \|(\nabla \boldsymbol{u}_i, \nabla \boldsymbol{b}_i)\|^2 \\ &\leqslant \left(\sum_{i=1}^n \|(\nabla \boldsymbol{u}_i, \nabla \boldsymbol{b}_i)\|^2 \right)^{1/2} \left[c_3 L(\|\nabla \boldsymbol{u}\| + \|\nabla \boldsymbol{b}\|) - \min\{\nu,\mu\} \left(\sum_{i=1}^n \|(\nabla \boldsymbol{u}_i, \nabla \boldsymbol{b}_i)\|^2 \right)^{1/2} \right] \\ &\leqslant \left(\sum_{i=1}^n \|(\nabla \boldsymbol{u}_i, \nabla \boldsymbol{b}_i)\|^2 \right)^{1/2} \left(c_3 L(\|\nabla \boldsymbol{u}\| + \|\nabla \boldsymbol{b}\|) - \min\{\nu,\mu\} c_1^{1/2} \frac{n}{L} \right), \end{aligned} \tag{28}$$

where we have used the Cauchy-Schwarz inequality in the second step, the 2D Lieb-Thirring inequality (10) in the fourth step, and Eq. (8) in the final step. The condition $\sum_{i=1}^n \lambda_i \leqslant 0$ is satisfied when

$$n \geqslant \frac{L^2}{\min\{\nu,\mu\}}\,(\|\nabla \boldsymbol{u}\| + \|\nabla \boldsymbol{b}\|), \tag{29}$$

where the constants c_1 and c_3 have been omitted. It follows that

$$N \leqslant \frac{L^2}{\min\{\nu,\mu\}}\,(\|\nabla \boldsymbol{u}\| + \|\nabla \boldsymbol{b}\|). \tag{30}$$

Let Re and Rm be defined as in Eq. (25), except that Ω and I are replaced by $\|\nabla \boldsymbol{u}\|$ and $\|\nabla \boldsymbol{b}\|$, respectively. In other words, $\epsilon_{\boldsymbol{u}}'$ and $\epsilon_{\boldsymbol{b}}'$ are, respectively, replaced by the actual dissipation rates $\epsilon_{\boldsymbol{u}}$ and $\epsilon_{\boldsymbol{b}}$. In terms of these more standard Reynolds numbers, Eq. (30) becomes

$$\begin{aligned} N &\leqslant \mathrm{Pr}\,\mathrm{Re}^{3/2} + \mathrm{Rm}^{3/2} \quad \text{for} \quad \mathrm{Pr} > 1, \\ N &\leqslant \mathrm{Re}^{3/2} + \mathrm{Pr}^{-1}\mathrm{Rm}^{3/2} \quad \text{for} \quad \mathrm{Pr} \leqslant 1. \end{aligned} \tag{31}$$

The estimates (27) and (31) can be expected to be sharp for moderate values of Pr. In the two extreme cases $\mathrm{Pr} \ll 1$ and $\mathrm{Pr} \gg 1$, they can become excessive for a number of reasons. Furthermore, in the Kraichnan picture, they can be excessive irrespective of the value of Pr. These issues, among other things, are discussed in the subsequent sections.

C. Discussion

Equations (27) and (31) can be expected on the basis of existing results in the literature for fluid systems with quadratic nonlinearities. For the 3D Navier-Stokes and 2D surface quasigeostrophic systems (which are quadratically nonlinear) N has previously been found to scale as $\mathrm{Re}^{9/4}$ and $\mathrm{Re}^{3/2}$, respectively. In the former case, Re is defined in terms of $\epsilon_{\boldsymbol{u}}' = \nu\Omega^2$, where Ω is given by Eq. (21) without the magnetic component. In the latter case, Re is defined in terms of $\epsilon_{\boldsymbol{u}} = \nu\|\nabla\theta\|^2 = \nu\|\nabla \boldsymbol{u}\|^2$, where θ is the potential temperature. The present results are consistent with these findings as one can expect like contributions to N from the two types of quadratic nonlinear interactions. It is interesting to note that the exponents $3/2$ and $9/4$ of the Reynolds numbers are characteristic of quadratic nonlinearities in 2D and 3D, respectively.

For extreme limits of the magnetic Prandtl number Pr, the dynamics can change in fundamental ways, and the estimates (27) and (31) can become excessive. In the regime $\mathrm{Pr} \ll 1$, which can be appropriately called the Navier-Stokes regime, the magnetic field becomes relatively far more diffusive and may not be treated on an equal footing with the velocity field. More precisely, the small scales of the magnetic field are relatively inactive, so that the Lorentz force essentially operates at large scales. The quadratic coupling between $\boldsymbol{u}$ and $\boldsymbol{b}$ becomes crippled at small scales, and the direct energy transfer is primarily due to the vortex stretching mechanism alone. As this mechanism is absent in 2D, the dynamical system is like the usual 2D Navier-Stokes system, for which N has been known to scale virtually linearly with the Reynolds number [14]. For 3D, however, the present treatment remains sound, albeit the magnetic field becomes largely insignificant. In this case, one expects $\mathrm{Pr}^{-1}\,\mathrm{Rm}^{3/2}$ to become negligible compared with $\mathrm{Re}^{3/2}$. In the limit $\mathrm{Pr} \to 0$, Eq. (27) correctly reduces to the result $N \leqslant \mathrm{Re}^{9/4}$ for 3D Navier-Stokes turbulence. Now, for the regime $\mathrm{Pr} \gg 1$, which may be called the passive advection regime, the velocity field becomes far more diffusive than the magnetic field and its small scales become relatively inactive. The magnetic field is effectively advected and stretched by a large-scale velocity field. This is very much like a linear advection problem, for which the present analysis is not expected to yield sharp estimates. This regime is discussed further in the next section.

The number of degrees of freedom N of a dynamical system can be considered a measure of its complexity. As far as complexity generation is concerned, the coupling between the velocity and magnetic fields complements rather than enhances the small-scale dynamics of each other. This is reflected through the fact that Re and Rm are not coupled in the estimates for N, each making an essentially independent contribution. An important implication is that the vortex stretching term and the source term (the curl of the Lorentz force term) in the vorticity equation do not enhance each other significantly in their vortex amplification.

The kinetic and magnetic Reynolds numbers Re and Rm used in this study are dynamic and unconventional. These nondimensional numbers emerge naturally from our analysis. There appear to be no obvious relations between Re and Rm and their conventional counterparts $L\|\boldsymbol{u}\|/\nu$ and $L\|\boldsymbol{u}\|/\mu$, particularly between Rm and $L\|\boldsymbol{u}\|/\mu$. Nonetheless, it is clear that unless $\epsilon_{\boldsymbol{u}}$ ($\epsilon'_{\boldsymbol{u}}$) and $\epsilon_{\boldsymbol{b}}$ ($\epsilon'_{\boldsymbol{b}}$) vanish (or grow without bound) in the small ν and μ limits, Re and Rm remain asymptotically the same as the conventional Reynolds numbers. In this case, our results remain intact when expressed in terms of the conventional Reynolds numbers.

Finally, there remains the question whether the approximations $\Omega \approx \|\nabla\boldsymbol{u}\|$ and $I \approx \|\nabla\boldsymbol{b}\|$ hold in 3D. These approximations would allow us to replace the nonstandard Reynolds numbers in Eq. (27) by more standard ones, defined in terms of $\|\nabla\boldsymbol{u}\|$ and $\|\nabla\boldsymbol{b}\|$. This question is highly challenging. Nonetheless it appears to be both theoretically and numerically tractable. On the theoretical side, there exists some evidence for a positive answer. First, it can be seen that the qualitative arguments in Ref. [1] in support of the approximation $\Omega \approx \|\nabla\boldsymbol{u}\|$ for 3D Navier-Stokes turbulence, where Ω is defined as in Eq. (21) with $\boldsymbol{b}_i = 0$, can be applied to the present case. The idea behind these arguments is that weakly unstable and stable Lyapunov modes may not be strongly spatially correlated with $\nabla\boldsymbol{u}$, therefore playing a moderating role on $\nabla\boldsymbol{u}$ in the spatial average definition of Ω, thus giving rise to the possibility $\|\nabla\boldsymbol{u}\| \approx \Omega \ll \|\nabla\boldsymbol{u}\|_\infty$. Second, we have seen earlier in this section that for 2D, (Ω, I) and $(\|\nabla\boldsymbol{u}\|, \|\nabla\boldsymbol{b}\|)$ can be used interchangeably in the estimation of N, essentially rendering the same results. The implication is that in 2D, $\Omega \approx \|\nabla\boldsymbol{u}\|$ and $I \approx \|\nabla\boldsymbol{b}\|$. While the same conclusion may not be made for 3D without further investigation, there are no reasons why that should not be the case. On the numerical side, it appears feasible to (at least) compute the most unstable mode $(\boldsymbol{u}_1, \boldsymbol{b}_1)$ and a neutral mode (i.e., one with a vanishing Lyapunov exponent). These modes are not sufficient to determine Ω and I with precision, but nonetheless provide a basis for a quantitative sense of the magnitudes of these quantities.

IV. ENERGY SPECTRA

We now deduce from the results in the preceding section constraints on power-law scalings of the energy inertial range. We consider the case $\mathrm{Pr} = 1$, for which the derived estimates for N can be sharp, and briefly discuss the regimes $\mathrm{Pr} \ll 1$ and $\mathrm{Pr} \gg 1$. Both the 2D and 3D cases are handled simultaneously since they differ in minor details due to the dependence of N on (Ω, I) for 3D and on $(\|\nabla\boldsymbol{u}\|, \|\nabla\boldsymbol{b}\|)$ for 2D.

Consider the Fourier representation of $(\boldsymbol{u}, \boldsymbol{b})$

$$(\boldsymbol{u},\boldsymbol{b}) = \sum_{\boldsymbol{k}} (\boldsymbol{u}_{\boldsymbol{k}}, \boldsymbol{b}_{\boldsymbol{k}}) \exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}, \tag{32}$$

where $\boldsymbol{k}$ is the wave vector and $\boldsymbol{u}_{\boldsymbol{k}}(t)$ and $\boldsymbol{b}_{\boldsymbol{k}}(t)$ are the Fourier transforms of $\boldsymbol{u}(\boldsymbol{x},t)$ and $\boldsymbol{b}(\boldsymbol{x},t)$, respectively. The incompressibility condition requires $\boldsymbol{u}_{\boldsymbol{k}}\cdot\boldsymbol{k} = 0 = \boldsymbol{b}_{\boldsymbol{k}}\cdot\boldsymbol{k}$ while the reality condition requires $(\boldsymbol{u}_{\boldsymbol{k}}, \boldsymbol{b}_{\boldsymbol{k}}) = (\boldsymbol{u}_{-\boldsymbol{k}}, \boldsymbol{b}_{-\boldsymbol{k}})^*$, where the asterisk denotes the complex conjugate. Let k_d be the dissipation wave number, so that $(\boldsymbol{u}_{\boldsymbol{k}}, \boldsymbol{b}_{\boldsymbol{k}})$ decays rapidly (presumably exponentially) for $k = |\boldsymbol{k}| > k_d$. By ignoring the contribution from the modes in the exponential tail, we can write

$$(\boldsymbol{u},\boldsymbol{b}) \approx \sum_{k\leqslant k_d} (\boldsymbol{u}_{\boldsymbol{k}}, \boldsymbol{b}_{\boldsymbol{k}}) \exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}. \tag{33}$$

By comparing Eq. (33) with Eq. (19) we obtain

$$(\boldsymbol{u},\boldsymbol{b}) \approx \sum_{i=1}^{N} \rho_i(t)(\boldsymbol{u}_i, \boldsymbol{b}_i) \approx \sum_{k\leqslant k_d} (\boldsymbol{u}_{\boldsymbol{k}}, \boldsymbol{b}_{\boldsymbol{k}}) \exp\{i\boldsymbol{k}\cdot\boldsymbol{x}\}. \tag{34}$$

As in previous studies on Navier-Stokes [1,14] and surface quasigeostrophic [16] turbulence and Burgers flows [15], N is assumed to approximate the number of Fourier modes within the wave number radius k_d. This means that the sum on the right-hand side of Eq. (34) has about N terms. This assumption is both physically plausible and mathematically sound. The reason for this is that since volume elements in the Lyapunov subspace (the space spanned by the above N-dimensional Lyapunov basis) contract under the dynamics, all other N-dimensional volume elements do too, including those in the subspace spanned by the first N Fourier modes.

Hence this subspace, like its Lyapunov counterpart, is adequate to accommodate the solutions. In other words, the sum on the right-hand side of Eq. (34) need not have more than N terms.

Consider the general power-law energy spectra $E(k) = Ck^{-\alpha}$ in the inertial range $k_0 \ll k \ll k_d$, where the dimensional parameter C and the exponent α are to be determined. For Pr = 1, we have

$$\epsilon \approx \nu \int_{k_0}^{k_d} k^2 E(k)\, dk = \nu C \int_{k_0}^{k_d} k^{2-\alpha}\, dk \approx \frac{\nu C}{3-\alpha} k_d^{3-\alpha}, \tag{35}$$

where we have assumed $\alpha < 3$. The number N' of Fourier modes within the wave number radius k_d is given by $N' \approx (Lk_d)^2$ for 2D and $N' \approx (Lk_d)^3$ for 3D. It follows that

$$N' \approx L^2 \left(\frac{(3-\alpha)\epsilon}{\nu C} \right)^{2/(3-\alpha)} \quad \text{for 2D}, \tag{36}$$

$$N' \approx L^3 \left(\frac{(3-\alpha)\epsilon}{\nu C} \right)^{3/(3-\alpha)} \quad \text{for 3D}. \tag{37}$$

By comparing N' with the derived estimates for N we obtain

$$\frac{\epsilon}{\left(\epsilon_{\boldsymbol{u}}^{1/2} + \epsilon_{\boldsymbol{b}}^{1/2}\right)^{(3-\alpha)/2}} \lesssim C \nu^{(3\alpha-5)/4} \quad \text{for 2D}, \tag{38}$$

$$\frac{\epsilon}{\left(\epsilon'^{1/2}_{\boldsymbol{u}} + \epsilon'^{1/2}_{\boldsymbol{b}}\right)^{(3-\alpha)/2}} \lesssim C \nu^{(3\alpha-5)/4} \quad \text{for 3D}, \tag{39}$$

where constant prefactors of order unity have been omitted.

In the limit $\nu \to 0$, the right-hand sides of Eqs. (38) and (39) vanish if $\alpha > 5/3$ and diverge if $\alpha < 5/3$, only remaining positive and finite for the critical value $\alpha = 5/3$. This means that for 2D, spectra steeper than $k^{-5/3}$ can be ruled out, unless ϵ vanishes as a power law of ν. This result holds even if the estimate for N is excessive. The same conclusion is valid for 3D, provided that $\epsilon'_{\boldsymbol{b}} < \infty$ and $\epsilon'_{\boldsymbol{b}} < \infty$. Now, if ϵ remains bounded and nonzero or vanishes less rapidly than a power law in ν, then the left-hand sides of Eqs. (38) and (39) remain so (again for $\epsilon'_{\boldsymbol{b}} < \infty$ and $\epsilon'_{\boldsymbol{b}} < \infty$ in 3D), thereby requiring $\alpha = 5/3$, provided that the estimates for N are optimal. With $\alpha = 5/3$, the parameter C is given by $C \approx \epsilon/(\epsilon_{\boldsymbol{u}}^{1/2} + \epsilon_{\boldsymbol{b}}^{1/2})^{2/3} \approx \epsilon^{2/3}$ for 2D (and also for 3D if $\epsilon \approx \epsilon'_{\boldsymbol{u}} + \epsilon'_{\boldsymbol{b}}$). Hence we can write $E(k) \propto \epsilon^{2/3} k^{-5/3}$, thus recovering the classical spectrum by the present method. Finally, the regime $\alpha < 5/3$, which includes the Iroshnikov and Kraichnan spectrum $k^{-3/2}$, is permissible, simply corresponding to excessive estimates for N. In this picture, strong Alfvén waves undermine nonlinear effects, giving rise to more stable solutions or equivalently reducing N. This is consistent with a recent study [16] suggesting that a reduction in nonlinear effects allows for shallower spectra to develop.

From $C \approx \epsilon^{2/3}$ and $\alpha = 5/3$, one can immediately deduce that

$$k_d \propto \frac{\epsilon^{1/4}}{\nu^{3/4}}, \tag{40}$$

which is another classical result by Kolmogorov's phenomenology.

The above calculations and conclusions apply to the special case Pr = 1 and are expected to hold for moderate Pr. In the Navier-Stokes regime (i.e., Pr $\ll 1$), this result remains valid for 3D but not for 2D. For the passive advection regime (i.e., Pr $\gg 1$), the theory of Batchelor [31], which predicted the scaling k^{-1} for the viscous-advective range between the viscous dissipation wave number and the magnetic diffusion wave number, is applicable. A rigorous version of this prediction can be found in Ref. [32].

In passing, we briefly elaborate on the Iroshnikov [7] and Kraichnan [8] theory, which predicted a $k^{-3/2}$ inertial range. This prediction has gained considerable support over the years, nonetheless, the spectral scaling of the inertial range of MHD turbulence has always been a matter of debate. Furthermore, the Iroshnikov and Kraichnan theory has not been widely accepted without challenge [12,13]. In any case, the $k^{-3/2}$ spectrum was proposed for turbulence with strong Alfvén waves and does not contradict our result (and the classical prediction), which has been derived in broad generalities. As argued by Kraichnan, strong Alfvén waves can decrease the energy transfer (suppression of turbulence or reduction of nonlinear effects), thereby giving rise to his $k^{-3/2}$ spectrum, which is slightly shallower than the classical spectrum of Kolmogorov. A similar turbulence suppression by an external uniform magnetic field was later considered in detail by Moffatt [33]. Moffatt's result is consistent with Kraichnan's theory as the turbulence in this theory could be considered being driven by a strong external magnetic field at large scales. While detailed numerical studies with high resolutions appear necessary to determine the precise conditions under which what spectra become realizable, the case Pr $\ll 1$ discussed earlier is clearly in favor of the Kolmogorov spectrum. In a recent study, Beresnyak [4] has demonstrated the realizability of this spectrum by numerically integrating the governing equations for $Z^{\pm}$ with a forcing that provides a constant energy injection rate at large scales.

V. REGULARITY

Whether the 3D and 2D MHD equations possess global smooth solutions is one of the outstanding open problems in mathematical fluid mechanics. Owing to the coupling between the induction and Navier-Stokes or Euler equations, mathematical results for the MHD system are often weaker than those for the Navier-Stokes or Euler equations. Like the 3D Navier-Stokes system, it is not known whether solutions to the 3D MHD equations starting from smooth initial conditions remain smooth for all time. For 2D, the answer to this problem is positive (just like the 2D Navier-Stokes or surface quasigeostrophic equations). However, unlike the 2D Navier-Stokes case, the global smoothness of solutions becomes questionable as soon as either $\nu = 0$ or $\mu = 0$, despite the fact that the presence of a strong magnetic field has been observed both experimentally and numerically to have regularizing effects on solutions. This is due to the suppression of turbulence (reduction of energy transfer) by Alfvén waves as suggested by the theory of Iroshnikov [7] and Kraichnan [8].

In any case, it is easy to recognize that the smoothness of $\boldsymbol{u}$ alone is sufficient for the system's smoothness since, once $\boldsymbol{u}$ remains smooth, the magnetic field, being advected and stretched by the smooth velocity and velocity gradient fields, must remain smooth also. Therefore the smoothness of $\boldsymbol{u}$ guarantees that of $\boldsymbol{b}$ and hence of the system as a whole. From the regularizing effects of the magnetic field discussed above, one may expect the reverse to hold also, that is, the smoothness of $\boldsymbol{b}$ leads to the smoothness of $\boldsymbol{u}$. However, this is not known, except for the 2D case, which we now discuss in detail. For simplicity of presentation we consider the case $\nu = \mu = 0$, that is, the ideal MHD system. The same conclusion holds for more general situations with either $\nu > 0$ or $\mu > 0$, albeit the proof becomes more involved.

The ideal 2D MHD system can be conveniently written in the form

$$\omega_t + J(\psi,\omega) = J(a,\Delta a), \tag{41}$$

$$a_t + J(\psi,a) = 0. \tag{42}$$

Here ω is the vorticity, ψ the stream function, a the magnetic potential, and $J(\cdot,\cdot)$ is the Jacobian. Note that a is a materially conserved quantity whose variance is known to undergo an inverse transfer. This transfer is in the opposite sense to that for other scalars (passive or active) and can be understood from the total energy conservation law. We would like to emphasize that in 3D no conserved positive-definite quadratic quantity is known to undergo an inverse transfer.

Now consider a fluid particle trajectory emanating from $\boldsymbol{x}_0$ at $t = 0$ and reaching some $\boldsymbol{x}$ at a later time $t = T$. The vorticity $\omega(\boldsymbol{x},T)$ is given in terms of $\omega_0 = \omega(\boldsymbol{x}_0,0)$ and the source term $J(a,\Delta a)$ by

$$\omega(\boldsymbol{x},T) = \omega_0 + \int_0^T J(a,\Delta a)\, dt, \tag{43}$$

where the integral is along the particle trajectory. Equation (43) means that if the magnetic field remains smooth up to $t = T$ [i.e., $|J(a,\Delta a)| < \infty$ for $t \leqslant T$] then $|\omega(\boldsymbol{x},T)| < \infty$. By the regularity criterion of Caflisch, Klapper, and Steele [34], which is the MHD analog of the celebrated Beale-Kato-Majda criterion [35] for the Euler equations, no singularity (of any type) of the velocity field can develop for $t \leqslant T$. So a smooth solution of the 2D MHD system persists as long as the magnetic field remains smooth.

On the other hand, as discussed above, if the velocity field remains smooth up to $t = T$ then so does the magnetic field. So a smooth solution of the 2D MHD system persists as long as the velocity field remains smooth. Thus, the smoothness of one field implies that of the other and is sufficient for global regularity. We would like to mention that the smoothness of solutions for all time does not mean the solutions remain bounded as $t \to \infty$. In particular, the velocity gradient may grow without bound.

The above analysis suggests that one dissipation mechanism, either mechanical or magnetic, should be capable of regularizing the whole system. However, such a result is still unavailable due to the limitation of available tools from the theory of nonlinear partial differential equations. That said, if strong enough hyperdissipation is applied to either the momentum or the induction equation, a proof of global regularity is possible. This proof is somewhat technical and will be presented elsewhere.

VI. CONCLUDING REMARKS

We have studied incompressible MHD turbulence in both 2D and 3D by using a dynamical systems approach, which has been employed in recent studies of fluid turbulence [1,14–17]. The new approach allows us to recover key predictions (in the form of bounds) in the literature on the basis of Komolgorov's phenomenology. These predictions include the number of degrees of freedom, the slope of the energy spectrum in the inertial range, and the dissipation wave number, all of which are intimately related. Estimates for the number of degrees of freedom N have been derived in terms of key dynamical quantities and physical parameters. In 2D, it has been found that N scales as $\mathrm{Pr}\,\mathrm{Re}^{3/2} + \mathrm{Rm}^{3/2}$ for $\mathrm{Pr} > 1$ and as $\mathrm{Re}^{3/2} + \mathrm{Pr}^{-1}\,\mathrm{Rm}^{3/2}$ for $\mathrm{Pr} \leqslant 1$, where Pr is the magnetic Prandtl number and Re and Rm are, respectively, the kinetic and magnetic Reynolds numbers. These Reynolds numbers are dynamic in nature, defined in terms of the time-dependent kinetic and magnetic energy dissipation rates, viscosity and magnetic diffusivity, and the system size. In 3D, N scales as $(\mathrm{Pr}\,\mathrm{Re}^{3/2} + \mathrm{Rm}^{3/2})^{3/2}$ for $\mathrm{Pr} > 1$ and as $(\mathrm{Re}^{3/2} + \mathrm{Pr}^{-1}\,\mathrm{Rm}^{3/2})^{3/2}$ for $\mathrm{Pr} \leqslant 1$, where the Reynolds numbers are similarly defined. These results have been argued to be consistent with those in the literature for fluid systems with quadratic nonlinearities. In particular, the result for 3D appropriately reduces to that for 3D Navier-Stokes turbulence as $\mathrm{Pr} \to 0$. For $\mathrm{Pr} \approx 1$, the derived estimates for N implies the bound $\alpha \leqslant 5/3$ for the exponent α of the energy spectrum $E(k) = Ck^{-\alpha}$ in the inertial range. The value $\alpha = 5/3$ corresponds to optimal bounds for N. In this case, $E(k)$ reduces to the classical form $E(k) \propto \epsilon^{2/3}k^{-5/3}$, which extends to the Kolmogorov dissipation wave number $k_d \propto \epsilon^{1/4}/\nu^{3/4}$.

For the usual Navier-Stokes turbulence, the reduction of dimension from three to two removes the vortex stretching mechanism, effectively rendering the small-scale dynamics approximately linear. The reduction in the scaling of N with the Reynolds number is from highly superlinear (namely $\mathrm{Re}^{9/4}$) to virtually linear. For MHD turbulence, the dimension reduction also removes the vortex stretching term. However, the quadratic coupling between the velocity and magnetic fields via the Lorentz force and magnetic stretching terms does maintain certain aspects of quadratic nonlinearity for the system as a whole. This makes it possible to apply Kolmogorov's phenomenological theory to derive the classical spectrum $k^{-5/3}$. Our alternative method has recovered this result, among other things.

It is remarkable that the respective 2D and 3D MHD turbulence may not be necessarily "more turbulent" than 2D surface quasigeostrophic and 3D Navier-Stokes turbulence, as was implied by Kraichnan's theory of energy transfer suppression. This theory is based on the physics of Alfvén waves, which undermines nonlinear effects, and has gained considerable support from both theoretical and numerical investigation [36] and, to some extent, observational data from the solar wind [37]. These waves are "invisible" to

Kolmogorov's phenomenology, which therefore unsurprisingly fails to account for the Kraichnan spectrum. In the present approach, Alfvén waves' effects render overestimates for N and α. It is not clear how to capture these effects to the precise extent that sharp estimates for N and α can be obtained.

The direct energy transfer in 2D can be better understood by examining how it takes place at the modal level. For this purpose, one can ignore the velocity self-advection term (the simultaneous conservation of energy and enstrophy by the advection term prohibits a direct energy transfer) and focus on the detailed behavior of the energy conversion, which plays a key role in the direct energy transfer. As dynamo action necessarily drains kinetic energy off the fluid and weakens the advecting velocity field, the direct energy transfer is not as self-sustained as that in 3D Navier-Stokes or 2D surface quasigeostrophic turbulence. This is evident from the dynamical nature of the interacting triads responsible for the energy transfer (and conversion). These triads are linear in the velocity field, each involving one mode from $\boldsymbol{u}$ and two modes from $\boldsymbol{b}$. It turns out that the dynamics of these triads are quite simple and mathematically assessable. In fact, one can show that dynamo triads (those converting kinetic into magnetic energy) are associated with a direct magnetic energy flux, while antidynamo triads (those converting magnetic into kinetic energy) are associated with an inverse magnetic energy flux [38]. For a persistent direct energy flux, the $\boldsymbol{u}$ mode in a dynamo triad, upon losing energy, must be replenished. Such an energy recharge can be accomplished only through antidynamo interactions with other $\boldsymbol{b}$ modes, thereby involving some inverse energy transfer and resulting in a reduced direct energy flux. A quantitative analysis of these triads is given in Ref. [38].

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