Critical points of the O(n) loop model on the martini and the 3-12 lattices

Chengxiang Ding
Physics Department, Anhui University of Technology, Maanshan 243002, People's Republic of China

Zhe Fu and Wenan Guo*
Physics Department, Beijing Normal University, Beijing 100875, People's Republic of China
(Received 14 March 2012; published 19 June 2012)

We derive the critical line of the O(n) loop model on the martini lattice as a function of the loop weight n basing on the critical points on the honeycomb lattice conjectured by Nienhuis [Phys. Rev. Lett. **49**, 1062 (1982)]. In the limit $n \to 0$ we prove the connective constant $\mu = 1.750\,564\,5579\ldots$ of self-avoiding walks on the martini lattice. A finite-size scaling analysis based on transfer matrix calculations is also performed. The numerical results coincide with the theoretical predictions with a very high accuracy. Using similar numerical methods, we also study the O(n) loop model on the 3-12 lattice. We obtain similarly precise agreement with the critical points given by Batchelor [J. Stat. Phys. **92**, 1203 (1998)].

DOI: 10.1103/PhysRevE.85.062101 PACS number(s): 05.50.+q, 64.60.Cn, 64.60.F−, 75.10.Hk

Introduction. The O(n) loop model [1] originates from the high-temperature expansion of the O(n) spin model [2]. It can be considered a model describing a nonintersecting loop gas. On lattices with coordination number three, the partition function is very simple:

$$Z = \sum_{\mathcal{G}} x^b n^l, \tag{1}$$

where the sum is over all configurations of nonintersecting loops denoted as $\mathcal{G}$; x is the weight of a bond (an edge occupied by loop segments), or an occupied vertex, and n is the weight of a loop. b is the number of bonds or occupied vertices, and l is the number of loops.

Generally speaking, there is a high-temperature phase with dilute loops and a low-temperature phase with dense loops. At the transition point x_c, the longest loop grows to infinity and begins to percolate the system. The critical properties of this transition are universal, which are well described by the Coulomb gas theory [3]. However, the determination of the critical points of this model on various lattices remains to be treated case by case. O(n) critical lines have been found on the honeycomb lattice [4–6], the square lattice [7], and the triangular lattice [8]. In addition to this transition point, there are several other branches of critical behavior, for example, "branch 0," which describes a higher critical point, as reported in Refs. [8–10].

In the $n \to 0$ limit, the critical O(n) loop model describes long polymers in a good solvent or self-avoiding walks (SAWs) [11]. The study of the O(n) loop model has led to a wealth of information on the configuration properties of SAWs [12]. The number of configurations of SAWs in k steps, that is, C_k scales as [13]

$$C_k \sim A\mu^k k^{\gamma-1} \tag{2}$$

for large k. A is a constant. $\gamma = 43/32$ is an universal critical exponent, which can be obtained via the Coulomb gas theory [4]. μ is the connective constant which is lattice dependent, and equals to $1/x_c$ of the $n \to 0$ loop model. Although the studies on the SAWs have advanced a lot since it was introduced [14], the values of μ for most of two-dimensional lattices are found numerically [15–19]. The conjectured critical line of the O(n) loop model on the honeycomb lattice [4] provides $\mu = \sqrt{2+\sqrt{2}}$ on the honeycomb lattice.

The critical line of the honeycomb O(n) loop model was found by an exact mapping on a Potts model [4] and by using the Bethe ansatz [5,6]:

$$x_c^2 = \frac{1}{2+\sqrt{2-n}}. \tag{3}$$

This result was known to be true for the Ising case ($n = 1$) [20], and was recently proved for the case $n = 0$ by Duminil-Copin and Smirnov [21]. The phase diagram inferred from this result has been well verified by different numerical methods [22,23]. Batchelor derived the critical points of the O(n) loop model on the 3-12 lattice by mapping the honeycomb loop model to the 3-12 lattice [24]:

$$\left(\frac{x_c^2 + x_c^3}{1 + x_c^3 n}\right)^2 = \frac{1}{2+\sqrt{2-n}}. \tag{4}$$

Making use of this result, Batchelor obtained $\mu = 1.711041\ldots$ for the 3-12 lattice, which coincides with the result previously found by Jensen and Guttmann [18] using other methods. This was made rigorous by the result of Duminil-Copin and Smirnov [21]. In the present paper we shall provide some independent numerical results for the critical point of the O(n) loop model on the 3-12 lattice, and on its phase behavior.

Inspired by Batchelor's work, we studied the O(n) loop model on the martini lattice. This lattice was first proposed by Scullard [25] in the study of percolation. The percolation threshold and the critical points of the q-state Potts model [26,27] on this lattice are known exactly [28], but the critical points of the O(n) loop model are not known yet. In this paper we derive the critical points of the O(n) loop model as a function of n on the martini lattice. In particular, in the limit $n \to 0$, we prove the connective constant $\mu = 1.750\,564\,5579\ldots$ of SAWs on the martini lattice. In addition, we build the transfer matrix (TM) and apply a finite-size

*Corresponding author: waguo@bnu.edu.cn

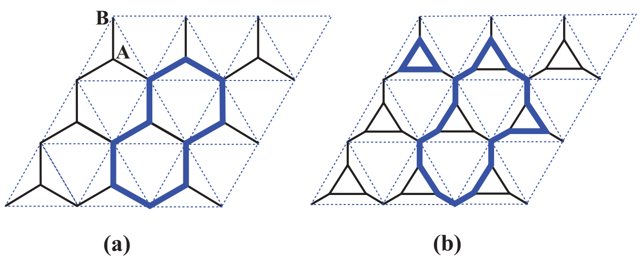

FIG. 1. (Color online) The O(n) loop model on the honeycomb lattice (a) and the martini lattice (b).

scaling analysis for a numerical study of the model on the martini and the 3-12 lattice.

Critical points of O(n) loop model on the martini lattice. Consider a honeycomb lattice with two sublattices A and B. A loop configuration $\mathcal{G}$ also denotes the occupations of sublattice A, as shown in Fig. 1(a). We rewrite the partition function (1) in the following way:

$$Z_h = \sum_{\mathcal{G}} x^v n^l = \sum_{\mathcal{G}} (x^2)^{v_A} 1^{V_A - v_A} n^l, \qquad (5)$$

where $v = b$ is the number of vertices visited by loop segments, V_A and $v_A = v/2$ are the number of vertices and the number of visited vertices of sublattice A, respectively. Thus the weight of a visited A vertex shown in Fig. 2(a) is x^2, the weight of an empty A vertex shown in Fig. 2(d) is 1.

Now consider the O(n) loop model on the martini lattice, which is constructed by replacing the "star" (an A vertex) shown in Fig. 2(d) by the structure shown in Fig. 2(e). Each occupied (empty) A vertex corresponds to two possible occupations on that structure, as shown in Figs. 2(b) and 2(c) [Figs. 2(e) and 2(f)]. Thus, any given configuration $\mathcal{G}$ of loops on the honeycomb lattice maps to the sum of 2^{V_A} possible loop configurations on the martini lattice. Let x be the weight of a bond for the martini loop model, the partition function of the loop model on the martini lattice can be obtained by summing on $\mathcal{G}$:

$$Z_m = (1 + x^3 n)^{V_A} \sum_{\mathcal{G}} \left(\frac{x^3 + x^4}{1 + x^3 n} \right)^{v_A} n^l. \qquad (6)$$

Mapping $(x^3 + x^4)/(1 + x^3 n) \to x^2$, we obtain Z_h in (5) multiplied by a trivial factor. It follows the critical points x_c of the O(n) loop model on the martini lattice:

$$\frac{{x_c}^3 + {x_c}^4}{1 + {x_c}^3 n} = \frac{1}{2 + \sqrt{2 - n}}. \qquad (7)$$

This result is in agreement with an existing result for the O(1) loop model, which is equivalent to the high temperature expansion of the Ising model model [29] with $x = \tanh K^I$, where K^I is the coupling of Ising spins sitting on the vertices of the martini lattice. According to (7), $K_c^I = 0.749\,790\,959\,036\ldots$, which coincides with the exactly known critical point of the $q = 2$ Potts model on the martini lattice [28].

As another special case, we prove the connective constant of the SAWs on the martini lattice. The substitution $\mu = 1/x_c$ in (7) for $n = 0$ determines μ as the solution of

$$\frac{1}{\mu^3} + \frac{1}{\mu^4} = 1 - \frac{\sqrt{2}}{2}, \qquad (8)$$

which yields $\mu = 1.750\,564\,557\,897\ldots$.

We may further generalize the above results by allowing bonds on the small triangles in Fig. 2(e) to have weight (x_t) different from those on the remaining ones (x_s). Following the mapping described above, we thus obtain a critical line in the x_t versus x_s plane for a given n:

$$\frac{{x_s}^2 x_t + {x_t}^2 {x_s}^2}{1 + {x_t}^3 n} = \frac{1}{2 + \sqrt{2 - n}}. \qquad (9)$$

For the 3-12 lattice, the critical line of the generalized model is

$$\frac{x_s\left(x_t + x_t^2\right)}{1 + x_t^3 n} = \frac{1}{\sqrt{2 + \sqrt{2 - n}}}. \qquad (10)$$

Finite-size scaling and transfer matrix calculation. Consider the lattice (the 3-12 or the martini lattice) wrapped on a cylinder with circumference L. The magnetic correlation function of the O(n) spin model is translated as the probability that two sites at a distance r are linked by a single loop segment [3], $g_r = Z'/Z$, where $Z' = \sum_{\mathcal{G}'} x^b n^l$, and $\mathcal{G}'$ denotes the configurations that connect sites 0 and r by precisely one single loop segment. In our transfer-matrix analysis of the finite-size-scaling behavior, it is sufficient to substitute the configurations $\mathcal{G}'$ that connect any site of row 0 to any site of a row at a distance r as measured in the length direction of the cylinder. The exponential decay of g_r at large distances is determined by the magnetic gap in the eigenvalue spectrum of the transfer matrix. The scaled magnetic gap $X_h(x,L) = \frac{L\zeta}{2\pi}\ln(\Lambda^{(0)}/\Lambda^{(1)})$, where $\Lambda^{(0)}$ and $\Lambda^{(1)}$ are the largest eigenvalue of the TM for Z and Z', respectively. ζ is a geometrical factor determined by the ratio between the unit of L and the thickness of a row added by the transfer matrix. Another scaled gap $X_t(x,L) = \frac{L\zeta}{2\pi}\ln(\Lambda^{(0)}/\Lambda^{(2)})$ describes the exponential decay of the energy-energy correlation with $\Lambda^{(2)}$ the second eigenvalue of the TM for Z.

The TM techniques of the O(n) loop model are well described in the literature (e.g., see [9]). The procedure of sparse matrix decomposition for the martini and the 3-12 lattice equals that for the honeycomb lattice with the adding units suitably chosen [22]. For further details see [30,31].

According to the finite-size scaling [32] and the conformal invariance [33] theory, the scaled gap $X_i(x,L)$, in the vicinity of the critical point, satisfies

$$X_i(x,L) = X_i + a(x - x_c)L^{y_t} + buL^{y_u} + \cdots, \qquad (11)$$

where $X_i (i = h,t)$ is the magnetic and the temperature scaling dimension, respectively; y_t is the thermal exponent; u denotes

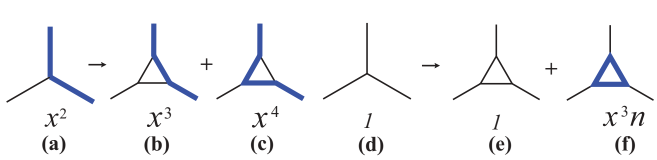

FIG. 2. (Color online) Mapping of the vertex configurations. The weights of the different vertices are shown. The same weights apply to versions rotated by $\pm 2\pi/3$.

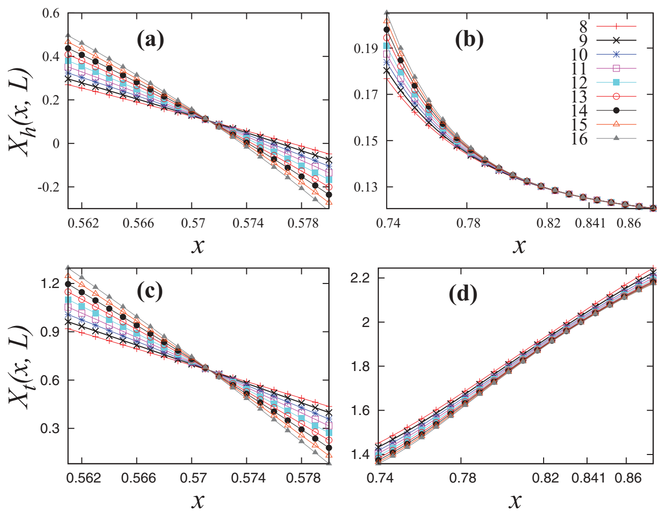

FIG. 3. (Color online) Scaled gaps $X_h(x,L), X_t(x,L)$ vs x for the O(n) loop model on the martini lattice with $L = 8$ to 16. (a) and (c): $n = 0$; (b) and (d): $n = 2$, with $x_c = 0.840896$ indicated. Lines connecting data points are added to guide the eye.

the leading irrelevant field, y_u is the associated irrelevant exponent. a,b are unknown constants. Such behavior is illustrated in Figs. 3(a) and 3(c) for the case $n = 0$. The critical point is estimated by numerically solving x in the equation $X_h(x,L) = X_h(x,L-1)$, with system sizes up to $L = 16$. The solution $x_c(L)$ converges to the critical value x_c as

$$x_c(L) = x_c + a'uL^{y_u - y_t} + \cdots, \tag{12}$$

where a' is an unknown constant. The numerical estimations and the theoretical predictions of the critical points of the martini lattice are listed in Table I. Our numerical estimations coincide with the theoretical predictions in a very high accuracy.

The universal values of X_i and the conformal anomaly c of the two-dimensional O(n) loop model are exactly known as [3,33]

$$c = 1 - \frac{6(g-1)^2}{g}, \qquad X_h = 1 - \frac{1}{2g} - \frac{3g}{8}, \quad X_t = \frac{4-2g}{g}, \tag{13}$$

where $n = -2\cos(\pi g)$, $1 \leqslant g \leqslant 2$.

At criticality, $X_i(L)$ converges as follows to X_i with increasing system size L,

$$X_i(L) = X_i + b'L^{y_u} + \cdots, \tag{14}$$

where b' is an unknown constant. The free energy density $f(L) = \zeta \ln \Lambda_L^{(0)}/L$ scales as [34,35]

$$f(L) = f(\infty) + \frac{\pi c}{6L^2}. \tag{15}$$

We then calculate $f(L), X_t(L)$, and $X_h(L)$ for a sequence of systems up to size $L = 16$ at the critical points (7). Fitting the data according to (14) and (15), we obtain the scaling dimensions X_h, X_t and the conformal anomaly c, which are also listed in Table I. Our numerical estimations are in agreement with the theoretical predictions with a high accuracy. For X_h, X_t, our results are also consistent with the Monte Carlo results for $n \geqslant 1$ [23,36].

When n approaches 2, $y_t = 2 - X_t \to 0$. The corrections to scaling due to the leading irrelevant field become relatively strong, thus the precision of our numerical estimation decreases. At $n = 2$, y_t is exactly 0, so that intersecting points between the curves $X_i(x,L)$ and $X_i(x,L-1)$ may be absent, as already suggested by (11), and as indeed observed in Figs. 3(b) and 3(d). Therefore, we cannot numerically determine the critical point x_c in the usual way. However, c and X_h, X_t are still estimated at the theoretical critical point.

Similar analysis is also performed to the 3-12 lattice. Theoretical predictions and numerical estimations of critical points for several values of n, which agree in a high accuracy, are listed in Table II. From Tables I and II we can see that the values of X_h, X_t and c for a two-dimensional O(n) loop model on the martini lattice coincide with those of the 3-12 lattice, as expected by the hypothesis of universality.

Conclusion. We derived the critical points of the O(n) loop model on the martini lattice basing on the conjectured critical points on the honeycomb lattice [4]. In the limit $n \to 0$, the connective constant of the SAWs on the martini lattice is proved to be $\mu = 1.750\,564\,5579\ldots$. Moreover, we performed a finite-size scaling analysis based on numerical TM calculations. Our numerical estimations agree with the theoretical predictions, within a margin that can reach 10^{-9} (for $n = 0, 0.25$). This rather high precision may be related to the vanishing of the leading irrelevant field in the Nienhuis result [4] for the critical line.

The critical points of the O(n) loop model on the 3-12 lattice derived by Batchelor are also verified.

TABLE I. Critical points x_c, conformal anomaly c, magnetic and temperature scaling dimensions X_h, X_t of the two-dimensional O(n) loop model on the martini lattice. (T = theoretical prediction, N = numerical estimation.)

n	x_c(T)	x_c(N)	c(T)	c(N)	X_h(T)	X_h(N)	X_t(T)	X_t(N)
0	0.571244285	0.571244285(1)	0	0	0.1041667	0.104166(1)	2/3	0.666668(2)
0.25	0.584248605	0.584248605(1)	0.1300704	0.1300705(2)	0.1100192	0.1100193(1)	0.7395254	0.739526(1)
0.5	0.598867666	0.59886768(2)	0.2559499	0.255950(1)	0.1154420	0.1154420(1)	0.8177559	0.817756(1)
0.75	0.615559079	0.6155590(1)	0.3788781	0.3788783(3)	0.1204452	0.1204452(1)	0.9035105	0.9035104(2)
1.0	0.635024224	0.63502422(1)	0.5	0.500000(1)	0.125	0.12500000(1)	1	1.00000(1)
1.25	0.658437850	0.65843786(1)	0.6205051	0.6205053(2)	0.1290128	0.1290127(1)	1.1126008	1.1126007(1)
1.5	0.688067393	0.68806739(2)	0.7418425	0.741842(1)	0.1322435	0.1322434(1)	1.2518912	1.25189(1)
1.75	0.729662053	0.729664(3)	0.8662562	0.8662563(1)	0.1339623	0.13396(1)	1.4457176	1.445718(1)
2.0	0.840896415	–	1	1.000000(1)	0.125	0.1250000(1)	2	2.00000(1)

TABLE II. Critical points x_c, conformal anomaly c, magnetic and temperature scaling dimensions X_h, X_t of the two-dimensional O(n) loop model on the 3-12 lattice. (T = theoretical prediction, N = numerical estimation.)

n	x_c(T)	x_c(N)	c(T)	c(N)	X_h(T)	X_h(N)	X_t(T)	X_t(N)
0	0.584439429	0.584439429(1)	0	0	0.1041667	0.104167(1)	2/3	0.666668(1)
0.25	0.601034092	0.601034092(1)	0.1300704	0.1300705(2)	0.1100192	0.1100193(1)	0.7395254	0.739526(1)
0.5	0.620240607	0.62024060(1)	0.2559499	0.255950(3)	0.1154420	0.115442(1)	0.8177559	0.817756(1)
0.75	0.642967899	0.6429678(1)	0.3788781	0.3788783(2)	0.1204452	0.1204452(1)	0.9035105	0.903510
1.0	0.670697664	0.67069766(1)	0.5	0.499999(1)	0.125	0.1250000(1)	1	1.000000(1)
1.25	0.706102901	0.70610291(1)	0.6205051	0.620505(1)	0.1290128	0.1290127(1)	1.1126008	1.112600(1)
1.5	0.754845016	0.754845(1)	0.7418425	0.741842(1)	0.1322435	0.1322435(1)	1.2518912	1.25189(1)
1.75	0.833205232	0.83320(2)	0.8662562	0.86626(1)	0.1339623	0.13396(1)	1.4457176	1.445718(2)
2.0	1.172534677	–	1	0.999999(1)	0.125	0.1250000(1)	2	2.00000(1)

The conformal anomaly, the magnetic and the temperature scaling dimensions of the O(n) models on the two lattices are numerically calculated. The estimations coincide with the theoretical predictions, as expected according to the universality hypothesis.

Acknowledgments. We thank Professor F. Y. Wu and Professor H. W. J. Blöte for a critical reading of the manuscript and valuable suggestions. This work is supported by the NSFC under Grant No. 11175018, the NCET-08-0053, and the HSCC of Beijing Normal University.

[1] E. Domany, D. Mukamel, B. Nienhuis, and A. Schwimmer, Nucl. Phys. B **190**, 279 (1981).
[2] H. E. Stanley, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green, Vol. 3 (Academic, London, 1987).
[3] B. Nienhuis, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. Lebowitz, Vol. 11 (Academic, London, 1987).
[4] B. Nienhuis, Phys. Rev. Lett. **49**, 1062 (1982).
[5] R. J. Baxter, J. Phys. A **19**, 2821 (1986).
[6] M. T. Batchelor and H. W. J. Blöte, Phys. Rev. Lett. **61**, 138 (1988); Phys. Rev. B **39**, 2391 (1989).
[7] M. T. Batchelor, B. Nienhuis, and S. O. Warnaar, Phys. Rev. Lett. **62**, 2425 (1989).
[8] Y. M. M. Knops, B. Nienhuis, and H. W. J. Blöte, J. Phys. A **31**, 2941 (1998).
[9] H. W. J. Blöte and B. Nienhuis, J. Phys. A: Math. Gen. **22**, 1415 (1989).
[10] B. Li, W.-A. Guo, and H. W. J. Blöte, Phys. Rev. E **78**, 021128 (2008).
[11] P. G. de Gennes, Phys. Lett. A **38**, 339 (1972).
[12] B. Duplantier, J. Stat. Phys. **54**, 581 (1989).
[13] J. M. Hammersley, Proc. Cambridge Philos. Soc. **53**, 642 (1957).
[14] P. J. Flory, J. Chem. Phys. **17**, 303 (1949).
[15] S. E. Alm, J. Phys. A **38**, 2055 (2005).
[16] M. E. Fisher and M. F. Sykes, Phys. Rev. **114**, 45 (1959).
[17] I. Jensen, J. Phys. A **37**, 11521 (2004).
[18] I. Jensen and A. J. Guttmann, J. Phys. A **31**, 8137 (1998).
[19] S. E. Alm and R. Parviainen, J. Phys. A **37**, 549 (2004).
[20] I. Syozi, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green, Vol. 1 (Academic, London, 1972).
[21] H. Duminil-Copin and S. Smirnov, Ann. Math. **175**, 1653 (2012).
[22] H. W. J. Blöte and B. Nienhuis, Physica A **160**, 121 (1989).
[23] Y. J. Deng, T. M. Garoni, W.-A. Guo, H. W. J. Blöte, and A. D. Sokal, Phys. Rev. Lett. **98**, 120601 (2007).
[24] M. T. Batchelor, J. Stat. Phys. **92**, 1203 (1998).
[25] C. R. Scullard, Phys. Rev. E **73**, 016107 (2006).
[26] R. B. Potts, Proc. Cambridge Philos. Soc. **48**, 106 (1952).
[27] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).
[28] F. Y. Wu, Phys. Rev. Lett. **96**, 090602 (2006).
[29] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).
[30] H. W. J. Blöte and M. P. Nightingale, Physica A **112**, 405 (1982).
[31] H. W. J. Blöte and M. P. Nightingale, Phys. Rev. B **47**, 15046 (1993).
[32] For reviews, see e.g., M. P. Nightingale, in *Finite-Size Scaling and Numerical Simulation of Statistical Systems*, edited by V. Privman (World Scientific, Singapore, 1990); M. N. Barber, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz, Vol. 8 (Academic, New York, 1983).
[33] J. L. Cardy, J. Phys. A **17**, L385 (1984).
[34] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986).
[35] I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).
[36] C. X. Ding, Y. J. Deng, W.-A. Guo, and H. W. J. Blöte, Phys. Rev. E **79**, 061118 (2009).