Density of zeros of the ferromagnetic Ising model on a family of fractals

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We studied distribution of zeros of the partition function of the ferromagnetic Ising model near the Yang-Lee edge on a family of Sierpinski gasket lattices whose members are labeled by an integer b ($2 \leqslant b < \infty$). The obtained exact results on the first seven members of this family show that, for $b \geqslant 4$, associated correlation length diverges more slowly than any power law when distance δh from the edge tends to zero, $\xi_{\rm YL} \sim \exp\,[\ln(b)\sqrt{|\ln(\delta h)|/\ln(\lambda_0)}]$, λ_0 being a decreasing function of b. We suggest a possible scenario for the emergence of the usual power-law behavior in the limit of very large b when fractal lattices become almost compact.

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I. INTRODUCTION

A close connection between the zeros of the partition function in the complex plane of an appropriate variable and the onset of phase transition was first pointed out by Yang and Lee [1]. They showed that all of the zeros of the partition function of a nearest-neighbor ferromagnetic Ising model lie on the unit circle in the complex field activity plane $y = \exp(-2H/k_BT)$. In the thermodynamic limit these zeros are expected to condense, yielding a limiting density of zeros $g(h'',T)$ ($h'' = H''/k_BT$, where H'' denotes the imaginary part of the complex field $H = H' + iH''$). For any temperature T above the critical temperature there exists a pair $\pm i H_0(T)$ of zeros lying closest to the real H axis, commonly referred to as Yang-Lee (YL) edge singularities. As these nonanalytic points exert the most direct influence on the model thermodynamic behavior for real H and T, it is very important to understand the nature of these singularities.

Determination of the limiting density of zeros is generally a highly nontrivial problem and little is known about its behavior near the YL edge, let alone about its actual form in the whole region of interest. It is widely accepted, however, that the density of zeros near the edge exhibits a power-law behavior, $g \sim |\delta h|^\sigma$, as $\delta h = h'' - h_0(T) \to 0$, where σ is the YL edge critical exponent. Since in this case exists only one relevant variable [2], all other YL critical exponents can be expressed in terms of σ. In particular, the correlation length $\xi_{\rm YL}$ critical exponent $\nu_{\rm YL}$, which describes the spatial decay of the two-spin correlation function near the edge, $\xi_{\rm YL} \sim |\delta h|^{-\nu_{\rm YL}}$, can be related to σ: $\sigma = d\nu_{\rm YL} - 1$, where d is the space dimensionality. It has been suggested [2,3] that the value of this exponent depends only on the space dimensionality and is independent of temperature for all T above the critical temperature. When $d = 1$, the problem can be solved exactly, which yields $\sigma = -1/2$. Despite the fact that $g(h'',T)$ is not known exactly for two-dimensional ferromagnetic Ising model, it has been predicted $\sigma = -1/6$ in $d = 2$ [4]. It is believed that σ maintains its mean-field value $\sigma = 1/2$ above the upper critical dimension $d = 6$ (see Ref. [5] for a recent review of the application of YL formalism to equilibrium and nonequilibrium phase transitions).

The original YL circle theorem [1] is independent of the topological structure of the underlying lattice and should also apply to appropriate models of certain nonhomogeneous systems, like models of diluted ferromagnets. This stimulated a considerable research interest for studies of the ferromagnetic Ising model on a variety of hierarchical graphs [6,7]. There are several reasons for such an interest. First, many interesting exact results can be obtained on the finitely ramified structures, while analogous problems on the standard homogenous spaces are usually rather difficult. Lacking translation invariance, self-similar graphs may be used to model some aspects of nonhomogeneous and disordered systems. One can also expect that such studies may reveal the influence of different geometrical and topological characteristics of the underlying graph on the nature of critical behavior in general.

Several studies performed so far on deterministic fractals show that density of zeros exhibits a scaling form near the edge, which is more complicated than a pure power law. Using a decimation approach, the exact recursion relations were constructed, and nontrivial values of ν_{YL} and σ were obtained, mainly numerically [6]. Subsequently, this method was extended to allow an exact description of the YL edge singular behavior, first, to the case of "quasilinear" fractal lattices [7] and, more recently, to self-similar lattices having better connectivity [8]. It was shown that the form of these singularities depends on the way the lattice coordination number fluctuates from site to site of the lattice and that it may differ markedly from the usual power-law form. Surprisingly, in some cases it was found that these singularity forms coincide with those for the simple zero-field Gaussian model on the same structure. This motivated us to explore this problem more systematically by considering the ferromagnetic Ising model on a class of Sierpinski-type fractal lattices (see Fig. 1). Each member of this family can be labeled by an integer b, $2 \leqslant b < \infty$, which represents lattice spatial scaling factor. This family often has been used in the past to study critical behavior of some interesting statistical mechanical models, especially in the context of a possible fractal to homogenous space critical

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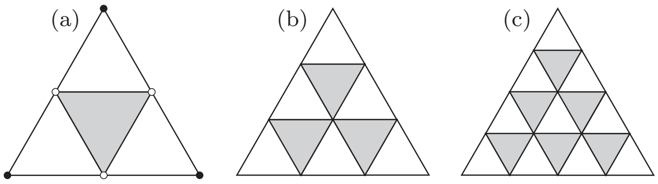

FIG. 1. Schematic representation of the r-th level Sierpinski type fractal with spacial scaling factor $b = 2$ (a), $b = 3$ (b), and $b = 4$ (c). To construct an r-th order triangle with $b = 2$, for example, one has to join three $r - 1$-th upward-pointed triangles as shown in (a).

behavior crossover, because both fractal and spectral lattice dimension tends to 2 for large b, while fraction $(b - 2)/(b + 4)$ of sites having coordination number 6 tends to 1 in this limit. These studies revealed, however, that there is no a simple connection between the limiting critical behavior on fractals (i.e., when $b \to \infty$) and those on standard two-dimensional lattices. For example, an exact analysis of linear and branched polymer models on this class of lattices [9] showed that some critical exponents do not tend to their two-dimensional counterparts.

In this paper, we show that YL correlation length for $b = 4,5,6,7$ diverges more slowly than any power law, $\xi_{\mathrm{YL}} \sim \exp[\ln(b)\sqrt{|\ln(\delta h)|/\ln(\lambda_0)}]$, where $\lambda_0(b)$ is a decreasing function of b. At first glance it is difficult to reconcile such a behavior with the common power-law divergence. It seems, however, that $\lambda_0(b)$ tends to 1 for large b, which indicates that the quoted asymptotic form may be inapplicable in the limit $b \to \infty$.

An elegant solution, which relies on the existence of a star-triangle transformation for the Ising model on the usual Sierpinski gasket with spatial scaling factor $b = 2$, has been found [6]. Unfortunately, it is difficult to extend this approach to the case of lattices with the spatial scaling factor $b > 2$. Recently, we used the method of recursion relations for conditional partition functions to study this problem on $b = 3$ lattice [8]. As it was established, however, these relations are singular at pertinent fixed points, and this fact prevented the standard fixed point analysis. Our approach was based on the existence of an invariant manifold, which allowed us to make an asymptotic expansion of appropriate variables near the fixed point. In this way, we have been able to show that the YL correlation length in this case displays a logarithmic rather than the usual power-law behavior.

In this paper we shall extend this analysis to fractals having $b > 3$. Since recursion relations for $b > 3$ are very cumbersome, we shall illustrate our approach on the simple $b = 2$ case in Sec. II A. The main motive for this is not, of course, to rederive known results using a different method but rather to introduce notation and establish some general intermediate results which we use throughout the paper. In Sec. II C we give a detailed account of analytical and numerical results for the $b = 4$ case, which seems to be representative of the whole $b \geqslant 4$ lattice family. Then, in Sec. II D, we present very precise numerical results for $b = 5,6,7$ lattices, which are based on exact recursion relations and numerical insights about structure of the invariant manifolds. The conclusions are summarized in Sec. III.

II. YANG-LEE EDGE PROBLEM FOR THE ISING MODEL ON SIERPINSKI FRACTALS

The Ising model on the $b = 2$ gasket, for real values of magnetic field and other interaction parameters, has been studied in detail in Ref. [10] using a decimation renormalization group transformation. Since decimation transformations are not well suited for studies of the YL edge problem, it has been studied using a block-spin [11] renormalization group transformation on the same lattice [6]. Some interesting exact results for the YL edge problem on the $b = 3$ gasket have been obtained recently [6]. Let us note, however, that the $b = 2$ and $b = 3$ cases are not generic in the sense that the correlation length in the first case displays the standard power-law singularity while it follows a logarithmic law in the second case. On the other hand, it turns out that correlation lengths of the Ising model on $b = 4,5,6,7$ lattices have a qualitatively similar singular behavior near the edge, which differs from both power and logarithmic law. Independent of the type, all these singularities can be examined within the approach that we are going to describe.

A. $b = 2$ gasket

Consider the nearest-neighbor ferromagnetic Ising model with a uniform magnetic field on an r-th level triangle shown in Fig. 1(a). The partition function $Z^{(r)}$ of the model can be written as a simple combination of only four conditional partition functions $Z_1^{(r)} = Z^{(r)}(+,+,+)$, $Z_2^{(r)} = Z^{(r)}(-,+,+)$, $Z_3^{(r)} = Z^{(r)}(+,-,-)$, and $Z_4^{(r)} = Z^{(r)}(-,-,-)$, where $Z^{(r)}(+,+,+)$, for example, denotes the partition function of an r-th level triangle with three corner spins [represented by black circles in Fig. 1(a)] being fixed in the "up state." One can then, by keeping fixed states of these three spins, and summing over 2^3 states of three interior spins [open circles in Fig. 1(a)] of an r-th level triangle, write down recursion relations for the above partition functions. Being homogenous, the obtained system of recursion relations can be simplified if one introduces the reduced variables, for example, in the following way: $z_2 = Z_2/Z_1$, $z_3 = Z_3/Z_1$, and $z_4 = Z_4/Z_1$. In terms of these variables, one can write

$$z_2' = \frac{\mathfrak{z}_2}{\mathfrak{z}_1}, \qquad z_3' = \frac{\mathfrak{z}_3}{\mathfrak{z}_1}, \qquad z_4' = \frac{\mathfrak{z}_4}{\mathfrak{z}_1}, \tag{1}$$

where $\mathfrak{z}_i$ $(i = 1,2,3,4)$ depend on z_2, z_3, z_4, and on the magnetic field variable $y = \exp(-2h)$,

$$\begin{aligned}
\mathfrak{z}_1 &= y^3 + 3y^2z_2^2 + 3yz_2^2z_3 + z_3^3,\\
\mathfrak{z}_2 &= y^3z_2 + y^2z_2^3 + 2y^2z_2z_3 + 2yz_2z_3^2 + yz_2^2z_4 + z_3^2z_4,\\
\mathfrak{z}_3 &= y^3z_2^2 + 2y^2z_2^2z_3 + y^2z_3^2 + yz_3^3 + 2yz_2z_3z_4 + z_3z_4^2,\\
\mathfrak{z}_4 &= y^3z_2^3 + 3y^2z_2z_3^2 + 3yz_3^2z_4 + z_4^3.
\end{aligned} \tag{2}$$

To obtain an r-th order partition function $z_i^{(r)}$ $(i = 2,3,4)$ one has to iterate the above recursion relations r times, starting with the following initial conditions: $z_2^{(0)} = xy$, $z_3^{(0)} = xy^2$, and $z_4^{(0)} = y^3$, where $x = \exp(-4K)$ (here $K = J/kT > 0$ stands for a standard ferromagnetic interaction strength between each nearest-neighbor pair of spins).

Now one can express derivatives of the free energy in terms of z_i and some scaled derivatives of the original variables Z_i.

For example, the average magnetization $\mathcal{M}$ per spin is given by

$$\mathcal{N}\mathcal{M} = \frac{t_1 + 3t_2 + 3t_3 + t_4}{1 + 3z_2 + 3z_3 + z_4}, \quad (3)$$

where we have omitted the iteration index r on the left- and right-hand side of the above relation; here $\mathcal{N}$ represents the number of sites of an r-th level lattice, $\mathcal{N} \equiv \mathcal{N}^{(r)} = 3(3^r + 1)/2$, and t_i denote the scaled derivatives $t_i^{(r)} = (\partial Z_i^{(r)}/\partial h)/Z_1^{(r)}$, $i = 1,2,3,4$. Using the original recursion relations for partition functions Z_i, one can construct an exact system of recursion relations for t_i, which can be used further to reveal the asymptotic behavior of density of zeros g near the edge [1], $g \propto \mathrm{Re}(\mathcal{M})$. In a similar way, one can construct a two-spin correlation function in the external magnetic field $\mathcal{G} =< S_1 S_2 > - < S_1 >< S_2 >$ between two outer spins of an r-th level triangle, lying on a mutual distance $R = 3^r$, in terms of $z_2^{(r)}$, $z_3^{(r)}$, and $z_4^{(r)}$

$$\mathcal{G} = 4\frac{z_3 - z_2^2 - z_2 z_3 - z_3^2 + z_4 + z_2 z_4}{(1 + 3z_2 + 3z_3 + z_4)^2}, \quad (4)$$

where we omitted the iteration index r. As is expected, one can show that this function decays exponentially, $\mathcal{G} \sim \exp(-R/\xi_{\mathrm{YL}})$ for $R \gg \xi_{\mathrm{YL}}$, where ξ_{YL} is the correlation length.

As we quoted above, for any $T > 0$ there exists a strip $|H''| < H_0(T)$ inside of which the density of zeros vanishes. The limiting value $H_0(T)$ can be determined numerically, by study of the magnetization (3) and the correlation function (4). To calculate magnetization one has to iterate (2) together with the corresponding system of recursion relations for scaled derivatives t_i. If one starts with a sufficiently small value of an imaginary magnetic field $H = iH''$, such that H'' lies inside the strip of $H_0(T)$, one finds $\mathcal{M} = 0$ for large r. Then, by increasing H'' in small steps, one shall cut the edge H_0 of the strip, which will be characterized by a finite value of $\mathcal{M}$. To make better control of the onset of $\mathcal{M} \neq 0$, it is convenient to add a small real part H' to the magnetic field. During iterations the value of H' should be decreased gradually by approaching the edge more and more closely. Numerically, one can also note that the correlation length ξ_{YL} grows without bounds when approaching the limits of the strip.

Let us note that the system of recursion relations (2) has a number of different fixed points. Here we are looking for the YL fixed points only, i.e., those fixed points that can be reached using the above quoted initial conditions. Both numerical and analytical analyses reveal that, for $H' = 0$ and $H'' \to H_0$, z_2, z_3, and z_4 iterate toward the fixed point

$$z_2^* = -iy_0^{1/2}, \quad z_3^* = -y_0, \quad z_4^* = iy_0^{3/2}, \quad (5)$$

where $y_0 = \exp(-2ih_0)$ depends on the value of the temperature parameter $x(T)$; there exists also an equivalent fixed point with coordinates being complex conjugate of (5). In particular, for $x = 1/2$ we have found $h_0 = 0.384\,389\,080\ldots$. But, as it can be easily verified, recursion relations (2) are singular at this point (because $\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3$, and $\mathfrak{z}_4$ vanish for $z_2 = z_2^*, z_3 = z_3^*$, and $z_4 = z_4^*$), which does not allow us to make a common fixed-point analysis. This puzzle is quite similar to the one already encountered in our earlier studies of the YL edge problem for ferromagnetic Ising model on fractals. As detailed in Refs. [7,8], the basic step in a search for the asymptotic form of relevant variables near the edge is the observation that all iterates of these variables lie on an (invariant) manifold. The situation is analogous for recursion relations (1): As we first noticed numerically, and afterward corroborated analytically, one can introduce a variable, for example, $\delta z = z_3 - z_3^*$, which is small near the fixed point in such a way that the remaining two variables can be expressed as some suitable expansions over δz

$$\begin{aligned} z_2 &= z_2^* + c_1\delta z + c_2(\delta z)^2 + c_3(\delta z)^3 + \ldots, \\ z_4 &= z_4^* + d_1\delta z + d_2(\delta z)^2 + d_3(\delta z)^3 + \ldots, \end{aligned} \quad (6)$$

where c_i and d_i $(i = 1,2,3\ldots)$ are some complex parameters which depend on y_0. These parameters obey a set of algebraic equations which follow from the conditions that two successive iterates, i.e., $z_2, \delta z$ and $z_2', \delta z'$ as well as $z_4, \delta z$ and $z_4', \delta z'$, satisfy (6) with the *same values* of c_i and d_i. The existence of an invariant manifold, as well as the form of expansion (6) is not obvious. In fact, using the insights from the numerical study of recursion relations at the fixed point, we propose the ansatz (6), which will be justified *a posteriori*.

The expansion coefficients c_i and d_i can be calculated numerically for a given value of the interaction parameter $x(K)$ and the associated critical value y_0 of $y(x)$. As we noted above, they satisfy a set of polynomial equations, which allow us to find the first few coefficients explicitly: $c_1 = iy_0^{-1/2}/2$, $d_1 = -3iy_0^{1/2}/2$, $c_2 = ia_2 y^{-3/2}$, $d_2 = ib_2 y_0^{-1/2}$, $c_3 = ia_3 y^{-5/2}$, $d_3 = ib_3 y^{-3/2}$, where new coefficients a_2, b_2, a_3, b_3 are real and can be determined as suitable solutions of some simpler algebraic equations. For example, we have found that a_2 and b_2 satisfy the simple relation $b_2 = 3a_2$, and a_2 represents a solution of the equation

$$\begin{aligned} &262\,144\, a_2^6 + 102\,400\, a_2^4 + 10\,240\, a_2^3 \\ &+192\, a_2^2 - 96\, a_2 - 5 = 0. \end{aligned} \quad (7)$$

Since this equation has six solutions, we can select the proper one by making comparison with the value that follows from numerical iterations of (2) at the edge. In this way, we have identified the positive solution of (7), $a_2 = 0.085\,095\,609$, as the appropriate one. Higher-order coefficients are subject to rather cumbersome equations and they will not be given here. Nevertheless, these coefficients can be calculated with arbitrary numerical precision (we have calculated, for example, $a_3 = 0.028\,611\,062$, $b_3 = 0.110\,815\,562$), which can be regarded as a justification of the validity of the form (6) along the invariant manifold.

Having determined the form (6), we can use it to find the proper form of recursion relations, which is expected to be valid near the edge. Thus, taking into account that near the fixed point (5) variables z_2 and z_4 follow the asymptotic behavior (6), and making an expansion of the third recursion relation near the fixed point, we get a very simple recursion relation for the deviation δz

$$\delta z' = c\,\delta z + O(\delta z^2), \quad (8)$$

where c is a real constant which can be expressed in terms of first few expansion coefficients (numerically we have found

$c = 0.534\,172\,974$). As a consequence of this, $z_3^{(r)}$ approaches its fixed point value $z_3^* = -y_0$ in the following way: $z_3^{(r)} - z_3^* \sim c^r$. Now one can see that the two remaining variables z_2 and z_4 follow the same asymptotic behavior, $z_i^{(r)} - z_i^* \sim c^r$, $i = 2,4$. Then a simple analysis of (4) reveals that, for $h = ih_0$, the correlation function displays a similar behavior $\mathcal{G} \sim c^{2r}$ ("critical slowing down").

In fact, the quoted asymptotic relations also hold for a finite but sufficiently small values of $\delta h = h'' - h_0$, provided $r \lesssim r_0$, where $r_0 \gg 1$ is the number of iterations one can make along the invariant manifold before going away from it [starting with initial conditions in which $y = y_0 \exp(-2i\delta h)$]. Thus, the value of r_0 depends on δh [see formula (10)]. On the other hand, for $r > r_0$ correlation function decreases much faster: $\mathcal{G} \sim \kappa^{2^r} \equiv \exp(-2^r/\xi_{\rm YL})$, where parameter $\kappa < 1$ depends on δh and $\xi_{\rm YL} = -1/\ln[\kappa(\delta h)]$ denotes the YL correlation length. Then an asymptotic matching, $r \sim r_0$, leads to $\xi_{\rm YL} = 2^{r_0}$, as could be expected on the finite-size scaling grounds.

The number r_0 can be determined from the asymptotic behavior of scaled derivatives $t_i^{(r)} = (\partial Z_i^{(r)}/\partial h)/Z_1^{(r)}$. Indeed, for a small δh and $r < r_0$ we can write

$$z_i^{(r)}(\delta h) = \frac{Z_i^{(r)}(\delta h)}{Z_1^{(r)}(\delta h)} \approx \frac{z_i^{(r)}(0) + t_i^{(r)}\delta h}{1 + t_1^{(r)}\delta h} \approx z_i^* + \left(t_i^{(r)} - z_i^* t_1^{(r)}\right)\delta h. \tag{9}$$

On the other hand, our study of an exact system of recursion relations for scaled derivatives, $t_i^{(r)} = (\partial Z_i^{(r)}/\partial h)/Z_1^{(r)}$, reveals that, for $r < r_0$, their leading asymptotic behavior has the form $t_i^{(r)} \sim \lambda_t^r$, where $\lambda_t = 5.379\,953\,310$ plays the role of "thermal eigenvalue" (in the terminology of the renormalization group theory). Thus, r_0 can be estimated from the condition that the second (growing) term on the right-hand side of (9) should be of the order of the first one, $|t_i^{(r_0)} - z_i^* t_1^{(r_0)}||\delta h| \sim \lambda_t^{r_0}|\delta h| \sim |z_i^*|$, i.e.,

$$\lambda_t^{r_0}\delta h \sim O(1), \quad \text{or} \quad r_0 \sim |\ln(\delta h)|/\ln(\lambda_t). \tag{10}$$

This means that the asymptotic behavior of the YL correlation length has the power-law form $\xi_{\rm YL} \sim (\delta h)^{-\nu_{\rm YL}}$ with

$$\nu_{\rm YL} = \ln(2)/\ln(\lambda_t) = 0.411\,193. \tag{11}$$

Leading asymptotic behavior near the edge of the magnetization per site, and thus of the density of zeros $g \propto \mathrm{Re}(\mathcal{M})$, can be derived from (3) as follows: $g \sim \mathcal{N}^{-1} t_i^{(r_0)} \sim 3^{-r_0}\lambda_t^{r_0}$, which means that g diverges following the power law

$$g \sim (\delta h)^{\sigma}, \quad \sigma = \ln(3/\lambda_t)/\ln(\lambda_t) = -0.347\,105. \tag{12}$$

Let us note here that the above-quoted values of σ and $\nu_{\rm YL}$ coincide with corresponding exact values that have been obtained earlier using block-spin transformations [6]. The advantage of the approach we outlined in this section is, however, that it can be extended to the cases where the methods based on star-triangle transformation are seem to be inapplicable (in particular, to gaskets having spatial scaling factor $b > 2$).

B. $b = 3$ gasket

This case was studied recently in the context of a relation between YL ferromagnetic Ising model criticality and the criticality of the Gaussian model on nonhomogenous fractal structures [8]. For the sake of completeness, we quote here the main results. Thus, it has been shown that the YL correlation length near the edge diverges following a logarithmic law rather than the usual power-law behavior,

$$\xi_{\rm YL} \sim \ln^{\Phi}(\delta h), \quad \Phi = \ln(3/2)/\ln(2). \tag{13}$$

It has also been demonstrated that density of zeros near the edge displays very sharp divergence, which is modulated by the presence of a weaker logarithmic term,

$$g \sim |\ln(\delta h)|^{-\Psi}/\delta h, \quad \Psi = \ln(6)/\ln(2). \tag{14}$$

C. $b = 4$ gasket

To study YL critical behavior of the ferromagnetic Ising model on gaskets with spatial scaling factor $b > 3$ we shall use an extension of the approach we described above. Thus, as in the case $b = 2$, partition function of the model on an r-th order triangle can be written as a combination of four conditional partition functions defined in Sec. II A. To obtain recursion relations for these partition functions, one has to sum over $2^{(b-1)(b+4)/2}$ states of interior spin variables, keeping the states of three outer spins fixed. Since the number of relevant states increases very quickly with b, one has to use computer facilities to obtain requisite relations. Given that resulting recursion relations are rather cumbersome for $b > 3$, we will not present them here but they will be available on request. On the other hand, the initial conditions that we introduced in Sec. II A, as well as the general formulas (3) and (4) for the magnetization and correlation function, remain unchanged, except that $\mathcal{N}$ now denotes the number of sites of an r-th order triangle with spatial scaling factor b,

$$\mathcal{N} = [2^{r+1}(b+1) + (b+4)(b+1)^r b^r]/[2^r(b+2)]. \tag{15}$$

Since it turns out that the singular behavior near the edge of the Ising model on the $b = 4,5,6,7$ lattices is qualitatively the same, we shall give here some details of the analytical and numerical analyses for the simplest case, $b = 4$, and provide only the main results for $b > 4$.

As mentioned above, the resulting recursion relations for the restricted partition functions are rather cumbersome even in the $b = 4$ case, so they will be omitted here. Both the numerical and the analytical analyses show that, for a given value of $x = \exp(-4K)$, there exists a pair $\pm ih_0$ of purely imaginary values of H/k_BT such that $z_j^{(r)}$ iterates toward the YL critical point

$$z_2^* = y_0^{2/3}(1 - i\sqrt{3})/2, \quad z_3^* = -y_0^{4/3}(1 + i\sqrt{3})/2, \quad z_4^* = -y_0^2, \tag{16}$$

or to the second (equivalent) fixed point having values that are complex conjugate of (16). For example, we have found numerically that this fixed point can be reached for $x = 1/2$ and $h_0 = 0.224\,844\,628\ldots$. As in the case $b = 2$, we have found that both the numerator and the denominator of the corresponding recursion relations vanish at (16), which means that it represents a singular fixed point. Following the approach outlined in Sec. II A, one can show that this fixed point lies on an invariant manifold. Using the insights based on numerical studies, it seems that an expansion of type (6) still holds,

$$z_2 = z_2^* + e_1\delta z + e_2(\delta z)^2 + e_3(\delta z)^3 + \dots,$$
$$z_3 = z_4^* + f_1\delta z + f_2(\delta z)^2 + f_3(\delta z)^3 + \dots, \tag{17}$$

where $\delta z = z_4 + y_0^2$ is small near the critical point and parameters e_i, f_i can be determined from the condition that "renormalized" variables z_2' and z_3' have the same expansion (17) over $\delta z' = z_4' + y_0^2$. Using these conditions and recursion relations one can, in principle, find requisite algebraic equations for expansion coefficients. To handle tedious algebraic manipulations we used facilities provided by the known MATHEMATICA package. For example, we have found that the coefficient e_1 should satisfy a cumbersome quadratic equation which we give in the appendix [see Eq. (A1)]. It is interesting, however, that the rather messy solution of this equation can be cast into the very simple form: $e_1 = -y_0^{-4/3}(1 - i\sqrt{3})/6$. Once we found e_1, the expansion coefficient f_1 can be determined from Eq. (A2) of the appendix, $f_1 = y_0^{-2/3}(1 + i\sqrt{3})/3$. In particular, for the above-quoted values $x = 1/2$ and $y_0 = y(x,h_0) = 0.836\,861\,304 - 0.547\,414\,977\,i$, one finds $e_1 = -0.320\,823\,359 + 0.090\,462\,607\,i$ and $f_1 = 0.091\,323\,506 + 0.660\,382\,057\,i$, which is in excellent agreement with the values obtained by direct study of pertinent recursion relations. In a similar way, after a number of algebraic transformations, we get simple expressions for e_2 and f_2: $e_2 = 5y_0^{-10/3}(1 - i\sqrt{3})/72$ and $f_2 = -5y_0^{-8/3}(1 + i\sqrt{3})/72$. For the above-quoted value of y_0, numerical values of e_2 and f_2 are in agreement with those obtained from iterations of recursion relations. We have not been able to find some simple closed forms for the higher-order expansion coefficient, and so we report here only the numerical values $e_3 = 0.422\,859\,654 + 0.380\,131\,181\,i$ and $f_3 = 0.307\,884\,985 - 0.455\,064\,135\,i$, which are determined from pertinent algebraic constraints and the above quoted value of y_0. Anyway, the presented analytical and numerical findings can be regarded as a justification of the expansion form (17).

Having specified the form (17), we can use it to find the proper form of recursion relations, which is expected to be valid near the edge. In particular, taking into account that near the fixed point (16) variables z_2 and z_3 follow the asymptotic behavior (17), and making an expansion of recursion relation $z_4' = z_4'(z_2,z_3,z_4)$ near this fixed point, we get an exact asymptotic recursion relation for $\delta z = z_4 - z_4^*$,

$$\delta z' = \tfrac{2}{5}\delta z + O(\delta z^2). \tag{18}$$

As a consequence of this, along the invariant manifold one finds $z_i^{(r)} - z_i^* \sim (2/5)^r$, $i = 2,3,4$, implying that the correlation function (4) for $h = i\,h_0(x)$ follows the similar behavior, $\mathcal{G} \sim (2/5)^{2r}$. In fact, the same asymptotic behavior holds for a finite but small $\delta h = h'' - h_0$, provided $r \lesssim r_0$, where r_0 depends on δh. It turns out that this correlation function decreases much faster for $r > r_0$, $\mathcal{G} \sim \kappa^{4^r} = \exp(-4^r/\xi_{\rm YL})$, where $\xi_{\rm YL} = -1/\ln[\kappa(\delta h)]$ is the YL correlation length. These two regions can be connected by an asymptotic matching, which yields the estimates $\xi_{\rm YL} \sim 4^{r_0}$. To express this asymptotic relation in terms of δh, we use the approach that we outlined in Sec. II A. Thus, we also studied an exact system of recursion relations for the scaled derivatives $t_i^{(r)} = (\partial Z_i^{(r)}/\partial h)/Z_1^{(r)}$. It is convenient to express these recursion relations in a matrix form, with matrix elements being some function of $z_i^{(r)}$. Using the above results for the asymptotic behavior of $z_i^{(r)}$ along the invariant manifold, one can show that these matrix elements grow as $\sim(5/2)^{5r}$. A condition of self-consistency for the asymptotic behavior of $t_i^{(r)}$ then implies that its leading term should have the form $(5/2)^{r^2}$. This allows us to write

$$t_i^{(r)} \sim (5/2)^{r^2}\lambda_1^r, \quad r < r_0 \gg 1, \quad i = 1,2,3,4, \tag{19}$$

where we also indicated the form of the next-to-leading term in the asymptotic form of $t_i^{(r)}$. We have not been able to extract the exact value of λ_1, but we calculated it numerically. To estimate $r_0 = r_0(\delta h)$ we can still use formula (9) and arguments presented after it. Thus, taking only dominant term of $t_i^{(r)}$, we get

$$(5/2)^{r_0^2}\delta h \sim O(1). \tag{20}$$

This result, together with the above estimate $\xi_{\rm YL} \sim 4^{r_0}$, reveals that the YL correlation length increases as $\delta h \to 0$ more slowly than any power law but faster than (13)

$$\xi_{\rm YL} \sim \exp\left[\ln(4)\sqrt{|\ln(\delta h)|/\ln(5/2)}\right]. \tag{21}$$

Let us note that the same leading asymptotic behavior follows the correlation length of a simple Gaussian model on the same structure [12].

To corroborate our analytical findings, and to provide some further insight into the critical behavior of the model, we also studied it numerically. First, we examined behavior of scaled derivatives $t_i^{(r)}$ as a function of the iteration index r ($r < r_0$). In particular, we focused our attention on the ratio $\delta(r) = \ln(t_4^{(r+1)}/t_4^{(r)})/2r$, which should have the form $\delta(r) = \ln(5/2) + \text{const}/r$ for $r \gg 1$, providing (19) holds. To test it, we iterated the exact system of the recursion relation $t_i^{(r)}$ using very large numerical precision. In this way, we have obtained, for example, $\delta(50) = 0.921\,042\,871\,463\,727$, $\delta(51) = 0.920\,949\,692\,256\,088$, $\delta(52) = 0.920\,860\,096\,864\,128$, which is not far from the theoretical value $\ln(5/2) = 0.916\,290\,731\,874\,2$. In order to extrapolate the finite-r values of $\delta(r)$ to $r \to \infty$, we use the simple sequential fit, $\delta(r) = \delta(\infty) + A/r^\alpha$, where parameters $\delta(\infty)$, α, and A can be estimated from any three consecutive values of $\delta(r)$. In particular, for the quoted values of $\delta(r)$ one finds $\delta(\infty) = 0.916\,290\,731\,873\,6$ which differs from $\ln(5/2)$ just on the 12th significant digit! For the same values of $\delta(r)$ one gets $\alpha = 0.999\,999\,999\,8$, which also provides clear numerical support of validity of the asymptotic form (19). As in the case of Gaussian model [12], the subdominant term λ_1^r of this form is a decreasing function of r, $\lambda_1 = 0.643\,343$. Although this value differs from the corresponding value for the Gaussian model [12], note that this entails a change in the form of corrections to scaling terms only.

We have also studied the correlation function (4) and its correlation length near the edge numerically. The overall picture of the logarithm of the correlation length as a function

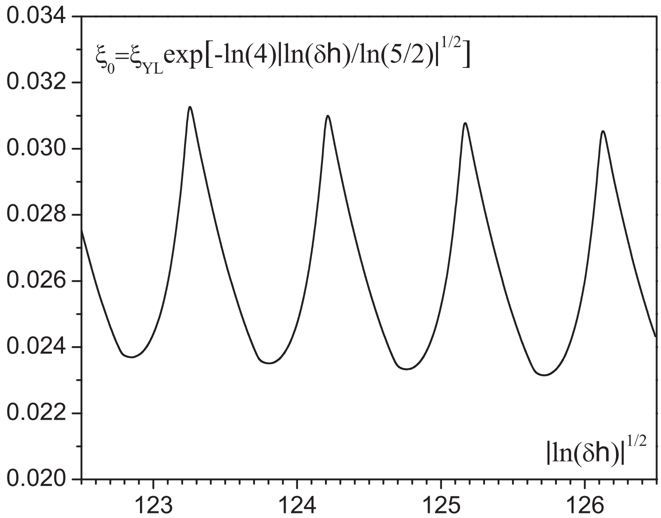

FIG. 2. Scaled correlation length (critical amplitude) $\xi_0 = \xi_{YL}/\exp[\ln(4)|\ln(\delta h)/\ln(5/2)|^{1/2}]$ as a function of $|\ln(\delta h)|^{1/2}$. The correlation length ξ_{YL} is computed using formula (4) and an exact system of recursion relations. Our numerical estimate of the period τ ($\tau \approx 0.9572$) is in very good agreement with the theoretical prediction $\tau = \sqrt{\ln(5/2)}$.

of $|\ln(\delta h)|^{1/2}$ looks like a straight line with superimposed oscillations. A similar sort of some small oscillations have been noticed earlier in many different situations, especially in the context of fractal self-similarity or discrete scale invariance (see, e.g., Ref. [13]). In Fig. 2 we presented oscillatory behavior of the correlation length critical amplitude ξ_0, i.e., scaled correlation length $\xi_0 = \xi_{YL}/\exp[\ln(4)|\ln(\delta h)/\ln(5/2)|^{1/2}]$ as a function of $|\ln(\delta h)|^{1/2}$. According to the theory of log-periodic corrections to power-law scaling [13], period τ of these oscillations is determined by "thermal" eigenvalue λ_t, $\tau = \ln(\lambda_t)$. If we rewrite the asymptotic law (20) in the form (10), $r_0 \sim |\ln(\delta h)|^{1/2}/\ln(5/2)^{1/2}$, we conclude that ξ_0 should be a periodic function in $|\ln(\delta h)|^{1/2}$ with period $\tau = \sqrt{\ln(5/2)}$. These findings are in excellent agreement with numerical data presented in Fig. 2. It is also interesting to note that a very small change of the exact values of parameters appearing in (21) can destroy the steady behavior of ξ_0 presented in this figure.

The singular behavior of the density of zeros g near the edge can be deduced from formula (3). Indeed, keeping only dominant terms in this relation, we get $g \sim t_i^{(r_0)}/\mathcal{N} \sim (5/2)^{r_0^2}\, 10^{-r_0}$, where $\mathcal{N} \sim 10^{r_0}$ describes the asymptotic site number growth of an r_0-th order triangle [see formula (15) for $b=4$]. This, together with asymptotic relation (20), reveals that density of zeros diverges near the edge in the following way:

$$g \sim \frac{\exp[-\ln(10)\sqrt{|\ln(\delta h)|/\ln(5/2)}]}{\delta h}. \qquad (22)$$

D. $b > 4$ gaskets

Yang-Lee edge singularity for the ferromagnetic Ising model on fractals with $b > 4$ can be, in principle, studied in a similar way. Since the corresponding system of recursion relations is very large in these cases, their analytical analysis would be a very involved procedure. For this reason, for the $b = 5,6,7$ cases we shall give numerical results only. Note, however, that the presented results are obtained from a very precise numerical analysis of the corresponding exact systems of recursion relations and that they are accurate to (at least) seven significant digits.

TABLE I. Values of critical magnetic fields $h = ih_0$ calculated for the fixed value $x = 1/2$ of the interaction strength $x = \exp(-4K)$. The presented values of growth constants λ_0 and λ_1, $b > 4$, are also calculated for these specific values of h_0 and x; it is expected, however, that they remain unchanged along the critical line $h_0 = h_0(x)$.

b	h_0	λ_0	λ_1
4	0.289 636 078	5/2	$6.433\,433 \times 10^{-1}$
5	0.262 841 748	2.005 661 588	$1.626\,075 \times 10$
6	0.241 967 570	1.846 528 451	$1.099\,184 \times 10^2$
7	0.224 844 628	1.771 693 310	$4.579\,340 \times 10^2$

As this analysis reveals, in all these cases the YL singular behavior is qualitatively similar to the $b = 4$ example. In particular, we have found that YL fixed points have an unchanged form (16), while the asymptotic form of the scaled derivatives $t_i^{(r)}$, $r < r_0$, is quite similar to (19),

$$t_i^{(r)} \sim \lambda_0^{r^2}\lambda_1^{r}, \quad \text{and, therefore,} \quad \lambda_0^{r_0^2}\lambda_1^{r_0}\delta h \sim O(1), \qquad (23)$$

where the values of the growth constants λ_0 and λ_1 are given in Table I. An analysis of asymptotic behavior of the correlation function (4) at criticality $h = ih_0(x)$ shows that its correlation length follows the expected behavior $\xi_{YL} \sim b^r$, yielding the general asymptotic behavior

$$\xi_{YL} \sim \exp[\ln(b)\sqrt{|\ln(\delta h)|/\ln(\lambda_0)}], \quad b > 3. \qquad (24)$$

In the same spirit, the singular behavior of the density of zeros near the edge can be expressed in terms of b and λ_0 only,

$$g \sim \frac{\exp[-\ln[b(b+1)/2]\sqrt{|\ln(\delta h)|/\ln(\lambda_0)}]}{\delta h}. \qquad (25)$$

Although the growth constants λ_1 have no influence on the leading asymptotic behavior, we believe that they should be useful for a more precise description of the correlation length, especially for larger values of b. Indeed, one can notice a general trend that λ_0 slowly decreases while λ_1 rather quickly increases as a function of b (see Table I). Then there exists a range of values δh over which the second term of (23) is, in fact, dominant [i.e., $\lambda_1^{r_0}\delta h \sim O(1)$, instead of $\lambda_0^{r_0^2}\delta h \sim O(1)$], which entails a power-law behavior of the correlation length. Though such a power-law will, after all, cross to the true asymptotic law (24) for sufficiently small δh, this example points out the necessity of using much care when interpreting numerical results on finite lattices.

III. CONCLUSION

We studied the YL edge singularity problem for the ferromagnetic Ising model on a two-dimensional class of Sierpinski fractal lattices by using exact complex-valued recursion relations. These relations are singular at pertinent YL fixed points, preventing us from making a common fixed-point analysis. Instead, we applied an asymptotic matching analysis to explore singular behavior of the correlation length and density of zeros near the edge. Exact results, based on an asymptotic expansion of relevant variables along the invariant manifolds, are presented for lattices having $b = 2$ and $b = 4$ and numerically exact results for lattices with $4 < b < 8$.

We have shown that the YL correlation length for $b > 3$ grows more slowly than power law, $\xi_{YL} \sim \exp[\ln(b)\sqrt{|\ln(\delta h)|/\ln(\lambda_0)}]$, while the power-law divergence of the density of zeros near the edge is modulated by the presence of a weaker multiplicative term (25). This singular behavior is qualitatively similar to the one found earlier for a zero-field Gaussian model (corresponding to the statistics of equally weighted random paths) on the same class of lattices. In fact, we have demonstrated that these two models share the same class of universality in the $b = 3$ and $b = 4$ cases. The same coincidence has been noticed earlier—in the case of models situated on fractals with nonuniform coordination number. In this sense, our results for $b > 4$ present some exceptions from earlier findings. It is well known, however, that these two models on homogenous lattices (and even fractal lattices having uniform coordination number) belong to different universality classes, and it is a real surprise that they can share the same universality class in some cases; further work is needed in order to clarify these points.

We presented here results for only the first seven members of the lattice family. It is tempting to speculate what can happen for larger values of the spatial scaling factor b. For large b the fraction $(b-2)/(b+4)$ of lattice sites having coordination number 6 tends to 1, and one can expect that in this limit fractal lattices mimic increasingly a compact triangular lattice. Since the YL singularity in the latter case has the power-law form, it seems difficult to recover this form if general formula (23) is valid for every $b > 3$. Note, however, that the growth constant λ_0 decreases with b. A rough estimate of $\lambda_0(\infty)$, based on data presented in Table I, indicates that $\lambda_0(b)$ should be close to 1 for large b. This could provide a sign of the inapplicability of (21) in the limit $b \to \infty$. Under same assumptions, the asymptotic form (23) crosses to a form of type (10), which entails a power-law behavior.

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APPENDIX

To indicate the form of the algebraic constraints imposed on the expansion coefficients, we give here two equations which determine parameters e_1 and f_1 entering asymptotic formulas (17). They are expressed in terms of y and the fixed point of (16), using the approach described in the main text. Although these equations are rather cumbersome, one can verify that they allow the simple solutions $e_1 = -y^{-4/3}(1 - i\sqrt{3})/6$ and $f_1 = y^{-2/3}(1 + i\sqrt{3})/3$, where y denotes a point on the critical line $y_0 = y(x, h_0)$. Direct numerical analysis of recursion relations near the edge, performed for $x = 1/2$ and associated value of $y = y_0$, corroborates these analytical findings.

$$
\begin{aligned}
&52y + 29\,900y^2 + 1\,315\,600y^3 + 1\,0623\,470y^4 + 20\,801\,200y^5 + 10\,623\,470y^6 + 1\,315\,600y^7 + 29\,900y^8 + 52y^9\\
&\quad - (2 + 5\,200y + 460\,460y^2 + 6\,249\,100y^3 + 19\,315\,400y^4 + 15\,452\,320y^5 + 3\,124\,550y^6 + 131\,560y^7 + 650y^8)z_2^*\\
&\quad + (650 + 131\,560y + 3\,124\,550y^2 + 15\,452\,320y^3 + 19\,315\,400y^4 + 6\,249\,100y^5 + 460\,460y^6 + 5\,200y^7 + 2y^8)z_2^{*2}\\
&\quad + 3y^2e_1[23 + 7150y - 32\,890y^2 - 4\,061\,915y^3 - 18\,572\,500y^4 - 17\,866\,745y^5 - 4\,029\,025y^6 - 182\,390y^7 - 949y^8\\
&\quad + 3yz_2^*(1 + 1\,625y + 32\,890y^2 - 1\,562\,275y^3 - 13\,520\,780y^4 - 21\,246\,940y^5 - 7\,811\,375y^6 - 624\,910y^7 - 7\,475y^8\\
&\quad - 3y^9) - 3yz_2^{*2}(247 + 20\,930y - 411\,125y^2 - 8\,209\,045y^3 - 21\,544\,100y^4 - 12\,810\,655y^5 - 1\,743\,170y^6 - 42\,250y^7\\
&\quad - 77y^8)] + 9y^2e_1^2\big[y(y-1)(1 + 2\,276y + 166\,726y^2 + 1\,729\,001y^3 + 3\,660\,541y^4 + 1\,729\,001y^5 + 166\,726y^6\\
&\quad + 2\,276y^7 + y^8) + y(299 + 50\,830y + 904\,475y^2 + 2\,414\,425y^3 - 742\,900y^4 - 2\,187\,185y^5 - 427\,570y^6\\
&\quad - 12\,350y^7 - 25y^8)z_2^* - (25 + 12\,350y + 427\,570y^2 + 2\,187\,185y^3 + 742\,900y^4 - 2\,414\,425y^5\\
&\quad - 904\,475y^6 - 50\,830y^7 - 299y^8)z_2^{*2}\big] = 0.
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
&y(1+y)(1+83y+y^2)\big(1 + 3z_2^{*2}e_1 - 3z_2^*f_1\big) + 9y(4+14y+y^2)\big(z_2^* + 3z_2^{*2}f_1 + 3e_1y^2\big)\\
&\quad + 9(1+14y+4y^2)\big(z_2^{*2} - 3z_2^*e_1y^2 + 3f_1y^2\big) = 0.
\end{aligned}
\tag{A2}
$$

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