

# Hierarchical maximum entropy principle for generalized superstatistical systems and Bose-Einstein condensation of light

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A principle of hierarchical entropy maximization is proposed for generalized superstatistical systems, which are characterized by the existence of three levels of dynamics. If a generalized superstatistical system comprises a set of superstatistical subsystems, each made up of a set of cells, then the Boltzmann-Gibbs-Shannon entropy should be maximized first for each cell, second for each subsystem, and finally for the whole system. Hierarchical entropy maximization naturally reflects the sufficient time-scale separation between different dynamical levels and allows one to find the distribution of both the intensive parameter and the control parameter for the corresponding superstatistics. The hierarchical maximum entropy principle is applied to fluctuations of the photon Bose-Einstein condensate in a dye microcavity. This principle provides an alternative to the master equation approach recently applied to this problem. The possibility of constructing generalized superstatistics based on a statistics different from the Boltzmann-Gibbs statistics is pointed out.

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## I. INTRODUCTION

Superstatistics represents a statistics of canonical statistics and allows one to consider stationary states of nonequilibrium systems with fluctuations of an intensive parameter  $\beta$  [1]. Though usually considered as an inverse temperature,  $\beta$  can be interpreted in a more general way [2,3]. A superstatistical system comprises a set of subsystems, or cells, each having the Gibbs canonical distribution determined by  $\beta$ . An essential feature of the superstatistical system is sufficient spatiotemporal scale separation, so that  $\beta$  fluctuates on a much larger time scale than the typical relaxation time of the local dynamics in a cell. Superstatistics can be given a basis by the theory of hyperensembles [4,5].

The distribution of  $\beta$  can be considered as a function of some additional control parameters [6]. However, in ordinary superstatistics, the intensive parameter fluctuates, but the control parameters are constant. Considering the control parameter fluctuations has led very recently to the generalization of superstatistics—"statistics of superstatistics," or "generalized superstatistics" [7]. Generalized superstatistics is the statistics of generalized superstatistical systems. A generalized superstatistical system comprises a set of nonequilibrium superstatistical subsystems and can be associated with a generalized hyperensemble, an ensemble of hyperensembles. Compared with an ordinary superstatistical system, a generalized superstatistical system is characterized by the existence of the third, upper level of dynamics in addition to the two levels of dynamics existing in each superstatistical subsystem. This is reflected in the existence of a fluctuating vector control parameter on which both the intensive parameter distribution and the density of energy states depend. Significantly, generalized superstatistics can be used for nonstationary nonequilibrium systems. It was applied to branching processes and pair production in a neutron star magnetosphere [7].

The main problem of generalized superstatistics is the determination of the intensive parameter distribution, characterizing the superstatistical dynamics in each subsystem, and the control parameter distribution, characterizing the dynamics of the system as a whole. The aim of this paper is to develop the maximum entropy principle that can be used to solve the above problem.

The paper is organized as follows: In Sec. II the hierarchical maximum entropy principle for generalized superstatistical systems is formulated and the canonical, intensive parameter, and control parameter distributions are consecutively determined. In Sec. III this principle is applied to Bose-Einstein condensation of light and fluctuations of the number of ground-mode photons are considered. In Sec. IV the main conclusions are given.

## II. HIERARCHICAL MAXIMUM ENTROPY

A generalized superstatistical system is conveniently thought of as a set of superstatistical subsystems, each in turn made up of a set of cells. There are three levels of dynamics in this system: the first, lower level of fast dynamics in a cell, the second, middle level of superstatistical dynamics in a subsystem, and the third, upper level of global dynamics in the whole system. The levels are arranged in increasing order of dynamical time scale so that the shortest time scale corresponds to the lower level. The local dynamics in a cell is characterized by an energy  $E$ , the superstatistical dynamics in a subsystem is characterized by an intensive parameter  $\beta$ , and the global dynamics in the whole system is characterized by a control parameter  $\xi$ , which may be a multidimensional vector.

The system hierarchy is formed as a result of the sufficient time-scale separation between different levels of dynamics. This allows us to formulate the maximum entropy principle for the generalized superstatistical system as a principle of hierarchical entropy maximization. More specifically, the entropy should be maximized first for each cell, second for each subsystem, and finally for the whole system.

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### A. Local dynamics

Though the existence of the Gibbs canonical distribution at the lower dynamical level is postulated in superstatistics, it is reasonable to explicitly obtain this distribution from the maximum entropy principle. This trivial derivation will allow us to readily observe an analogy between the dynamics at different hierarchical levels of a generalized superstatistical system.

Choose a superstatistical subsystem of the generalized superstatistical system. A fixed value of the control parameter  $\xi$  corresponds to this subsystem, but the intensive parameter  $\beta$  may still fluctuate. Choosing the subsystem also fixes the density of energy states:

$$g(E|\xi) = \frac{\partial \Gamma(E|\xi)}{\partial E}, \quad (1)$$

where  $\Gamma(E|\xi)$  is the number of states with energy less than  $E$ . In integrals with  $d\Gamma(E|\xi)$ , integration over  $E$  will be performed,  $d\Gamma(E|\xi) = g(E|\xi)dE$ .

To consider the local dynamics, choose a cell of the subsystem. Then  $\beta$  also becomes fixed, but the energy  $E$  is not fixed and is characterized by a probability distribution  $\rho(E|\beta, \xi)$ . To find the distribution maximizing the Boltzmann-Gibbs-Shannon entropy

$$S[E](\beta|\xi) = - \int \rho(E|\beta, \xi) \ln \rho(E|\beta, \xi) d\Gamma(E|\xi)$$

under the normalization condition  $N[E](\beta|\xi) = 1$  and the mean energy constraint  $U[E](\beta|\xi) = U(\beta|\xi)$ , where

$$N[E](\beta|\xi) = \int \rho(E|\beta, \xi) d\Gamma(E|\xi),$$

$$U[E](\beta|\xi) = \int E \rho(E|\beta, \xi) d\Gamma(E|\xi),$$

we should consider the condition of zero variation,  $\delta L_1 = 0$ , for the Lagrange function

$$L_1(v_1, \beta, \xi) = S[E](\beta|\xi) - (v_1 - 1)N[E](\beta|\xi) - \beta U[E](\beta|\xi).$$

Then we arrive at the Gibbs canonical distribution

$$\rho_G(E|\beta, \xi) = \frac{e^{-\beta E}}{Z(\beta|\xi)},$$

where

$$Z(\beta|\xi) = \int e^{-\beta E} d\Gamma(E|\xi) \quad (2)$$

is the partition function. The entropy is

$$S[E](\beta|\xi) = v_1(\beta|\xi) + \beta U(\beta|\xi), \quad (3)$$

where the mean energy

$$U(\beta|\xi) = - \frac{\partial v_1(\beta|\xi)}{\partial \beta} \quad (4)$$

is expressed via the Massieu function

$$v_1(\beta|\xi) = \ln Z(\beta|\xi). \quad (5)$$

### B. Superstatistical dynamics

Now consider the superstatistical dynamics of the chosen subsystem. This dynamics is characterized by the fluctuating intensive parameter  $\beta$  that determines the properties of cells of the subsystem. To find the intensive parameter distribution  $f(\beta|\xi)$ , we should maximize the entropy of the joint probability distribution of  $E$  and  $\beta$ , given  $\xi$ . It is written as [5,8]

$$S[E, \beta](\xi) = S[\beta](\xi) + \int S[E](\beta|\xi) f(\beta|\xi) d\beta, \quad (6)$$

where

$$S[\beta](\xi) = - \int f(\beta|\xi) \ln f(\beta|\xi) d\beta \quad (7)$$

is the entropy associated with  $f(\beta|\xi)$ , and  $S[E](\beta|\xi)$  is given by Eq. (3). The normalization condition for  $f(\beta|\xi)$  is  $N[\beta](\xi) = 1$ , where

$$N[\beta](\xi) = \int f(\beta|\xi) d\beta.$$

In addition, we may impose a set of  $n$  constraints given by an  $n$ -dimensional vector equality

$$M[\beta](\xi) = M(\xi), \quad (8)$$

where

$$M[\beta](\xi) = \int m(\beta|\xi) f(\beta|\xi) d\beta, \quad (9)$$

and  $m(\beta|\xi) = [m_1(\beta|\xi), \dots, m_n(\beta|\xi)]$  and  $M(\xi) = [M_1(\xi), \dots, M_n(\xi)]$  are  $n$ -dimensional vectors specifying, respectively, the form and values of the constraints. Each  $M_i(\xi)$  is the mean of  $m_i(\beta|\xi)$  over the fluctuating  $\beta$ , given  $\xi$ . We consider  $M[\beta](\xi)$  as some general constraint vector, but it may be composed of the constraints used in ordinary superstatistics, e.g., the mean values of energy, entropy, square of entropy, energy divided by temperature, or logarithm of the partition function [4,6,9,10].

Also define an  $n$ -dimensional vector Lagrange multiplier  $\mu = (\mu_1, \dots, \mu_n)$ , where each  $\mu_i$  is the Lagrange multiplier corresponding to the constraint  $M_i[\beta](\xi) = M_i(\xi)$ . We then have the following Lagrange function:

$$L_2(v_2, \mu, \xi) = S[E, \beta](\xi) - (v_2 - 1)N[\beta](\xi) - \mu \cdot M[\beta](\xi).$$

By  $a \cdot b = \sum a_i b_i$ , we denote the scalar product of some vectors  $a$  and  $b$ . The condition  $\delta L_2 = 0$  yields the intensive parameter distribution

$$\tilde{f}(\beta|\mu, \xi) = \frac{Z(\beta|\xi)}{\tilde{Y}(\mu, \xi)} \exp[-\mu \cdot m(\beta|\xi) + \beta U(\beta|\xi)], \quad (10)$$

where the partition function

$$\tilde{Y}(\mu, \xi) = \int Z(\beta|\xi) \exp[-\mu \cdot m(\beta|\xi) + \beta U(\beta|\xi)] d\beta \quad (11)$$

is determined from the normalization condition for  $\tilde{f}(\beta|\mu, \xi)$ .

Note that  $\tilde{f}(\beta|\mu, \xi)$  and  $\tilde{Y}(\mu, \xi)$  still depend on the Lagrange multiplier  $\mu$ . The implicit dependence of  $\mu$  on the control parameter  $\xi$ ,

$$\mu = \mu(\xi), \quad (12)$$

is determined from

$$M(\xi) = -\frac{\partial \tilde{v}_2(\mu, \xi)}{\partial \mu}, \quad (13)$$

where

$$\tilde{v}_2(\mu, \xi) = \ln \tilde{Y}(\mu, \xi) \quad (14)$$

is the Massieu function, and  $\partial/\partial\mu = (\partial/\partial\mu_1, \dots, \partial/\partial\mu_n)$  is the  $n$ -dimensional gradient operator. Equations (13) and (14) are analogous to Eqs. (4) and (5), respectively. Thus, given the constraints (8), the intensive parameter distribution (10), partition function (11), and Massieu function (14) depend only on  $\beta$  and  $\xi$ :

$$f(\beta|\xi) = \tilde{f}(\beta|\mu(\xi), \xi), \quad (15)$$

$$Y(\xi) = \tilde{Y}(\mu(\xi), \xi), \quad v_2(\xi) = \tilde{v}_2(\mu(\xi), \xi). \quad (16)$$

We may either first set the constraint vector  $M(\xi)$  and then find  $\mu(\xi)$  from the maximum entropy principle, or vice versa. This is in full analogy with the case of the dynamics in a cell, when we may first set the mean energy  $U(\beta)$  and then find the corresponding intensive parameter  $\beta$ , or set  $\beta$  and then find  $U(\beta)$ , which is more common. Incidentally, this duality allows one to alternatively formulate superstatistics by introducing the fluctuations of  $U(\beta)$  instead of those of  $\beta$  [11]. Note that the control parameter  $\xi$  has a more general nature than  $\beta$ , since  $\beta$  is exactly a Lagrange multiplier, while  $\xi$ , though controlling the Lagrange multiplier  $\mu$ , may not coincide with  $\mu$ . The analogy between  $\beta$  and  $\xi$  will be complete if we choose  $\mu(\xi) = \xi$ .

It follows from Eqs. (3), (5)–(10), (12), and (14)–(16) that the entropy associated with the superstatistical subsystem is

$$S[E, \beta](\xi) = v_2(\xi) + \mu(\xi) \cdot M(\xi). \quad (17)$$

It is analogous to Eq. (3).

Thus, the intensive parameter distribution for the superstatistical subsystem is given by Eq. (15). The superstatistical distribution

$$\rho(E|\xi) = \int \rho_G(E|\beta, \xi) f(\beta|\xi) d\beta$$

has the form

$$\rho(E|\xi) = \frac{1}{Y(\xi)} \int \exp\{-\beta[E - U(\beta|\xi)] - \mu(\xi) \cdot m(\beta|\xi)\} d\beta, \quad (18)$$

with the normalization condition  $\int \rho(E|\xi) d\Gamma(E|\xi) = 1$ .

Ordinary superstatistics is a special case of generalized superstatistics: an ordinary superstatistical system is a generalized superstatistical system without fluctuations of the control parameter  $\xi$ . Therefore, we can easily obtain the intensive parameter distribution  $f = f(\beta|\mu)$  for this system by formally removing  $\xi$  from Eq. (10) and from subsidiary Eqs. (1), (2), (4), (5), (8), (9), (11), (13), and (14). It is consistent with the distributions obtained earlier [6,8,10].

### C. Global dynamics

Consider the third level of dynamics. We should find the probability distribution  $c(\xi)$  of the fluctuating control

parameter  $\xi$ . This distribution is normalized,  $N[\xi] = 1$ , where

$$N[\xi] = \int c(\xi) d\xi.$$

The entropy of the joint probability distribution of  $E$ ,  $\beta$ , and  $\xi$  is determined by analogy with the entropy associated with a superstatistical subsystem [cf. Eq. (6)]:

$$S[E, \beta, \xi] = S[\xi] + \int S[E, \beta](\xi) c(\xi) d\xi, \quad (19)$$

where

$$S[\xi] = - \int c(\xi) \ln c(\xi) d\xi \quad (20)$$

is the entropy associated with the control parameter distribution  $c(\xi)$ , and  $S[E, \beta](\xi)$  is given by Eq. (17). We may impose a set of  $m$  additional constraints by analogy with Eqs. (8) and (9):

$$K[\xi] = K, \quad (21)$$

where

$$K[\xi] = \int k(\xi) c(\xi) d\xi,$$

and  $k(\xi) = [k_1(\xi), \dots, k_m(\xi)]$  and  $K = (K_1, \dots, K_m)$  are  $m$ -dimensional vectors specifying, respectively, the form and values of the constraints. Each  $K_i$  is the mean of  $k_i(\xi)$  over the fluctuating  $\xi$ .

The Lagrange function is

$$L_3(v_3, \kappa) = S[E, \beta, \xi] - (v_3 - 1)N[\xi] - \kappa \cdot K[\xi],$$

where we have defined an  $m$ -dimensional vector Lagrange multiplier  $\kappa = (\kappa_1, \dots, \kappa_m)$ , where each  $\kappa_i$  is the Lagrange multiplier corresponding to the constraint  $K_i[\xi] = K_i$ . The condition  $\delta L_3 = 0$  yields the control parameter distribution

$$c(\xi, \kappa) = \frac{Y(\xi)}{X(\kappa)} \exp[-\kappa \cdot k(\xi) + \mu(\xi) \cdot M(\xi)], \quad (22)$$

where the partition function is

$$X(\kappa) = \int Y(\xi) \exp[-\kappa \cdot k(\xi) + \mu(\xi) \cdot M(\xi)] d\xi,$$

and  $Y(\xi)$  is defined by Eq. (16). By analogy with Eq. (13), we can rewrite the constraints (21) as follows:

$$K = -\frac{\partial v_3(\kappa)}{\partial \kappa}, \quad (23)$$

where

$$v_3(\kappa) = \ln X(\kappa)$$

is the Massieu function, and  $\partial/\partial\kappa = (\partial/\partial\kappa_1, \dots, \partial/\partial\kappa_m)$  is the  $m$ -dimensional gradient operator. It remains to find the entropy (19) at the maximum point [cf. Eqs. (3) and (17)]:

$$S[E, \beta, \xi] = v_3(\kappa) + \kappa \cdot K.$$

Thus, the intensive parameter distribution  $c(\xi) \equiv c(\xi, \kappa)$  is given by Eq. (22), with the Lagrange multiplier  $\kappa$  determined from Eq. (23). By Eqs. (18) and (22), we get that the generalized superstatistical distribution

$$\sigma(E) = \int \rho(E|\xi) g(E|\xi) c(\xi) d\xi$$

has the form

$$\sigma(E) = \frac{1}{X(\kappa)} \int \exp\{-\beta[E - U(\beta|\xi)] - \mu(\xi) \cdot [m(\beta|\xi) - M(\xi)] - \kappa \cdot k(\xi)\} g(E|\xi) d\beta d\xi,$$

with the normalization condition  $\int \sigma(E) dE = 1$ .

### III. BOSE-EINSTEIN CONDENSATION OF LIGHT

Recently, thermalization of light in a dye microcavity has been observed [12]. In this experiment, photons are confined in a curved-mirror optical microresonator filled with a dye solution. In the microresonator, absorption and reemission of photons by dye molecules results in thermalization of the photon gas. Since the free spectral range of the microresonator is comparable to the spectral width of the dye, the emission of photons with a fixed longitudinal number dominates. Therefore, the photon gas is effectively two dimensional, and thermalization of transverse photon states occurs. Moreover, Bose-Einstein condensation (BEC) of light has been experimentally observed in the described system [13,14]. This reflects the fact that a two-dimensional harmonically trapped ideal gas of massive bosons can undergo BEC [15–19]. In the case of the light BEC, the curvature of the mirrors provides a nonvanishing effective photon mass and at the same time induces a harmonic trapping potential for photons.

The problem of thermalization and fluctuations of the photon Bose-Einstein condensate has been considered very recently in Ref. [20]. The condensate exchanges excitations with a reservoir consisting of  $M$  dye molecules. The authors assume that the ground-state photon mode is coupled to the electronic transitions of a given number of dye molecules. This means that the sum  $X$  of the number of ground-mode photons,  $n$ , and that of excited dye molecules,  $X - n$ , is constant. To analyze this system, the authors use the master equation approach.

Note that if we are interested in the behavior of the fluctuating photon BEC after thermalization has occurred, we can obtain the corresponding probability distribution merely using the thermodynamic consideration. The population of the electronic states of dye molecules is quickly thermalized, with the characteristic time  $\sim 1$  ps at room temperature (see Refs. [21–25] for details). Since the typical fluorescence lifetime is  $\sim 1$ – $10$  ns, the emission of photons occurs from thermally equilibrated excited states. This apparent time-scale separation allows us to consider the above system as a generalized superstatistical system. Therefore, we can find the limiting probability distribution of the number of ground-mode photons by directly applying the hierarchical maximum entropy principle to this system.

For simplicity, consider the case of the ground-mode coupling and neglect the twofold polarization degeneracy by analogy with Ref. [20]. The whole system is then composed of two subsystems: the subsystem of the dye solution and the subsystem of the photon BEC. The control parameter characterizing the interaction of the subsystems is the fluctuating number of ground-mode photons,  $n$ . The subsystem of the dye solution in turn consists of  $M$  dye molecules, among which there are  $X - n$  excited molecules and  $M - X + n$  ground-state molecules. Obviously,  $0 \leq n \leq X \leq M$ . Each

molecule is in contact with a solvent, which plays the role of thermostat. In this sense, dye molecules resemble cells, but the inverse temperature  $\beta$  does not fluctuate. For  $f(\beta)$ , this formally corresponds to the conditions of normalization, a given mean, and zero variance. In what follows, we will not explicitly indicate the dependence of functions on  $\beta$ .

Let  $D_0(\varepsilon_0)$  and  $D_1(\varepsilon_1)$  be the density of rovibrational states for the ground,  $S_0$ , and first excited,  $S_1$ , singlet electronic state, respectively. Note that  $\varepsilon_i = E - E_i$ , where  $E_i$  is the lowest-energy substate of  $S_i$ , where  $i = 0, 1$ . Hence,  $D_i(\varepsilon) = 0$  for any  $\varepsilon < 0$ . The partition functions  $Z_0$  and  $Z_1$  corresponding, respectively, to the ground-state and excited dye molecules are

$$Z_i = e^{-\beta E_i} w_i, \quad (24)$$

where

$$w_i = \int_0^\infty e^{-\beta \varepsilon} D_i(\varepsilon) d\varepsilon, \quad i = 0, 1.$$

It follows from Eqs. (3)–(5) and (24) that the entropy for a ground-state molecule,  $s_0$ , and for an excited molecule,  $s_1$ , is

$$s_i = \ln w_i + \beta(u_i - E_i),$$

where

$$u_i = E_i - \frac{1}{w_i} \frac{dw_i}{d\beta} \quad (25)$$

is the corresponding mean energy.

Now consider the subsystem of all dye molecules. After enumerating them and denoting a ground-state molecule by 0 and an excited molecule by 1, we can write an  $M$ -digit binary number  $\eta = (\eta_1 \eta_2 \dots \eta_M)$  with  $M - X + n$  zeros and  $X - n$  unities such that the state of the  $k$ th dye molecule is given by the  $k$ th digit  $\eta_k$ . For any given  $\eta$ , the entropy of the corresponding combination of dye molecules is

$$s_{\eta|n} = (M - X + n)s_0 + (X - n)s_1.$$

The probability that  $\eta$  takes on a fixed value is

$$p_{\eta|n} = \binom{M}{X - n}^{-1} = \frac{(X - n)!(M - X + n)!}{M!}.$$

The entropy  $s_n^d$  of the subsystem of dye molecules is calculated using the discrete analogs of Eqs. (6) and (7), with  $S[E](\beta|\xi)$  and  $f(\beta|\xi)$  replaced by  $s_{\eta|n}$  and  $p_{\eta|n}$ , respectively:

$$s_n^d = s_{\eta|n} + \ln \binom{M}{X - n}.$$

The mean energy of the subsystem is

$$u_n^d = (M - X + n)u_0 + (X - n)u_1,$$

where  $u_0$  and  $u_1$  are defined by Eq. (25).

The entropy of the photon BEC is zero,  $s_n^{\text{ph}} = 0$ , since the absence of the polarization degeneracy is assumed. The total energy of the condensate is

$$u_n^{\text{ph}} = n\hbar\omega,$$

where  $\hbar\omega$  is the energy of a ground-mode photon.

Finally, consider the system as a whole. The control parameter  $n$  corresponding to the number of ground-mode photons is characterized by a normalized discrete probability distribution  $(\pi_0, \dots, \pi_X)$ , where  $\pi_n$  is the probability of  $n$



photons. For a fixed  $n$ , the energy and entropy of the system are given by  $U_n = u_n^d + u_n^{\text{ph}}$  and  $S_n = s_n^d + s_n^{\text{ph}}$ , respectively. Maximizing the entropy [see Eqs. (19) and (20)]

$$S = - \sum_{n=0}^X \pi_n \ln \pi_n + \sum_{n=0}^X \pi_n S_n$$

under the normalization condition

$$\sum_{n=0}^X \pi_n = 1 \tag{26}$$

and the mean energy constraint  $\sum \pi_n U_n = U$  yields

$$\pi_n = \frac{1}{Z} \binom{M}{X-n} w_0^{M-X+n} w_1^{X-n} \times \exp\{-\beta[(M-X+n)E_0 + (X-n)E_1 + n\hbar\omega]\}, \tag{27}$$

where  $Z$  is determined from Eq. (26). Dividing Eq. (27) by  $\pi_0$  and writing  $\hbar\omega_0 = E_1 - E_0$ , we obtain the probability distribution of the number of ground-mode photons in the form

$$\frac{\pi_n}{\pi_0} = \frac{X!(M-X)!}{(X-n)!(M-X+n)!} \left(\frac{w_0}{w_1}\right)^n e^{-\beta n\hbar(\omega-\omega_0)}. \tag{28}$$

This equation allows us to find  $\pi_0 = (\sum \pi_n / \pi_0)^{-1}$  and then calculate  $\pi_n$  for all positive  $n \leq X$ .

Thus, the long-run behavior of the photon BEC, when the probability distribution  $(\pi_0, \dots, \pi_X)$  becomes stationary, can be investigated using the hierarchical maximum entropy principle. The link with the result of the master equation approach can be readily observed via the Kennard-Stepanov law [20,26–30],

$$\frac{B_{10}(\omega)}{B_{01}(\omega)} = \frac{w_0}{w_1} e^{-\beta\hbar(\omega-\omega_0)}, \tag{29}$$

which relates the Einstein coefficients for stimulated emission,  $B_{10}(\omega)$ , and absorption,  $B_{01}(\omega)$ . Equation (29) allows us to rewrite Eq. (28) as

$$\frac{\pi_n}{\pi_0} = \frac{X!(M-X)!}{(X-n)!(M-X+n)!} \left[ \frac{B_{10}(\omega)}{B_{01}(\omega)} \right]^n,$$

which is identical to Eq. (10) of Ref. [20].

It seems interesting to use the described approach for studying the photon BEC fluctuations in more detail, e.g., for considering a more realistic situation of the polarization degeneracy and additional fluctuations of  $M$  and  $X$ .

#### IV. CONCLUSION

I have formulated the hierarchical maximum entropy principle for generalized superstatistical systems. Such systems comprise a set of nonequilibrium superstatistical subsystems, where each subsystem is made up of many cells, and are characterized by the three-level dynamical hierarchy formed as a result of the sufficient time-scale separation between different dynamical levels. By arranging these levels in increasing order of dynamical time scale and consecutively maximizing the entropy at each level, I have obtained first the Gibbs canonical distribution for each cell, second the intensive parameter distribution for each subsystem, and finally the control parameter distribution for the whole system. From these distributions, I have also found the superstatistical distribution for each subsystem and the generalized superstatistical distribution for the whole system.

I have applied this principle to Bose-Einstein condensation of light in a dye microcavity. Assuming the ground-mode coupling and neglecting the polarization degeneracy, I have obtained the long-run probability distribution of the fluctuating number of ground-mode photons. This distribution is consistent with the analogous result of the master equation approach.

Note that when the hierarchical maximum entropy principle is applied to a generalized superstatistical system, certain constraints should be imposed on a normalized distribution to obtain the canonical distribution at the lower dynamical level. However, the constraints imposed on the intensive and control parameter distributions may be quite general. I propose erasing such a distinction, viz., choosing some general constraints at the lower dynamical level and additionally considering a vector intensive parameter. This will result in the generalized superstatistics the local dynamics of which is described by a more general statistics than the usual Boltzmann-Gibbs statistics. Grand canonical statistics may be the simplest alternative.

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