First-principles derivation of static avalanche-size distributions

Pierre Le Doussal and Kay Jörg Wiese
CNRS-Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75005 Paris, France

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We study the energy minimization problem for an elastic interface in a random potential plus a quadratic well. As the position of the well is varied, the ground state undergoes jumps, called shocks or static avalanches. We introduce an efficient and systematic method to compute the statistics of avalanche sizes and manifold displacements. The tree-level calculation, i.e., mean-field limit, is obtained by solving a saddle-point equation. Graphically, it can be interpreted as the sum of all tree graphs. The 1-loop corrections are computed using results from the functional renormalization group. At the upper critical dimension the shock statistics is described by the Brownian force model (BFM), the static version of the so-called Alessandro-Beatrice-Bertotti-Montorsi (ABBM) model in the nonequilibrium context of depinning. This model can itself be treated exactly in any dimension and its shock statistics is that of a Lévy process. Contact is made with classical results in probability theory on the Burgers equation with Brownian initial conditions. In particular we obtain a functional extension of an evolution equation introduced by Carraro and Duchon, which recursively constructs the tree diagrams in the field theory.

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I. INTRODUCTION: MODEL AND METHOD

Complex systems, as well as systems with quenched disorder, often respond nonsmoothly, with jumps or avalanches, to a change in external parameters, as, e.g., an applied field. This is seen as Barkhausen noise in magnets [1], earthquakes in the motion of tectonic plates [2,3], wetting of a disordered substrate [4,5], dry friction [6], cracks in brittle material [7,8], vortices in superconductors [9,10], and many more.

Quite generally, these systems can be modeled by an elastic interface pinned by disorder [3,9–18,56]. In a previous work [19,20] we have obtained the probability distribution of the sizes of static avalanches for an elastic interface in a random pinning potential. The interface is parameterized by a one-component displacement field $u(x)$, where x is the d-dimensional internal coordinate. The interface is submitted to an additional quadratic well centered at w (e.g., a spring acting on it) and the ground state of the interface, $u(x;w)$, experiences discontinuous jumps as the center-of-well position w is varied. These static avalanches, also called shocks because of interesting connections with the Burgers equation in the limit $d=0$ [21–27], are characterized by their size; i.e., $S := \int_x u(x;w)$, where $\int_x \equiv \int d^d x$. In [19,20] we obtained the distribution $P(S)$ from a combination of graphical and analytical methods, first at tree, i.e., mean-field level, valid for $d \geqslant d_{\rm uc}$, where the upper critical dimension is $d_{\rm uc}=4$ for the usual short-range elasticity, and then to first order in a dimensional expansion in $\epsilon = 4-d$, by resumming all 1-loop corrections. This calculation was technically rather complicated as it required summing an infinite set of diagrams, both at the tree and 1-loop level. Further their nonanalytic dependence on w had to be extracted. Thus [19,20] contains some amount of heuristics in extrapolating formulas from small moments of $P(S)$ to arbitrary ones, while the final structure is a relatively simple self-consistent equation. This suggests that a simpler method should exist.

In this paper we present such a simple, complementary method. It is powerful and versatile enough to apply to many situations. It is extended in companion papers to (i) the depinning transition, for which the avalanche-size distribution was also studied numerically in [28]: There we predict and measure the distribution of velocities inside an avalanche [29–32]. (ii) Elastic objects where the displacements $u(x)$ have more than one component [31,33]. This new method accounts for the (relatively) simple structures unveiled in [20], via a saddle-point equation and dressed propagators. It also allows deriving a more precise picture of the structure of avalanches around the upper critical dimension $d_{\rm uc}$. In particular we find that their statistics at $d_{\rm uc}$ is given in the statics by a Brownian force model (BFM) which we study here, closely related to the so-called Alessandro-Beatrice-Bertotti-Montorsi (ABBM) model [34] that we also recently showed to describe interfaces at $d_{\rm uc}$ near depinning [29].

In the second part of this article we make connection to the work by Carraro and Duchon [35], as well as Bertoin [36]. These authors use methods of probability theory to study the Burgers equation with Brownian initial conditions, which is the $d=0$ limit of the BFM for the interface. Their description in terms of Lévy processes is extended to interfaces and we unveil a new connection between evolution equations for these Lévy processes in Burgers dynamics, and the mean-field theory for pinned interfaces.

Our model is defined by the standard energy for a disordered elastic interface:

$$\mathcal{H}[u] = \mathcal{H}_{\rm el}[u] + \int_x V(u(x),x), \tag{1}$$

$$\mathcal{H}_{\rm el}[u] = \frac{1}{2}\int_{xx'} g^{-1}_{xx'}[u(x)-w_x][u(x')-w_{x'}]\,, \tag{2}$$

where $g_{xx'} = \int_q g_q e^{iq(x-x')}$ and the elastic energy kernel is, in the simplest case of short-range elasticity, $g_q^{-1} = q^2 + m^2$ (we denote $\int_q \equiv \frac{d^dq}{(2\pi)^d}$). A quadratic external potential of curvature m^2 and centered at w_x has been added and acts as a large-scale (infrared) cutoff. In all cases $g_{q=0} = \int_{x'} g_{xx'} = m^{-2}$. $V(u,x)$ is a centered Gaussian random potential with correlator

$$\overline{V(u,x)V(u',x')} = R_0(u-u')\delta^d(x-x'). \tag{3}$$

At finite temperature one considers the canonical partition sum $\mathcal{Z} = \int \mathcal{D}[u]\, e^{-\mathcal{H}[u]/T}$ in a given disorder realization (sample).

Disorder averages are denoted by $\overline{\cdots}$, and thermal ones by $\langle\cdots\rangle$.

To study the statics of this model, one introduces replica $u_a(x)$, $a = 1,\ldots,n$ and considers the replicated action functional denoted $\mathcal{S}_{R_0}[u] \equiv \mathcal{S}[u] \equiv \mathcal{S}[\{u_a(x)\}]$:

$$\overline{\mathcal{Z}^n} = \int \prod_a \mathcal{D}[u_a] e^{-\mathcal{S}[u]},$$
$$\mathcal{S}[u] = \frac{1}{T}\sum_a \mathcal{H}_{\rm el}[u_a] - \frac{1}{2T^2}\sum_{ab}\int_x R_0(u_a(x) - u_b(x)). \tag{4}$$

The correlation functions of the disordered model are obtained from those of the replicated theory in the limit of $n \to 0$, implicit in all formulas below.

Let us now sketch the principle of the method, starting with a simple example. Since we are interested in probability distributions of observables, we need to compute averages of the form

$$\overline{\langle e^{\int_x \lambda_x u(x)}\rangle} = \lim_{n\to 0}\left\langle e^{\int_x \lambda_x u_1(x)}\right\rangle_{\mathcal{S}}, \tag{5}$$

where $u_1(x)$ designates one of the n replicas. One defines

$$\langle \mathbb{O}[u]\rangle_{\mathcal{S}} := \frac{\int \prod_a \mathcal{D}[u_a]\mathbb{O}[u]e^{-\mathcal{S}[u]}}{\int \prod_a \mathcal{D}[u_a]e^{-\mathcal{S}[u]}}, \tag{6}$$

and since for $n = 0$ the denominator equals one, it can be dropped. Equation (5) is a generating function from which one can, at least in principle, extract via Laplace inversion a probability distribution, here the distribution of the displacement field, i.e., $\mathcal{P}[\mathsf{u}] = \overline{\langle \prod_x \delta(\mathsf{u}(x) - u(x))\rangle}$ with a double average over sample and thermal realizations. Note that averages such as (5), and their multipoint generalizations discussed below, are also frequently studied as generating functions of the distribution of the velocity field in turbulence. Burgers turbulence, e.g., maps exactly to the present model at $d = 0$ with time $t = 1/m^2$ and velocity u [21,22,25–27,35–42].

We now recall a few basic facts from field theory. Let us consider $\Gamma[u]$, the *effective action* functional associated with the action $\mathcal{S}[u]$. Then the above average (5) can be expressed as

$$\langle e^{\int_x \lambda_x u_1(x)}\rangle_{\mathcal{S}} = e^{\int_x \lambda_x u^\lambda(x) - \Gamma[u^\lambda]}, \tag{7}$$

where $u^\lambda(x)$ extremizes the exponential, i.e., is the solution of the equation

$$\partial_{u_a(x)}\Gamma[u]|_{u=u^\lambda} = \lambda_x \delta_{a1}. \tag{8}$$

This property follows from the definition of the effective action as the Legendre transform of $W[\lambda] = \langle e^{\int_x \lambda_x^a u_a(x)}\rangle_{\mathcal{S}}$, i.e., from the relation $W[\lambda] + \Gamma[u] = \sum_a \int_x u_a(x)\lambda_x^a$.

The calculation of $\Gamma[u]$ can be performed in an $\epsilon = d_{\rm uc} - d$ expansion around the upper critical dimension $d_{\rm uc}$. To lowest order in this expansion one replaces $\Gamma[u]$ by the action $\mathcal{S}[u]$. The corresponding calculation yields the *tree-level* result

$$\langle e^{\int_x \lambda_x u_1(x)}\rangle_{\mathcal{S}}^{\rm tree} = e^{\int_x \lambda_x u^\lambda(x) - \mathcal{S}[u^\lambda]}, \tag{9}$$

$$\partial_{u_a(x)}\mathcal{S}[u]|_{u=u^\lambda} = \lambda_x \delta_{a1}. \tag{10}$$

It is written in terms of a tree-level extremum field $u^\lambda \equiv u^{\lambda,\rm tree}$ which extremizes (9). This precisely amounts to resumming *all tree diagrams* in the perturbation expansion in the nonlinear part of the action $\mathcal{S}_{R_0}$, i.e., in the disorder R_0, also known as the *mean-field* calculation. This is discussed in Sec. III and Appendix A. For the problem at hand it gives the correct result for probability distributions for $d \geqslant d_{\rm uc}$, if the renormalized disorder R is used in the action, rather than the bare one R_0, as discussed in [20] and again below. The corresponding action $\mathcal{S}_R$ is called the improved action. The precise definition of R is recalled in Sec. IV A2, and useful equivalent definitions can be found in Secs. II and III of Ref. [20] (with which present definitions and notations aim to be consistent).

Here we start with the tree calculation in Sec. II, by first defining the proper observable to compute, a generalization of (9), and deriving the saddle-point equation which resums all tree diagrams. In the following Sec. III, we give a graphical derivation and illustration of the saddle-point equations. We then compute in Sec. IV $\Gamma[u]$ to first order in ϵ, and analyze the resulting saddle-point equation, from which the avalanche-size distribution to 1-loop order is obtained. In Sec. V we study a simpler model where the force landscape is a Brownian motion, and we make the connection to Lévy processes and the Burgers equation. In Sec. VI we derive the generalized Carraro-Duchon equation which encodes the mean-field theory of interfaces.

The appendices contain details and extensions: In Appendix A we study a nonuniform deformation w_x. Appendix B derives useful formulas for the diagonalization of replica matrices. Appendix C calculates $\Gamma_1[u,v]$. Appendix D gives a diagrammatic interpretation of the loop corrections. Appendix E contains a detailed derivation of many-point correlations in the BFM. Appendices F and H recall the derivation of the Carraro-Duchon formula and its connection to the exact RG equations. In Appendix I we discuss the (near absence of) loop corrections in the BFM model and we prove that it is an attractive fixed point of the RG. Finally in Appendix J we recall how the statistics of shocks depends on their correlations.

II. TREE-LEVEL (MEAN-FIELD) CALCULATION

A. Avalanche observables

Let us recall the avalanche observables introduced in [20]. Unless stated otherwise the considerations below are valid in all generality (i.e., beyond mean field).

At $T = 0$, the minimal energy configuration of the interface $u(x;w)$, and its center of mass $u(w) := L^{-d}\int_x u(x;w)$, advances by jumps as the position of the center of the parabola, $w = w_x$ (taken uniform for now), is increased:

$$u(x;w) = \sum_i S_i(x)\theta(w - w_i), \tag{11}$$

$$u(w) = L^{-d}\sum_i S_i\theta(w - w_i). \tag{12}$$

Here w_i is the position, and $S_i := \int_x S_i(x)$ the total size of the ith shock. One defines $\rho(S) := \rho_0\, P(S) = \overline{\sum_i \delta(S - S_i)\delta(w - w_i)}$, where $\rho(S)$ is the shock-size density, $P(S)$ the shock-size probability distribution (normalized to unity), and ρ_0 the total shock density. From $\overline{u(w)} = w$ it follows that $\rho_0\langle S\rangle = L^d$ whenever all motion occurs in shocks (which is the case here in the limit of interest, $m \to 0$). We denote size moments as $\langle S^p \rangle := \int dS\, S^p P(S)$. It was also shown in [20] that for $S_0 \ll S$, $P(S)$ takes the general form

$$P(S) = \frac{\langle S \rangle}{S_m^2}\, p(S/S_m), \tag{13}$$

$$S_m := \frac{\langle S^2 \rangle}{2\langle S \rangle} = \frac{R'''(0^+)}{m^4}. \tag{14}$$

Here $S_m \sim m^{-d-\zeta}$ is the scale of the large avalanches, and $S_0 \ll S_m$ a microscopic cutoff. The function $p(s)$ is universal with $\int_0^\infty ds\, s p(s) = 1$ and $\int_0^\infty ds\, s^2 p(s) = 2$.

As detailed in [20] the shock-size moments can be extracted from the generating function:

$$\hat{Z}(\lambda) := L^{-d} \partial_w \overline{e^{\lambda L^d [u(w/2) - w - u(-w/2)]}}\,|_{w=0^+} = \frac{\langle e^{\lambda S} \rangle - 1 - \lambda \langle S \rangle}{\langle S \rangle}. \tag{15}$$

This formula follows from the fact that in a small window of width $w > 0$ the probability that there is a shock is $\rho_0 w$, in which case the field $u(w) - w$ jumps by S. We used that due to statistical translation invariance, we can without loss of generality consider the interval $]-w/2, w/2[$. Equation (15) can be compared to (5) with a uniform $\lambda_x = \lambda$: However, while (5) encodes only the *one-point* probability of $u(w)$ (say at $w = 0$), Eq. (15) depends on the *two-point* joint probability distribution of the field $u(w)$ at two values of w (denoted $-w/2$ and $w/2$), as required to study shocks. Furthermore, in $d = 0$, $m^2(w - u(w))$ is the velocity field of the decaying Burgers equation at space point w [21,22,25] and Eq. (15) is thus the generating function of the distribution of velocity differences at two points in space distant by w. We retain below this terminology of p-point distributions.

We now recall the results obtained at mean-field level in [20] by resumming the tree diagrams. There it was shown that $Z(\lambda) := \lambda + \hat{Z}(\lambda)$ satisfies the self-consistent equation

$$Z(\lambda) = \lambda + \frac{R'''(0^+)}{m^4} Z(\lambda)^2. \tag{16}$$

This quadratic equation is easily solved,

$$Z_{\rm tree}(\lambda) \equiv Z_{\rm MF}(\lambda) = \frac{1}{2S_m}\left(1 - \sqrt{1 - 4S_m\lambda}\right), \tag{17}$$

where

$$S_m = \frac{R'''(0^+)}{m^4} \tag{18}$$

is the characteristic avalanche size introduced in Eq. (14). Taylor-expanding Eq. (17) in λ, $Z(\lambda) = \lambda + S_m \lambda^2 + \ldots$, and comparing to the definition (15), allows us to identify $S_m = \frac{1}{2}\langle S^2 \rangle / \langle S \rangle$, also stated in Eq. (14).

Inverse Laplace transforming Eq. (17), one obtains the mean-field avalanche-size distribution,[1] valid for $d \geqslant d_{\rm uc}$:

$$p_{\rm tree}(s) \equiv p_{\rm MF}(s) = \frac{1}{2\sqrt{\pi} s^{3/2}} e^{-s/4}. \tag{19}$$

We now recover these results, and more, by introducing a method which does not use a graphical expansion.

B. Saddle-point equation

Here we show how to evaluate, at tree level, the slightly more general generating function for the joint probability of the field $u(x;w)$ at two "points" $w_x/2$ and $-w_x/2$. It corresponds to moving the center of the parabola from $-w_x/2$ to $w_x/2$. While this is not the most general nonuniform move, its symmetry simplifies the analysis below. For future convenience, we denote here the full disorder correlator (which contains loop corrections to all orders), by R; i.e., we consider the improved action $\mathcal{S}_R$. This is an improved tree approximation; i.e., it is the sum of all tree diagrams in R; in R_0 it is the sum of all tree diagrams plus those loop diagrams in R_0 correcting R itself.

Generalizing Eq. (9) requires us to introduce two sets of n replicated fields denoted u_a, v_a, $a = 1, \ldots, n$, subject to the same disorder. We find that[2]

$$\overline{\langle e^{\int_x \lambda_x [u(x;w/2) - w_x - u(x;-w/2)]} \rangle}^{\rm tree} = e^{-\mathcal{S}_\lambda[u^\lambda, v^\lambda]}, \tag{20}$$

where (dropping the superscript λ)

$$\begin{aligned} -\mathcal{S}_\lambda[u,v] = &\int_x \lambda_x [u_1(x) - w_x - v_1(x)] \\ &- \sum_a \int_{xx'} \frac{g^{-1}_{xx'}}{2T} \left\{ \left[u_a(x) - \frac{w_x}{2}\right]\left[u_a(x') - \frac{w_{x'}}{2}\right] \right. \\ &\left. + \left[v_a(x) + \frac{w_x}{2}\right]\left[v_a(x') + \frac{w_{x'}}{2}\right] \right\} \\ &+ \frac{1}{2T^2} \sum_{ab} \int_x [R(u_a(x) - u_b(x)) \\ &+ R(v_a(x) - v_b(x)) + 2R(u_a(x) - v_b(x))] \,. \end{aligned} \tag{21}$$

$u_a(x)$ and $v_a(x)$ are extrema of $\mathcal{S}_\lambda$, i.e., solution of

$$\begin{aligned} T\lambda_x \delta_{a1} = g^{-1}_{xx'}\left[u_a(x') - \frac{w_{x'}}{2}\right] - \frac{1}{T}\sum_c [R'(u_a(x) - u_c(x)) \\ + R'(u_a(x) - v_c(x))] \end{aligned} \tag{22}$$

together with a similar equation for v, obtained by $(u,\lambda,w) \leftrightarrow (v,-\lambda,-w)$. One can also write the functional derivative of (20),

$$\begin{aligned} &\partial_{w_y} \overline{\langle e^{\int_x \lambda_x [u(x;w/2) - w_x - u(x;-w/2)]} \rangle}^{\rm tree} \\ &= \left\{ -\lambda_y + \sum_a \int_{y'} \frac{g^{-1}_{yy'}}{2T} [u_a(y') - w_{y'} - v_a(y')] \right\} e^{-\mathcal{S}_\lambda[u^\lambda, v^\lambda]}. \end{aligned} \tag{23}$$

Due to the saddle-point equations (22), only the explicit dependence on w appears.

By parity, the solution of the saddle-point equation satisfies $v_a(x) = -u_a(x)$, which we use from now on. Since only

[1] These formulas correspond to an infinite density of avalanches. For discrete displacements u, as illustrated in Appendix J, ρ_0 is finite, and $Z(\lambda)$ is cut at large negative λ, equivalent to a small $S = S_{\rm min}$ for $P(S)$. This is further discussed in [20].

[2] Here $u(x;w)$ is a functional of the field w_x. For simplicity however we do not use the square bracket notation.

replica 1 is singled out, we look for a (replica symmetric) solution where all replicas $a \neq 1$ assume the same value. We denote

$$u_a(x) = u_1(x) - TU(x), \quad a \neq 1, \tag{24}$$

which at this stage is just a definition. The saddle-point equations for the two functions $u_1(x)$ and $U(x)$ become

$$\lambda_x T = g^{-1}_{xx'}\left[u_1(x') - \frac{w_{x'}}{2}\right] + \frac{1}{T}[R'(TU(x)) + R'(2u_1(x) - TU(x)) - R'(2u_1(x))], \tag{25}$$

$$\lambda_x T = T\int_{x'} g^{-1}_{xx'}U(x') - \frac{1}{T}R'(2u_1(x) - 2TU(x)) + \frac{1}{T}[2R'(2u_1(x) - TU(x)) - R'(2u_1(x))]. \tag{26}$$

Note that the second equation has been obtained by subtracting in (22) the equation for $a \neq 1$ from the one for $a = 1$. In both equations the sum over replica indices has been performed using (24), i.e., $\sum_c F(u_c) = F(u_1) + (n-1)F(u_1 - TU_1)$, and then setting $n \to 0$. We have also used that $R'(u)$ is an odd function with $R'(0) = 0$. Once these equations are solved, the solution can be used to compute the generating functions:

$$\overline{\langle e^{\int_x \lambda_x [u(x;w/2) - w_x - u(x;-w/2)]}\rangle}^{\text{tree}} = e^{-\mathcal{S}_\lambda},$$
$$\partial_{w_y} \overline{\langle e^{\int_x \lambda_x [u(x;w/2) - w_x - u(x;-w/2)]}\rangle}^{\text{tree}} = \left[-\lambda_y + \int_{y'} g^{-1}_{yy'}U(y')\right]e^{-\mathcal{S}_\lambda}, \tag{27}$$

with

$$-\mathcal{S}_\lambda := -\mathcal{S}_\lambda[u,-u] = \int_x \lambda_x(2u_1(x) - w_x) + \int_{xx'} g^{-1}_{xx'}[-U(x)(2u_1(x') - w_{x'}) + TU(x)U(x')] + \frac{1}{T^2}\int_x [2R(0) - 2R(TU(x)) + R(2u_1(x) - 2TU(x)) + R(2u_1(x)) - 2R(2u_1(x) - TU(x))]. \tag{28}$$

The limit $n = 0$ has been taken everywhere. This result is equivalent to the graphical summation of all tree diagrams, in terms of either R_0 or R, depending on whether bare or renormalized perturbation theory is used. It is valid for any T and w_x; hence in principle it allows us to compute at tree level a rather general 2-point correlation function of the field $u(x;w)$ at any temperature T.

In the absence of disorder, the saddle point is $u_1(x) - w_x/2 = TU(x) = T\int_{x'} g_{xx'}\lambda_{x'}$, and one obtains $-\mathcal{S}_\lambda = T\int_{x,x'} g_{xx'}\lambda_x\lambda_{x'}$. The tree formula (27) is then exact and corresponds to two copies with uncorrelated thermal fluctuations. In the presence of disorder, but for $\lambda_x = 0$, one must have $\mathcal{S}_\lambda = 0$. This is indeed the case, as the saddle point is then $U(x) = 0$ and $u_1(x) - \frac{w_x}{2} = 0$. When there could be several solutions to the saddle-point equation, the correct one should reduce to that one in the small-λ limit. The saddle-point solution can also be obtained order by order in λ from perturbation theory; i.e., a well-defined expansion of $u_1(x)$ and $U(x)$ in powers of λ must exist.

C. $T = 0$ limit of the saddle-point equations

We can now study the system at $T = 0$. Then $R(u)$ is nonanalytic; more precisely it exhibits a linear cusp in its second derivative, $R'''(0^+) > 0$. This cusp is related to the second moment of avalanche sizes, as shown in [20], via $R'''(0^+) = m^4\langle S^2\rangle/2\langle S\rangle$. We will recover this relation here.

In the $T \to 0$ limit we obtain a consistent solution assuming that U and u_1 are going to a finite limit, as we show now. Expanding (25) and (26) in powers of T we obtain to lowest order

$$\int_{x'} g^{-1}_{xx'}\left[u_1(x') - \frac{w_{x'}}{2}\right] + [R''(0) - R''(2u_1(x))]U(x) = 0, \tag{29}$$

$$\int_{x'} g^{-1}_{xx'}U(x') - R'''(2u_1(x))U(x)^2 = \lambda_x. \tag{30}$$

The generating functions are, omitting the thermal averages $\langle\ldots\rangle$, since we are studying $T = 0$,

$$\overline{e^{\int_x \lambda_x [u(x;w/2) - w_x - u(x;-w/2)]}}^{\text{tree}} = e^{-\mathcal{S}_\lambda}, \tag{31}$$

$$\partial_{w_y} \overline{e^{\int_x \lambda_x [u(x;w/2) - w_x - u(x;-w/2)]}}^{\text{tree}} = \left[-\lambda_y + \int_{y'} g^{-1}_{yy'}U(y')\right]e^{-\mathcal{S}_\lambda}, \tag{32}$$

$$-\mathcal{S}_\lambda = \int_x \left[u_1(x) - \frac{w_x}{2}\right]\left[2\lambda_x - \int_{x'} g^{-1}_{xx'}U(x')\right], \tag{33}$$

where (29) has been used to simplify $\mathcal{S}_\lambda$. These formulas are valid for arbitrary w_x. We now analyze these equations in several cases.

D. Uniform case: Avalanches of center of mass

Let us start with the simplest case of both $\lambda_x = \lambda$ and $w_x = w$ uniform, corresponding to the generating function (15). Then $u_1(x) = u_1$ and $U(x) = U$ satisfy

$$m^2(u_1 - w/2) + [R''(0) - R''(2u_1)]U = 0, \tag{34}$$

$$m^2U - R'''(2u_1)U^2 = \lambda, \tag{35}$$

$$L^{-d}\partial_w \overline{e^{\lambda L^d[u(w/2) - w - u(-w/2)]}}^{\text{tree}} = [-\lambda + m^2U]e^{-\mathcal{S}_\lambda}, \tag{36}$$

$$-\mathcal{S}_\lambda = L^d(2\lambda - m^2U)\left(u_1 - \frac{w}{2}\right). \tag{37}$$

These equations can be studied for any w.

We now consider the limit $w \to 0^+$ from which avalanche observables can be extracted. We look for a solution of the form

$$u_1 = y\frac{w}{2} + O(w^2), \tag{38}$$

which implies that $\mathcal{S}_\lambda = O(w)$. Hence

$$L^{-d}\partial_w \overline{e^{\int \lambda L^d[u(w/2) - w - u(-w/2)]}}^{\text{tree}}|_{w=0^+} = -\lambda + m^2U. \tag{39}$$

Assuming $y > 0$ we obtain from (34) and (35)

$$y\left(1 - \frac{2R'''(0^+)}{m^2}U\right) = 1, \tag{40}$$

$$m^2U - R'''(0^+)U^2 = \lambda. \tag{41}$$

Comparing the definition (15) and the result (39) we see that we can identify

$$Z(\lambda) \equiv \lambda + \hat{Z}(\lambda) = m^2 U. \tag{42}$$

Our self-consistent equation (41) is indeed the same as the one obtained in [20], namely (16). Its physical solution, which vanishes at $\lambda = 0$, is $m^2 U = Z(\lambda) = Z_{\rm MF}(\lambda)$ given in (17). Note that

$$y = \frac{1}{\sqrt{1 - 4S_m\lambda}} \tag{43}$$

is indeed positive for this solution. The breakdown at $\lambda \geqslant 1/(4S_m)$ signals that the Laplace transform of $P(S)$ does not exist beyond that value of λ, due to the exponential tail at large S in (19).

E. Nonuniform case: Local structure of avalanches

To obtain spatial information about avalanches one may consider both λ_x and w_x nonuniform. We specify $w_x = \tilde{w} f(x)$ and vary $\tilde{w}$ from $\tilde{w} = -\frac{w}{2}$ to $\tilde{w} = +\frac{w}{2}$ for a fixed $f(x)$. In the limit of small positive w we look for a solution of the form

$$u_1(x) = y(x)\frac{w}{2} + O(w^2). \tag{44}$$

Again one finds $\mathcal{S}_\lambda = O(w)$; hence

$$\partial_w \overline{e^{\int_x \lambda_x [u_x(w/2) - w_x - u_x(-w/2)]}}^{\rm tree}|_{w=0^+} = \int_x f(x)\left[-\lambda_x + \int_{x'} g_{xx'}^{-1} U(x')\right]. \tag{45}$$

The field $U(x)$ satisfies Eq. (30); i.e.,

$$\int_{x'} g_{xx'}^{-1} U(x') - R'''(y(x)w)U(x)^2 = \lambda_x. \tag{46}$$

Here we assume that $y(x) > 0$; a more general discussion is given in Appendix A. Then in the limit of $w \to 0^+$, one can replace $R'''(y(x)w) \to R'''(0^+)$ and (46) becomes a closed equation for $U(x)$, *independent* of $y(x)$,

$$\int_{x'} g_{xx'}^{-1} U(x') - R'''(0^+)U(x)^2 = \lambda_x. \tag{47}$$

This is a classical equation for a cubic field theory, which admits instanton solutions, from which local size distributions can be extracted, as discussed in [20]. Here we will not study again these applications, but simply make contact with the notations used there. For that purpose we identify

$$Z_x(\lambda) = \int_{x'} g_{xx'}^{-1} U(x'), \tag{48}$$

in terms of which the self-consistent equation becomes

$$Z_x(\lambda) = \lambda_x + R'''(0^+) \int_{x_1,x_2} g_{xx_1} g_{xx_2} Z_{x_1}(\lambda) Z_{x_2}(\lambda). \tag{49}$$

These are Eqs. (204) (in unrescaled form) and (F8) of [20]. The space-dependent generating function can then be written as

$$\partial_{w_y} \overline{e^{\int_x \lambda_x [u_x(w/2) - w_x - u_x(-w/2)]}}^{\rm tree}|_{w_y=0^+} = Z_y(\lambda) - \lambda_y := \hat{Z}_y(\lambda). \tag{50}$$

Note that a rescaled version of $U(x)$ was denoted $Y(x)$ in [20].

$Z_y(\lambda)$ is connected to local avalanche-size distributions. Assuming that for fixed L as $w \to 0^+$ the probability of a shock during a change of w_x is $\rho_0^f w$, we can write from Eq. (45)

$$\rho_0^f \left\langle e^{\int_x \lambda_x S_x} - 1 - \int_x \lambda_x S_x \right\rangle_f = \int_x f(x)\left[-\lambda_x + \int_{x'} g_{xx'}^{-1} U(x')\right]. \tag{51}$$

The subscript f reminds us that we use a nontrivial $f(x)$. In addition, one can show that $\overline{u_x(w/2) - u_x(-w/2)} = w_x$ and hence $\rho_0^f \langle S_x \rangle_f = 1$, which also implies $\rho_0^f \langle S \rangle_f = L^d$. Note that the size distribution, whose Laplace transform is given by (51), *a priori* depends on the function $f(x)$; the case $f(x) = 1$ was studied in [20].

F. Multipoint correlations of center-of-mass displacement

1. Discrete version

We now indicate how to compute, at tree level (i.e., as a sum of all tree diagrams), the correlations of the center-of-mass displacement field $u(w_i) - w_i$ at an arbitrary number of discrete points w_i. For this we introduce a generating function, parameterized by λ_i:

$$\overline{e^{L^d \sum_i \lambda_i [u(w_i) - w_i]}}^{\rm tree} = e^{L^d [\sum_i \lambda_i (u_{1i} - w_i) - \sum_{ai} \frac{m^2}{2T}(u_{ai} - w_i)^2 + \frac{1}{2T^2}\sum_{abij} R(u_{ai} - u_{bj})]}. \tag{52}$$

The fields u_{ai} are solutions of the saddle-point equations

$$m^2(u_{ai} - w_i) - \frac{1}{T}\sum_{cj} R'(u_{ai} - u_{cj}) = T\lambda_i \delta_{a1}. \tag{53}$$

As above, we look for a replica-symmetric saddle point $u_{ai} = u_i$ for $a \neq 1$. Define $u_i := u_{1i} - TU_i$, and subtract the equation for $a = 2$ from the one for $a = 1$:

$$m^2(u_{1i} - w_i) + \frac{1}{T}\sum_j [R'(u_{1i} - u_{1j} + TU_j) - R'(u_{1i} - u_{1j})] = T\lambda_i, \tag{54}$$

$$m^2 TU_i + \frac{1}{T}\sum_j [R'(u_{1i} - u_{1j} + TU_j) - R'(u_{1i} - u_{1j}) + R'(u_{1i} - u_{1j} - TU_i) - R'(u_{1i} - u_{1j} - TU_i + TU_j)] = T\lambda_i. \tag{55}$$

Note that in the second equation the terms $i = j$ can be excluded. Taking the limit $T \to 0$, we find

$$m^2(u_{1i} - w_i) + \sum_j R''(u_{1i} - u_{1j})U_j = 0, \qquad m^2 U_i + \sum_{j \neq i} R'''(u_{1i} - u_{1j})U_i U_j = \lambda_i. \tag{56}$$

One must solve these equations for U_i and u_{1i} and insert the result into

$$\overline{e^{L^d \sum_i \lambda_i [u(w_i) - w_i]}}^{\rm tree} = e^{L^d \sum_i (\lambda_i - \frac{1}{2}m^2 U_i)(u_{1i} - w_i)}, \tag{57}$$

which has been simplified using the saddle-point equations. Note that one recovers the 2-point equations (35) from the solution $u_{12} = -u_{11}$ and $U_2 = -U_1$ valid for $\lambda_2 = -\lambda_1$.

2. Continuous version

It is instructive to perform the same calculation in a continuum framework. We again restrict to the center of mass and compute

$$\overline{\langle e^{L^d \int_w \lambda(w)[u(w)-w)]}\rangle}^{\text{tree}} = e^{-\mathcal{S}_\lambda} \tag{58}$$

for a test function $\lambda(w)$, depending on w but uniform in space. Here and below $\int_w = \int_{-\infty}^{\infty} dw$. We now introduce[3] replica fields $u_{ai} \to u_a(w)$ and then extremize the action:

$$L^{-d} S_\lambda = \int_w \lambda(w)[u_1(w) - w] - \int_w \sum_a \frac{m^2}{2T}[u_a(w) - w]^2 + \frac{1}{2T^2}\int_{w,w'} \sum_{ab} R(u_a(w) - u_b(w')). \tag{59}$$

We will not repeat all the above manipulations. The saddle-point equations with respect to $u_a(w)$ in the limit $T = 0$ lead to

$$\begin{aligned} &m^2[u_1(w) - w] + \int_{w'} R''(u_1(w) - u_1(w'))U(w') = 0, \\ &m^2 U(w) + \int_{w'} R'''(u_1(w) - u_1(w'))U(w)U(w') = \lambda(w). \end{aligned} \tag{60}$$

Its solution is inserted into

$$-L^{-d} S_\lambda = \int_w \left[\lambda(w) - \frac{m^2}{2}U(w)\right][u_1(w) - w]. \tag{61}$$

The general analysis of these multipoint correlations and their discrete analog (56) is left for the future. Below we study them, in Sec. V and Appendix E, for a simpler model, where $R'''(u)$ is a constant, and in Sec. VI G for periodic disorder. We also discuss, in Sec. VI, another powerful method to generate multipoint correlations.

G. Displacement correlations for finite w: Absence of correlations for $d \geqslant d_{\rm uc}$

The two-point correlation function of the center-of-mass displacement,

$$\overline{e^{L^d \lambda [u(w/2) - w - u(-w/2)]}}^{\text{tree}} = e^{-\mathcal{S}_\lambda}, \tag{62}$$

can also be evaluated within tree level at finite $w > 0$ by setting $u_1 = yw/2$, and solving the saddle-point equations (35) for y and U, denoting y_w and U_w these solutions. This can be done, e.g., order by order in w. Here we give the small-w expansion up to $O(w^4)$. It is convenient to introduce the rescaled (dimensionless) variables $\tilde w$ and $\tilde\lambda$,

$$w = m^d S_m \tilde w, \quad \lambda = \tilde\lambda / S_m. \tag{63}$$

[3]The generalization with space dependence would involve $\lambda_x(w)$ and replica fields $u_a(x,w)$.

Defining $\tilde Z_w = m^2 S_m U_w$, one finds

$$\begin{aligned} -\frac{\mathcal{S}_\lambda}{(mL)^d} &= \tilde w(\tilde Z - \tilde\lambda) + m^{-\epsilon} R''''(0)\frac{\tilde w^2 \tilde Z^2}{2(1-2\tilde Z)^2} + \tilde w^3 \tilde Z^2 m^{-2\epsilon} \\ &\quad \times \left[R''''(0)^2 \frac{\tilde Z(1-\tilde Z)}{(1-2\tilde Z)^5} + \frac{R'''(0)R^{(5)}(0)}{6(1-2\tilde Z)^3}\right] \\ &\quad + O(\tilde w^4), \end{aligned} \tag{64}$$

$$\tilde Z \equiv \tilde Z_{\text{tree}}(\tilde\lambda) = \tfrac12 (1 - \sqrt{1 - 4\tilde\lambda}). \tag{65}$$

More generally, one can introduce the dimensionless rescaled renormalized disorder correlator,

$$R''(u) = A_d m^{\epsilon - 2\zeta} \tilde R''(u m^\zeta), \tag{66}$$

where $A_d = 1/(\epsilon \tilde I_2)$, and for short-ranged disorder $\epsilon = 4 - d$ and

$$\tilde I_2 = \int_k \frac{1}{(1+k^2)^2} \tag{67}$$

is an amplitude; for details see [17], and for generalization to long-range disorder [16,43,44]. It is known that as $m \to 0$, $\tilde R''(u)$ goes to a fixed-point function in any d, measured in [45]. Then, in any d, and within the tree approximation,

$$-\frac{\mathcal{S}_\lambda}{(mL)^d} = F(\tilde\lambda, \tilde w) := (y-1)\left(\tilde\lambda - \frac12 \tilde Z\right)\tilde w \tag{68}$$

takes a scaling form, obtained by eliminating $y \equiv y_w$ and $\tilde Z \equiv \tilde Z_w$ in the rescaled saddle-point equations:

$$\begin{aligned} &(y-1)A_d \tilde R'''(0)\frac{\tilde w}{2} + \frac{\tilde R''(0) - \tilde R''(y A_d \tilde R'''(0)\tilde w)}{\tilde R'''(0)}\tilde Z = 0, \\ &\tilde Z - \frac{\tilde R'''(y A_d \tilde R'''(0)\tilde w)}{\tilde R'''(0)}\tilde Z^2 = \tilde\lambda\,. \end{aligned} \tag{69}$$

While these equations for the tree approximation can be written for any $d < d_{\rm uc}$, they are expected to become exact as $d \to d_{\rm uc}$ and in $d > d_{\rm uc}$. For $d = d_{\rm uc} - \epsilon$, $\epsilon > 0$, the rescaled disorder $\tilde R''(u)$ is uniformly of order ϵ. In (69) the argument of the functions is $O(\epsilon)$ which justifies an expansion of the system for small w. This is because the scaling variable is

$$w = A_d \tilde R'''(0) m^{-\zeta} \tilde w \equiv S_m m^d \tilde w. \tag{70}$$

Hence near $d = d_{\rm uc}$ one can focus on (64). Since

$$m^{-\epsilon} R''''(0) = A_d \tilde R''''(0) = O(\epsilon), \tag{71}$$

$$m^{-2\epsilon} R'''(0) R^{(5)}(0) = O(\epsilon^2)\,, \tag{72}$$

we arrive (in terms of the unrescaled variables) for $w > 0$ at the result

$$\overline{e^{L^d \lambda [u(w/2) - w - u(-w/2)]}}^{\text{tree}} = e^{L^d w \hat Z(\lambda) + O(\epsilon)}. \tag{73}$$

The interpretation of this result is that, for $d \geqslant d_{\rm uc}$, *the increments in the displacements in the center of mass become uncorrelated*. It will further be discussed in Sec. V B in terms of *Lévy* processes.

Below $d_{\rm uc}$, in $d = d_{\rm uc} - \epsilon$, we expect correlations. They only exist on a distance $\tilde w = O(1)$, i.e., $w = O(\epsilon) m^{-\zeta}$, a very small layer as $\epsilon \to 0$. The above result (64) allows us to compute the first correction in ϵ; however one may also get contributions at one loop of the same order. This calculation is performed elsewhere.

III. GRAPHICAL INTERPRETATION

Here, we sketch a short graphical interpretation of the mean-field saddle-point equations. We work in $d = 0$ for simplicity. The results apply to the center-of-mass variable in any d, after restoring the necessary factors of L^d. For easier comparison with Sec. VI we use the notations

$$\Delta(u) = -R''(u), \quad t = \frac{1}{m^2}. \tag{74}$$

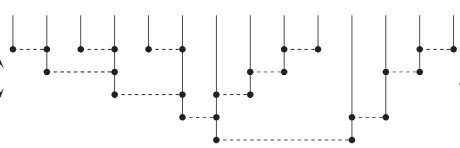
We define

$$e^{\mathbb{Z}(\lambda,w)} := \overline{e^{\lambda[u(w)-u(0)]}}. \tag{75}$$

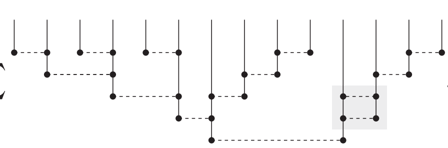
Thus $\hat{\mathbb{Z}}(\lambda,w) = \mathbb{Z}(\lambda,w) - w\lambda$ is the generating function of the connected moments $\overline{[u(w)-u(0)-w]^n}^c = t^n K^{(n)}(w)$, which in Eq. (41) of [20] were called the Kolmogorov cumulants. In [20], a graphical derivation of the recursion relation for the $O(w)$ term $Z(\lambda)$ was given, noting

$$\mathbb{Z}(\lambda,w) = Z(\lambda)w + O(w^2). \tag{76}$$

At tree level, $\mathbb{Z}^{\rm MF}(\lambda,w)$ is the sum of *all* connected tree diagrams. The generating function $Z^{\rm MF}(\lambda)$ as a function of the bare action can be written as a sum of *particularly simple* tree graphs, namely the ones of the form

$Z^{\rm MF}(\lambda) = \mathcal{K}\sum$  , (77)

where dotted lines represent the bare disorder Δ_0. Each graph represents correlations of the $\lambda[u(w_i) - w_i]$ fields at different points w_i (external lines on the top, each coming with a factor of λ). The linear operator $\mathcal{K}$ identifies w_i with w or 0, in order to build the Kolmogorov cumulants. The disorder vertex on the bottom contributes a factor of $\Delta_0(w) - \Delta_0(0)$ to $\mathbb{Z}(\lambda,w)$, which has been expanded to first order in w to obtain $Z(\lambda)$, hence it must be counted as $\Delta_0'(0^+)$ in (77). Equivalently, one can include a factor of $\partial_w|_{w=0}$ in $\mathcal{K}$. For more details of these graphical rules see [20], Sec. V C. They are also further used below in Sec. VI and Appendix D. The improved generating function $Z^{{\rm MF},R}(\lambda)$ is the same sum of tree diagrams with Δ at each vertex; hence, if reexpressed in terms of the bare disorder Δ_0, it is now a sum of graphs of the form

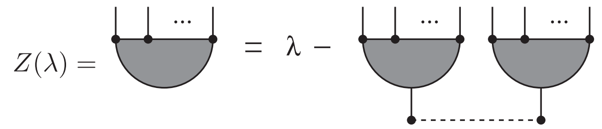
$Z^{{\rm MF},R}(\lambda) = \mathcal{K}\sum$  . (78)

It contains all loop corrections to the 2-point disorder vertex, while loop corrections to higher-point disorder vertices are neglected. Explicit formulas for low-order contributions can be found in [20].

The mean-field self-consistency equation for $Z(\lambda) = Z^{{\rm MF},R}(\lambda)$ reads

$$Z(\lambda) = \lambda - t^2\Delta'(0^+)Z(\lambda)^2. \tag{79}$$

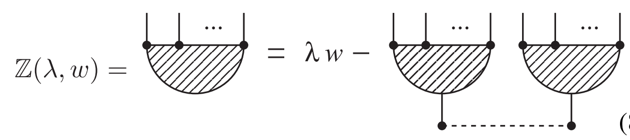
It is graphically written as [20]



. (80)

As indicated, the blob denotes $Z(\lambda)$ itself, while the lowest disorder vertex counts as a $\Delta'(0^+)$ and the lines entering the two blobs from below do not come with differentiations.[4] This self-consistent equation yields the desired sum of tree diagrams *with only one lower disorder vertex*, i.e., a term $\Delta(w) - \Delta(0)$ expanded into $w\Delta'(0^+)$, which is sufficient to obtain the $O(w)$ part in (76) as explained in [20].

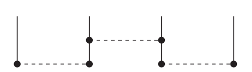
We now want to construct a recursion relation for $\mathbb{Z}(\lambda,w)$, which yields its *complete* w dependence. We thus need to sum *all* tree diagrams. To generate them, it *seems natural* to write



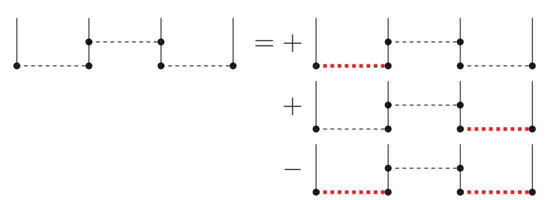
. (81)

Now the lower vertex is $\Delta(w) - \Delta(0)$ and the line entering a blob $\mathbb{Z}(\lambda,w)$ from below acts as derivative with respect to w.

However, a new difficulty arises: One may have two or more lower vertices, as, e.g., in

 . (82)

Unfortunately, the self-consistency equation (81) is then incorrect, as it leads to an overcounting since there are several ways to construct the same graph. Fortunately this can be corrected. Let us explain the source of the problem, and its correction on an example:



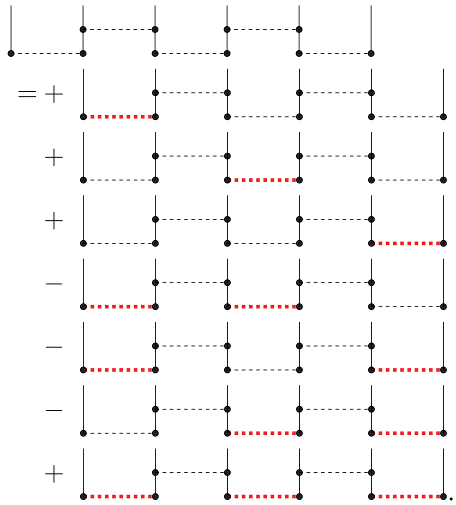
. (83)

On the left-hand side we have plotted the contribution to $\mathbb{Z}(\lambda,w)$, with the correct combinatorial factor. The first two terms on the right-hand side appear in a recursion relation of the form (81), plotting the added vertex in Eq. (81) red (thick in black-and-white). This leads to an overcounting, which can be corrected by subtracting the last term, which has *two* marked (red/thick) vertices.

---

[4] Note that in [20] a factor of $|\Delta'(0^+)| = -\Delta'(0^+)$ has been absorbed in λ, whereas here it is explicitly written, resulting in a seemingly opposite sign.

In the case of three lower vertices, the recursion reads


(84)

Now pairs of lower vertices are subtracted, leading to a cancellation of all terms, and consequently the *triplet* of lower vertices has to be added. This can be generalized, replacing the self-consistency condition (81) by

$$\lambda w = \lim_{\nu\to 0} \nu \ln\left(:e^{\nu[\Delta(w)-\Delta(0)]t^2\partial_w\partial_w}:\, e^{\frac{1}{\nu}\mathbb{Z}(\lambda,w)}\right). \qquad (85)$$

The dots around the first exponential function indicate that the derivatives act only on $\mathbb{Z}(\lambda,w)$, not on $\Delta(w)$. This can be written as

$$\lambda w = \lim_{\nu\to 0} \nu \ln\left(e^{\nu[\Delta(w)-\Delta(0)]t^2\partial_u\partial_u} e^{\frac{1}{\nu}\mathbb{Z}(\lambda,u)}\right)\Big|_{u=w}. \qquad (86)$$

The limit of $\nu \to 0$ selects the tree diagrams, and the ln selects a single connected component. Note that $e^{\nu[\Delta(w)-\Delta(0)]t^2\partial_u\partial_u}$ is defined by its series expansion in t; thus the limit $\nu \to 0$ is done term by term in the expansion in powers of t.

To proceed, we observe that independent of ν and for all functions $f(u)$ which are infinitely differentiable[5]

$$e^{\nu[\Delta(w)-\Delta(0)]t^2\partial_u\partial_u} f(u)|_{u=w} = \frac{1}{\sqrt{4\pi\nu[\Delta(w)-\Delta(0)]t^2}} \int_{-\infty}^{\infty} du\, e^{-\frac{(u-w)^2}{4\nu[\Delta(w)-\Delta(0)]t^2}} f(u). \qquad (87)$$

Inserting this relation into (86) yields

$$\lambda w = \lim_{\nu\to 0} \nu \ln\left(\frac{1}{\sqrt{4\pi\nu[\Delta(w)-\Delta(0)]t^2}} \times \int_{-\infty}^{\infty} du\, e^{-\frac{(u-w)^2}{4\nu[\Delta(w)-\Delta(0)]t^2}} e^{\frac{1}{\nu}\mathbb{Z}(\lambda,u)}\right). \qquad (88)$$

In the limit of $\nu \to 0$ to be taken here, the integral is dominated by its saddle point, and we get

$$w\lambda = \mathbb{Z}(\lambda,u) + [\Delta(0)-\Delta(w)]\, t^2\, [\partial_u \mathbb{Z}(\lambda,u)]^2\,, \qquad (89)$$

$$u = w + 2t^2\, [\Delta(w)-\Delta(0)]\, \partial_u \mathbb{Z}(\lambda,u). \qquad (90)$$

[5]The formula is valid for $\nu[\Delta(w)-\Delta(0)] > 0$; i.e., the integral should be evaluated for $\nu < 0$, and the limit to be taken is $\nu \to 0^-$.

The new variable u has to be eliminated between these two equations. Note that it is an independent variable not to be confused with $u(w)$.

We remark that passing from (86) to the self-consistent set of equations (89) and (90) is a quite common feature in tree-resummation problems. It also appears in the large-N resummation for the disorder itself, where the links are the 1-loop momentum integral, and which therefore are termed cactus-diagrams; see [46,47].

Below we devise another method to compute $\mathbb{Z}(\lambda,w)$ and we have checked to high orders, $\sim \lambda^{100}$, that (89) reproduces the solution Eq. (210), e.g., (212), as well as the lowest-order Kolmogorov cumulants (62)–(66) in [20].

One needs the derivative $\frac{du}{dw}$, obtained by deriving (90) with respect to w,

$$\frac{du}{dw} = \frac{1 + 2t^2\Delta'(w)\partial_u\mathbb{Z}(\lambda,u)}{1 + 2t^2\,[\Delta(0)-\Delta(w)]\,\partial_u^2\mathbb{Z}(\lambda,u)}. \qquad (91)$$

Deriving Eq. (89) with respect to w and using (91) yields the astonishingly rather similar equation,

$$\lambda = \partial_u\mathbb{Z}(\lambda,u) + \Delta'(w)t^2[\partial_u\mathbb{Z}(\lambda,u)]^2. \qquad (92)$$

We now make contact to the results given in Eq. (34). Since u and w are simply variables, and we are more interested in $\mathbb{Z}(\lambda,w)$ than in $\mathbb{Z}(\lambda,u)$, one can exchange their names, to obtain a second set of equations,

$$u\lambda = \mathbb{Z}(\lambda,w) + [\Delta(0)-\Delta(u)]t^2[\partial_w\mathbb{Z}(\lambda,w)]^2, \qquad (93)$$

$$w - u = 2t^2[\Delta(u)-\Delta(0)]\partial_w\mathbb{Z}(\lambda,w), \qquad (94)$$

$$\lambda = \partial_w\mathbb{Z}(\lambda,w) + \Delta'(u)t^2[\partial_w\mathbb{Z}(\lambda,w)]^2. \qquad (95)$$

Equation (95) is redundant, or can be used instead of (93). We have explicitly checked that up to order t^8, both expressions are correct.

Graphically, the interpretation of Eqs. (89)–(92) is rather different from that of Eqs. (93)–(95). To see this, recursively replace u in Eq. (89) by its value given by Eq. (90). This yields a perturbative expansion in t of λw, which can be read as a self-consistent equation for $\mathbb{Z}(\lambda,w)$. Graphically, it contains links made out of $[\Delta(w)-\Delta(0)]\,t^2$, which end in vertices made out of $\mathbb{Z}(\lambda,w)$. If n links enter into such a vertex $\mathbb{Z}(\lambda,w)$, it means to take n derivatives.

The picture is different when replacing recursively in Eq. (93) u by its value given by Eq. (94), thus constructing again a perturbative expansion in t for $\mathbb{Z}(\lambda,w)$. Apart from a single term $\mathbb{Z}(\lambda,w)$, all other terms are proportional to powers of $\partial_w\mathbb{Z}(\lambda,w)$, and no higher derivative of $\mathbb{Z}(\lambda,w)$ appears. The objects with more derivatives are $(n-2)$nd derivatives of the disorder $[\Delta(w)-\Delta(0)]t^2$, which have n outgoing lines which end either in a disorder, a $\partial_w\mathbb{Z}(\lambda,w)$ or λ. This will in more detail be discussed in [31].

Identifying

$$\partial_w\mathbb{Z}(\lambda,w) = m^2 U, \qquad (96)$$

$$u = 2u_1 \qquad (97)$$

shows that Eq. (95) is equivalent to Eq. (35), and (94) to (34). We still have to check expression (37). We know that $\hat{\mathbb{Z}}(\lambda,w) = \mathbb{Z}(\lambda,w) - \lambda w = -\mathcal{S}_\lambda$. Multiplying Eq. (94) with λ

and adding the result to (93) gives

$$\mathbb{Z}(\lambda,w) - \lambda w = [\Delta(u) - \Delta(0)]\, t^2\, [\partial_w \mathbb{Z}(\lambda,w)]^2 - 2t^2\lambda\, [\Delta(u) - \Delta(0)]\, \partial_w \mathbb{Z}(\lambda,w). \quad (98)$$

The right-hand side is nothing but $-\mathcal{S}_\lambda$, given in Eq. (37).

IV. BEYOND MEAN FIELD: 1-LOOP CALCULATION

A. Simpler example: 1-point probability

1. Perturbation around mean field

We start with the simpler case of a 1-point probability, e.g., as given by Eqs. (7) and (8). As we explain in detail below, in the dimensional expansion around mean field, the effective action can be written as

$$\Gamma[u] = \mathcal{S}[u] + \Gamma_1[u]. \quad (99)$$

Here $S[u] \equiv S_R[u]$ is the improved action, and $\Gamma_1[u]$ is "small" in a sense to be specified below. Hence we can expect that u^λ and $u^{\lambda,\text{tree}}$, the solutions of (8) and (10), are close to each others. Schematically we write

$$\mathcal{S}'[u^\lambda] + \Gamma_1'[u^\lambda] = \lambda, \quad (100)$$

$$\mathcal{S}'[u^{\lambda,\text{tree}}] = \lambda. \quad (101)$$

Inserting into Eqs. (7) and (8) and expanding to lowest order, i.e., in the differences $u^\lambda - u^{\lambda,\text{tree}} \sim O(\Gamma_1)$ and Γ_1, we find

$$\langle e^{\int_x \lambda_x u_1(x)}\rangle_S = \langle e^{\int_x \lambda_x u_1(x)}\rangle_S^{\text{tree}} e^{-\Gamma_1[u^{\lambda,\text{tree}}]}. \quad (102)$$

Hence to compute this generating function to lowest order around the tree result we only need to evaluate Γ_1 at the tree saddle point.

2. Effective action for the pinned interface

The (replica) effective action $\Gamma[u]$ associated with the (bare) action (4) of the pinned interface takes the form

$$\Gamma[u] = \mathcal{S}_{\text{el}}[u] - \sum_{p=2}^{\infty} \sum_{a_1,\ldots,a_p} \frac{1}{T^p p!} S^{(p)}[u_{a_1},\ldots,u_{a_p}]. \quad (103)$$

Here $\mathcal{S}_{\text{el}}[u] = \frac{1}{T}\sum_a H_{\text{el}}[u_a]$ arises from the elastic and quadratic well energy, defined in (2). The p-replica terms, $S^{(p)}$, are the pth cumulants of the renormalized disorder.[6] The local part [i.e., $u_a(x) = u_a$] of the second cumulant, $S^{(2)}[u_a,u_b] = L^d R(u_{ab})$, $u_{ab} = u_a - u_b$, defines the renormalized disorder correlator $R(u)$.

Let us now consider $T = 0$. As $m \to 0$, $R(u)$ flows to a fixed-point function $R = O(\epsilon)$ where $\epsilon = d_{\text{uc}} - d$. In general $\Gamma[u]$ can be computed in an expansion in powers of R, i.e., of ϵ. For $p = 2$ one has [17,25]

$$S^{(2)}[u_a,u_b] = \int_x \sum_{ab} R(u_{ab}(x)) + \frac{1}{2}\int_{xy} \left(g_{xy}^2 - \delta_{xy} g_z^2\right) \times \mathsf{R}''(u_{ab}(x))\mathsf{R}''(u_{ab}(y)) + O(R^3), \quad (104)$$

where here and below we denote $\mathsf{R}''(u) := R''(u) - R''(0)$. Purely nonlocal parts are thus $O(\epsilon^2)$ and higher. For $p \geqslant 3$, each $S^{(p)}$ is of order $O(R^p) = O(\epsilon^p)$. They were computed previously [20,48]; the result can be summarized by Eq. (99) with

$$\Gamma_1[u] = \frac{1}{2}\text{Tr}\ln\left(g^{-1}\delta_{ab} - W^1_{ab}\right) - \frac{1}{2}\text{Tr}\ln\left(g^{-1}\delta_{ab} - W^0_{ab}\right) + \frac{I_2}{4}\text{Tr}[(W^1)^2 - (W^0)^2], \quad (105)$$

$$W^\kappa_{ab,xy} = \frac{1}{T}\delta_{xy}\left[\delta_{ab}\sum_c R''(u_{ac}(x)) - \kappa R''(u_{ab}(x))\right], \quad (106)$$

where $I_n = \int_k g_k^n$. Equation (105) has the usual expression of a 1-loop effective action, $\frac{1}{2}\text{Tr}\ln \mathcal{S}''$, apart from a subtraction of the 1-loop graphs leading to $p+1$ replica terms, proportional to T (second term on first line) and of the $p = 2$ part, already taken into account in (104), thus expressing $\Gamma_1[u]$ as a functional of the renormalized instead of the bare disorder. [Note that as written Γ_1 also contains the bilocal $O(\epsilon^2)$ part of $p = 2$ in Eq. (104)]. Upon expanding the Tr ln, which acts both on replica and space indices, the $S^{(p)}$ are recovered to $O(W^p)$; see, e.g., formula (113) in [20]. Note that the two $O(W)$ terms cancel (using $n = 0$).

3. 1-loop probability distribution of displacements

Let us compute the 1-point generating function to lowest order beyond mean field,

$$\overline{\langle e^{\int_x \lambda_x u(x;w)}\rangle} = \overline{\langle e^{\int_x \lambda_x u(x;w)}\rangle}^{\text{tree}} e^{-\Gamma_1[\{u_a(x)\}]}. \quad (107)$$

Here $u_a(x)$ satisfies the tree-level saddle-point equation

$$\int_{x'} g^{-1}_{xx'}[u_a(x') - w_{x'}] - \frac{1}{T}\sum_c R'(u_{ac}(x)) = T\lambda_x\delta_{a1}. \quad (108)$$

Supposing that there are exactly two different fields u_1 and u_a for $a > 1$, this gives

$$\int_{x'} g^{-1}_{xx'}[u_1(x') - w_{x'}] + \frac{1}{T}R'(u_{12}(x)) = T\lambda_x, \quad (109)$$

$$\int_{x'} g^{-1}_{xx'}[u_2(x') - w_{x'}] - \frac{1}{T}R'(u_{21}(x)) = 0. \quad (110)$$

Note that in the first line we have dropped the term $\sim R'(0) = 0$, and the sign change comes from the factor of $(n-1)$ replicas. On the other hand, from the sum in the second line, only the term $c = 1$ survives, while all other terms are $\sim R'(0) = 0$.

As in Eq. (24), we now look for a solution $u_a = u_1 - TU$ for $a \neq 1$. Equation (109) and the difference between Eqs. (109) and (110) become

$$T\lambda_x = \int_{x'} g^{-1}_{xx'}[u_1(x') - w_{x'}] + \frac{1}{T}R'(TU(x)), \quad (111)$$

$$U_x = \int_{x'} g_{xx'}\lambda_{x'}. \quad (112)$$

In the limit of $T \to 0$, the first equation is expanded as

$$u_1(x) = w_x - R''(0)\int_{x'} g_{xx'}U(x') + O(T),$$
$$= w_x - R''(0)\int_y\int_z g_{xy}g_{yz}\lambda_z + O(T). \quad (113)$$

[6] Due to statistical tilt symmetry, the term $p = 1$ is a constant, dropped here.

This gives the tree contribution at $T=0$,

$$\overline{\langle e^{\int_x \lambda_x u(x;w)}\rangle}^{\text{tree}}_{T=0} = e^{\int_x \lambda_x w_x - \frac{1}{2}R''(0)\int_{xx'y}\lambda_x g_{xy} g_{yx'}\lambda_{x'}}. \quad (114)$$

It is a Gaussian distribution for the displacements at tree level [recall $R''(0)<0$].

Let us now compute the 1-loop corrections to mean field. Here we restrict to uniform $\lambda_x=\lambda$ and uniform $w_x=w$. Since we consider a 1-point function, $w=0$ can be chosen. The saddle point is uniform $u_a(x)=u_a$, and

$$\frac{\Gamma_1[u]}{L^d} = \frac{1}{2}\int_k \operatorname{tr}\ln\left(g_k^{-1}\delta_{ab}+M^1_{ab}\right) - \frac{1}{2}\int_k \operatorname{tr}\ln\left(g_k^{-1}\delta_{ab}+M^0_{ab}\right) + \frac{I_2}{4}\operatorname{tr}[(M^1)^2-(M^0)^2], \quad (115)$$

$$M^\kappa_{ab} = \frac{1}{T}\{\delta_{ab}[R''(u_{ac})-R''(u_{a1})]+\kappa R''(u_{ab})\}. \quad (116)$$

Here tr refers to replica indices, and we have used the saddle-point properties denoting by c any index with $c\neq 1$. More explicitly, M^κ is a replica matrix, where replica 1 is singled out, and which is symmetric in the other $n-1$ replicas $a\neq 1$. It thus has (for any n) four distinct components (denoting with a,b indices different from 1):

$$M^\kappa_{11} = \frac{1}{T}[R''(TU)+(\kappa-1)R''(0)], \quad (117)$$

$$M^\kappa_{1} = M^\kappa_{1a} = M^\kappa_{a1} = \frac{1}{T}\kappa R''(TU), \quad (118)$$

$$M^\kappa_{ab} = \delta_{ab}M^\kappa_c + M^\kappa, \quad (119)$$

$$M^\kappa_c = \frac{1}{T}[R''(0)-R''(TU)], \quad (120)$$

$$M^\kappa = \frac{1}{T}\kappa R''(0). \quad (121)$$

The field U is $U=\lambda/m^2$; see Eq. (112). The diagonalization of this matrix is performed in Appendix B. The eigenvalues are given in Eqs. (B5) and (B6). In the limit $n\to 0$ they are as follows:

(i) A (-2)-dimensional space with eigenvalue $\mu=M_c$; since M^κ_c is independent of κ, its contribution cancels between the two first lines of (115).

(ii) A 2-dimensional space with eigenvalues given in (B6). Since $\bar\mu^\kappa = M^\kappa_{11}+M^\kappa_c-M^\kappa=0$, the eigenvalues in (B6) are $\mu^\kappa=\pm\frac{1}{2}\sqrt{A^\kappa B^\kappa}$. The latter vanishes for $\kappa=1$, while for $\kappa=0$ one has $A^\kappa=B^\kappa$. Their contribution can be regrouped leading to

$$\frac{\Gamma_1[u]}{L^d} = -\frac{1}{2}\int_k\left[\ln\left(1-\frac{g_k^2}{T^2}[R''(TU)-R''(0)]^2\right) + \frac{g_k^2}{T^2}(R''(TU)-R''(0))^2\right]. \quad (122)$$

The limit of $T\to 0$ can then be taken unambiguously, i.e., independent of the sign of U:

$$\frac{\Gamma_1[u]}{L^d} = -\frac{1}{2}\int_k\left[\ln\left(1-R'''(0^+)^2 g_k^2U^2\right)+R'''(0^+)^2g_k^2U^2\right]. \quad (123)$$

This gives the final result for the characteristic function of the probability distribution of the center of mass $u(w=0)$ of the interface, to lowest order in $\epsilon=d_{\text{uc}}-d$:

$$\frac{1}{L^d}\ln\overline{\langle e^{\lambda L^d u(0)}\rangle} = -R''(0)\frac{\lambda^2}{2m^4} + \frac{1}{2}\int_k\left[\ln\left(1-R'''(0^+)^2\frac{g_k^2}{m^4}\lambda^2\right) + R'''(0^+)^2\frac{g_k^2}{m^4}\lambda^2\right]. \quad (124)$$

This result is in agreement with Eq. (G2) in [20]. At depinning, the distribution is different; see Eq. (42) of [49].

B. 2-point probabilities and avalanche-size distribution

1. General considerations

Following arguments similar to those in Sec. IV A1, the 2-point generating function can be computed as

$$\begin{aligned}&\overline{e^{\int_x\lambda_x[u(x;w/2)-w-u(x;-w/2)]}}\\ &= e^{\int_x\lambda_x[u_1^\lambda(x)-w-v_1^\lambda(x)]-\Gamma[u^\lambda,v^\lambda]}\\ &= \overline{e^{\int_x\lambda_x[u(x;w/2)-w-u(x;-w/2)]}}^{\text{tree}}\, e^{-\Gamma_1[u^{\lambda,\text{tree}},-u^{\lambda,\text{tree}}]}. \quad (125)\end{aligned}$$

In the second line, which is exact, u^λ,v^λ denote the saddle-point solutions obtained from $\Gamma[u,v]$. Here $\Gamma[u,v]=\mathcal{S}[u,v]+\Gamma_1[u,v]+O(\epsilon^2)$ is the effective action of $\mathcal{S}[u,v]$ given by (21), setting $\lambda=0$. The third line is only correct to 1-loop order (i.e., lowest order in $\epsilon=d_{\text{uc}}-d$), and requires the evaluation of Γ_1 at the tree-level saddle point; hence we can set $u^\lambda=u^{\lambda,\text{tree}}$, and $v^\lambda=v^{\lambda,\text{tree}}=-u^{\lambda,\text{tree}}$. Note that this symmetry property carries over to the effective action; hence one can set from the outset $\Gamma[u^\lambda,v^\lambda]\to\Gamma[u^\lambda,-u^\lambda]$. An important property is that the dependence on w of $\Gamma[u,v]$ is the same as the one of $\mathcal{S}[u,v]$, i.e., only through the elastic energy[7] in (21). We can thus derive (125) with respect to w, to obtain two alternative expressions:

$$\partial_{w_y}\overline{e^{\int_x\lambda_x[u(x;w/2)-w-u(x;-w/2)]}}|_{w\to 0} = -\lambda_y + \sum_a\int_{y'}\frac{g^{-1}_{yy'}}{T}u^\lambda_a(y') \quad (126)$$

$$= -\lambda_y + \sum_a\int_{y'}\frac{g^{-1}_{yy'}}{T}u_a^{\lambda,\text{tree}}(y') - \partial_{w_y}\Gamma_1\left[u_w^{\lambda,\text{tree}}\right]\Big|_{w\to 0}. \quad (127)$$

The first one (126) is exact in terms of the exact saddle point; hence only explicit derivatives with respect to w are needed. The second is expressed in terms of the tree saddle point and is true to 1-loop order. Γ_1 has no explicit w dependence; hence the last derivative acts only on the dependence on w of the tree solution (emphasized in the notation).

2. Computation of Γ_1

The general expression of $\Gamma_1[u,v]$ is given in Appendix C. Again we restrict to uniform $\lambda_x=\lambda$ and uniform $w_x=w$; thus

[7]This is because changing w results in a shift $(w_a^1(x)\to w_a^1(x)+w/2, w_a^2(x)\to w_a^2(x)-w/2)$ in the generating functional $W[w^1,w^2]$ associated with $S[u,v]$.

we only need the expression for $\Gamma_1[u,-u]$ at the uniform tree saddle point. Dropping the superscripts λ and tree, this is

$$\frac{\Gamma_1[u,-u]}{L^d} = \frac{1}{2}\mathrm{Tr}\ln\left(g^{-1}\delta_{ab}\mathbb{1} + \mathbb{M}^1_{ab}\right) - \frac{1}{2}\mathrm{Tr}\ln\left(g^{-1}\delta_{ab}\mathbb{1} + \mathbb{M}^0_{ab}\right) + \frac{I_2}{4}\mathrm{Tr}[(\mathbb{M}^1)^2 - (\mathbb{M}^0)^2], \tag{128}$$

with

$$\mathbb{M}^\kappa_{ab} = \begin{pmatrix} M^\kappa_{ab} & P^\kappa_{ab} \\ P^\kappa_{ab} & M^\kappa_{ab}\end{pmatrix}, \tag{129}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, \tag{130}$$

$$M^\kappa_{ab} = \frac{1}{T}\{\delta_{ab}[-R''(u_{a1}) + R''(u_{ac}) - R''(u_a+u_1) + R''(u_a+u_c)] + \kappa R''(u_{ab})\}, \tag{131}$$

$$P^\kappa_{ab} = \frac{1}{T}\kappa R''(u_a+u_b). \tag{132}$$

Here Tr refers to a trace over replica and u,v indices (space), and c to $c\neq 1$. More explicitly, the matrices M^κ and P^κ have for any n again each four distinct components (denoting a,b indices different from 1),

$$M^\kappa_{11} = \frac{1}{T}[R''(TU) - R''(2u_1) + R''(2u_1 - TU) + (\kappa-1)R''(0)], \tag{133}$$

$$M^\kappa_1 = M^\kappa_{1a} = M^\kappa_{a1} = \frac{\kappa}{T}R''(TU), \tag{134}$$

$$M^\kappa_{ab} = \delta_{ab}M^\kappa_c + M^\kappa, \tag{135}$$

$$M^\kappa_c = \frac{1}{T}[R''(0) - R''(TU) - R''(2u_1 - TU) + R''(2u_1 - 2TU)], \tag{136}$$

$$M^\kappa = \frac{\kappa}{T}R''(0), \tag{137}$$

and

$$P^\kappa_{11} = \frac{\kappa}{T}R''(2u_1), \tag{138}$$

$$P^\kappa_1 = P^\kappa_{1a} = P^\kappa_{a1} = \frac{\kappa}{T}R''(2u_1 - TU), \tag{139}$$

$$P^\kappa_{ab} = \delta_{ab}P^\kappa_c + P^\kappa, \tag{140}$$

$$P^\kappa_c = 0, \tag{141}$$

$$P^\kappa = \frac{\kappa}{T}R''(2u_1 - 2TU). \tag{142}$$

We recall that U and u_1 are solutions of (34).

The matrix $\mathbb{M}^\kappa$ is diagonalized in Appendix C for $n=0$. It has

(i) with multiplicity -4 the eigenvalue M^κ_c (since $P^\kappa_c = 0$). Since M^κ_c in (136) does not depend on κ, the contribution of these eigenvalues to (128) cancels between $\kappa=1$ and $\kappa=0$.

(ii) 2 eigenvalues each for $\sigma=\pm1$, of the form $\mu = \frac{1}{2}(\bar\mu_\sigma \pm \sqrt{A_\sigma B_\sigma})$, where $\bar\mu_\sigma$, A_σ, and B_σ will be calculated below.

Regrouping in Eq. (128) we get

$$\frac{\Gamma_1[u,-u]}{L^d} = \frac{1}{2}\int_k \sum_{\sigma=\pm1}\left\{\ln\left(\left[g_k^{-1} + \frac{1}{2}\bar\mu^1_\sigma\right]^2 - \frac{1}{4}A^1_\sigma B^1_\sigma\right) - \ln\left(\left[g_k^{-1} + \frac{1}{2}\bar\mu^0_\sigma\right]^2 - \frac{1}{4}A^0_\sigma B^0_\sigma\right) - \text{quadratic part}\right\}. \tag{143}$$

We find using formulas (B15), (B16) of Appendix B

$$\bar\mu^\kappa_\sigma = \frac{1}{T}[R''(2u_1 - 2TU) - R''(2u_1)](1-\sigma\kappa) = 2UR'''(2u_1)(\sigma\kappa - 1) + O(T), \tag{144}$$

$$A^1_\sigma = (\sigma-1)U^2R''''(2u_1)T + O(T^2), \tag{145}$$

$$B^1_\sigma = \frac{4}{T}[R''(0) + \sigma R''(2u_1)] + O(T^0), \tag{146}$$

$$A^0_\sigma = B^0_\sigma = \frac{1}{T}[2R''(2u_1 - TU) - R''(2u_1 - 2TU) + 2R''(TU) - R''(2u_1) - 2R''(0)], = \frac{1}{T}[2R''(TU) - 2R''(0) + O(T^2)]. \tag{147}$$

In Eq. (143) appears the product $A^0_\sigma B^0_\sigma$, which, contrary to each factor, has no ambiguity at $T=0$,

$$A^0_\sigma B^0_\sigma = 4R'''(0^+)^2U^2. \tag{148}$$

Putting everything together we obtain

$$\frac{\Gamma_1[u,-u]}{L^d} = \frac{1}{2}\int_k \left\{2\ln g_k^{-1} + \ln\left([g_k^{-1} - 2UR'''(2u_1)]^2 - 2U^2R''''(2u_1)[R''(2u_1) - R''(0)]\right) - 2\ln\left([g_k^{-1} - UR'''(2u_1)]^2 - R'''(0^+)^2U^2\right) - \text{quadratic part}\right\}, \tag{149}$$

where subtraction of terms quadratic in U, or equivalently in g_k, is indicated. Inserting into formula (125) this gives the correction to the tree expression for the 2-point generating function for arbitrary w, the distance between the points. It is expressed in terms of U and u_1 which, we recall, are solutions of (34). Since $u_1 \to 0^+$ as $w\to 0^+$ we check that $\Gamma_1[u,-u]$ indeed vanishes as $w\to 0^+$ as it should from (125). The linear term in w contains the information about avalanches as we discuss now.

3. Limit of $w \to 0^+$ and generating function of avalanche moments

We now want to use formula (127), i.e.,

$$\hat Z = L^{-d}\partial_w \overline{e^{L^d\lambda[u(w/2) - w - u(-w/2)]}}|_{w\to0^+} = -\lambda + m^2U - L^{-d}\partial_w\Gamma_1[u,-u]|_{w\to0^+}. \tag{150}$$

In Eq. (149) the dependence on w is only contained in U and u_1. Since $\Gamma_1[u,-u]$ vanishes for any U as $w\to0^+$, it is of the form $\Gamma_1[u,-u] = wf(U) + O(w^2)$; hence the dependence of U on w is not needed and we can consider U as its $w=0^+$ limit, i.e., the solution of $m^2U - R'''(0^+)U^2 = \lambda$. To obtain (150) we thus replace $u_1 = yw/2$, expanding to linear order in w using the $w=0^+$ limit given in (40), $y=[1-\frac{2R'''(0^+)}{m^4}m^2U]^{-1}$. We find $\hat{Z}=Z-\lambda$, with

$$Z=Z_{\rm tree}+yR''''(0^+)R'''(0^+)\int_k\frac{U^2}{[g_k^{-1}-2UR'''(0^+)]^2}+\frac{2U^2g_k}{g_k^{-1}-2UR'''(0^+)}-3g_k^2U^2 \tag{151}$$

with $Z_{\rm tree}=m^2U$. Specifying to $g_k^{-1}=k^2+m^2$ and rescaling $k\to km$ yields

$$Z=Z_{\rm tree}+\frac{R''''(0^+)m^{d-4}Z_{\rm tree}}{1-2S_mZ_{\rm tree}}\int_k\left[\frac{Z_{\rm tree}S_m}{(k^2+1-2Z_{\rm tree}S_m)^2}+\frac{1}{k^2+1-2Z_{\rm tree}S_m}-\frac{1}{k^2+1}-3\frac{Z_{\rm tree}S_m}{(1+k^2)^2}\right]+O(\epsilon^2), \tag{152}$$

where the terms appear in the same order as in (151). We have abbreviated the characteristic scale of avalanches

$$S_m=\frac{R'''(0^+)}{m^4} \tag{153}$$

already introduced in Eq. (14). $R''''(0^+)\sim\epsilon$ is the small expansion parameter, and as indicated subleading terms are of order ϵ^2. Equation (152) has the form

$$(Z-Z_{\rm tree})(1-2S_mZ_{\rm tree})=\epsilon\,\delta Z(Z_{\rm tree})+O(\epsilon^2), \tag{154}$$

and since $Z-Z_{\rm tree}\sim\epsilon$, it can be rewritten as

$$(Z-Z_{\rm tree})[1-S_m(Z+Z_{\rm tree})]=\epsilon\,\delta Z(Z)+O(\epsilon^2). \tag{155}$$

Rearranging gives

$$\begin{aligned}Z&=S_mZ^2+Z_{\rm tree}(1-S_mZ_{\rm tree})+\epsilon\,\delta Z(Z)+O(\epsilon^2)\\&=S_mZ^2+\lambda+\epsilon\,\delta Z(Z)+O(\epsilon^2).\end{aligned} \tag{156}$$

Explicitly, this is

$$Z=\lambda+S_mZ^2+R''''(0^+)m^{d-4}Z\times\int_k\left[\frac{ZS_m}{(k^2+1-2ZS_m)^2}+\frac{1}{k^2+1-2ZS_m}-\frac{1}{k^2+1}-3\frac{ZS_m}{(1+k^2)^2}\right]+O(\epsilon^2). \tag{157}$$

We see that ZS_m always appear together. It is therefore useful to introduce the dimensionless function $\tilde{Z}$ of the dimensionless argument λS_m,

$$\tilde{Z}(\lambda S_m):=Z(\lambda)S_m. \tag{158}$$

Inserting into the above equation yields

$$\tilde{Z}=\lambda+\tilde{Z}^2+\epsilon\tilde{I}_2R''''(0^+)m^{d-4}\frac{1}{\epsilon\tilde{I}_2}\int_k\left[\frac{\tilde{Z}^2}{(k^2+1-2\tilde{Z})^2}+\frac{\tilde{Z}}{k^2+1-2\tilde{Z}}-\frac{\tilde{Z}}{k^2+1}-3\frac{\tilde{Z}^2}{(1+k^2)^2}\right]+O(\epsilon^2), \tag{159}$$

where the combination

$$\alpha:=\epsilon\tilde{I}_2R''''(0^+)m^{d-4}\equiv\tilde{R}''''(0^+) \tag{160}$$

is the fourth derivative of the rescaled renormalized disorder, and $\tilde{I}_2$, defined in Eq. (67), is the (dimensionless) 1-loop integral used to eliminate the normalization of $\int_k$. The result (159) is equivalent to Eq. (151) of [20], noting that the force-force correlator used in Eq. (143) of [20] is $\Delta''(0)=-R''''(0^+)$.

The generalization to a more general elastic kernel is straightforward, and can be obtained replacing $k^2+1\to\tilde{g}_k^{-1}$ as detailed in Appendix E of [20], and for the contact-line experiment in [44].

C. Avalanche-size distribution

Let us recall the results for the normalized probability-distribution function $p(s)$, defined in Eq. (13), as obtained in [19,20] for standard elasticity: For $d\geqslant 4$, the tree or MF result is relevant. It reads

$$p_{\rm MF}(s)=\frac{1}{2\sqrt{\pi}s^{3/2}}e^{-s/4}. \tag{161}$$

For dimension d smaller than 4, loop corrections are relevant. The 1-loop result, obtained by inverse-Laplace transforming (159), is

$$p(s)=\frac{A}{2\sqrt{\pi}s^{\tau}}\exp\left(C\sqrt{s}-\frac{B}{4}s^{\delta}\right), \tag{162}$$

with exponents

$$\tau=\tfrac{3}{2}+\tfrac{3}{8}\alpha=\tfrac{3}{2}-\tfrac{1}{8}(1-\zeta_1)\epsilon, \tag{163}$$

$$\delta=1-\frac{\alpha}{4}=1+\frac{1}{12}(1-\zeta_1)\epsilon\ , \tag{164}$$

where $\alpha=-\frac{1}{3}(1-\zeta_1)\epsilon$ and $\zeta_1=1/3$ for the RF class, relevant to the present study. The constants A, B, and C depend on ϵ, and must satisfy the normalization conditions

$$\int_0^\infty ds\,sp(s)=1, \tag{165}$$

$$\int_0^\infty ds\,s^2p(s)=2. \tag{166}$$

At first order in ϵ they are

$$A=1+\tfrac{1}{8}(2-3\gamma_{\rm E})\alpha, \tag{167}$$

$$B=1-\alpha\left(1+\frac{\gamma_{\rm E}}{4}\right), \tag{168}$$

$$C=-\tfrac{1}{2}\sqrt{\pi}\alpha, \tag{169}$$

where $\gamma_{\rm E}=0.577216\ldots$ is Euler's constant.

V. UNCORRELATED AVALANCHES: THE BROWNIAN FORCE MODEL (BFM)

A. The BFM model

In this section we study the Brownian force model (BFM), which corresponds to a Gaussian bare disorder with a force correlator in Eq. (3) of

$$-R_0''(u)=-R_0''(0)+\sigma|u|\ ,\quad R_0'''(u)=\sigma\,{\rm sign}(u). \tag{170}$$

For a point, i.e., in $d=0$, $V'(u)$ performs a Brownian motion in u. The potential $V(u)$ is thus given by a so-called random acceleration process [50–52]. In the present framework we assume that the distribution of $V'(u)$ has statistical translational invariance; hence the model needs a regularization. It can for instance be defined in a periodic box $V(u+W) = V(u)$ with $W \to \infty$, the increments $V'(u_1) - V'(u_2)$ being those of the Brownian motion,

$$\overline{[V'(u_1) - V'(u_2)]^2} = \sigma |u_1 - u_2|. \tag{171}$$

The zero mode then has very large fluctuations; i.e., $R_0''(0) = O(W)$. The generalization to an interface is straightforward with $V'(u,x)$ being a set (indexed by x) of mutually uncorrelated Brownian motions along u. Note that a dynamical version of this model was studied in the context of nonequilibrium depinning [29,53]. In $d = 0$, it is known as the ABBM model (see [53] for a review).

A remarkable property of this model defined in the continuum is that it appears to be an exact fixed point of the FRG in any dimension d; i.e., the renormalized disorder correlator $R(u)$ (for its definition see Secs. IV A2, II and III of Ref. [20]) remains of the same form as (170). More precisely

$$R'''(u) = \sigma \, \text{sign}(u), \quad \tilde{R}'''(u) = \tilde{\sigma} \, \text{sign}(u), \tag{172}$$

where the rescaled disorder correlator was defined in (66). Its flow, i.e., its dependence on m, is given by the FRG equation

$$-m\partial_m \tilde{R}'''(u) = (\epsilon - \zeta)\tilde{R}'''(u) + \beta[R]'''(u). \tag{173}$$

The β function, taken for $u > 0$, contains only higher derivatives which vanish for (172), and this *to any loop order*. This property is detailed in Appendix I, together with a stability analysis, which shows that this fixed point is attractive. More precisely, it is at least linearly attractive up to 2-loop order. The roughness exponent for the BFM can be read off from Eq. (173) to be $\zeta = \epsilon = 4 - d$. Hence $\sigma = A_d \tilde{\sigma}$ with $A_0 = 1/4$.

At this stage, this remarkable property is not rigorously established for arbitrary d. In fact, some of the statements have to be qualified; see Appendix I. It should be considered as a (quite solid) conjecture. In $d = 0$, however, there exists some theorems, discussed below, which strengthen the case.

In $d = 0$, this model has been studied in the context of the 1D Burgers equation [35,36,40]. Let us recall the connection. It is convenient to denote space by w and consider the time-dependent velocity field $v \equiv v_t(w)$ satisfying the Burgers equation

$$\partial_t v + \tfrac{1}{2}\partial_w v^2 = \nu \partial_w^2 v \tag{174}$$

in the inviscid limit $\nu = 0^+$. It is solved via the Cole-Hopf transformation [54]

$$v_t(w) = \frac{1}{t}[w - u(w)] = \hat{V}'(w), \tag{175}$$

where $u(w)$ realizes the minimum of

$$\hat{V}(w) = \min_u \left[\frac{1}{2t}(u-w)^2 + V(u)\right]. \tag{176}$$

Hence this is exactly the disordered model in $d = 0$ with the (Burgers) "time" $t = 1/m^2$, taken at temperature $T = \nu/2 = 0^+$ (minimization condition), identical to the inviscid limit. At initial time $t = 0$, $\hat{V} = V$, hence the initial velocity field is $v_{t=0}(w) = V'(w)$. The Burgers velocity correlator thus equals the renormalized disorder correlator,

$$t^2 \, \overline{v_t(w_1)v_t(w_2)} = -R''(w_1 - w_2), \tag{177}$$

with $R = R_0$ at $t = 0$ (i.e., $m = \infty$).

Until now, these statements were completely general. The BFM corresponds to a choice of a random initial velocity $v_{t=0}(w)$ with the same increments as the Brownian motion. As a consequence, $\tilde{R}'''(u)$ is for all times given by (172).

B. Shocks in the BFM and Lévy processes

If we admit that there are no loop corrections for the BFM, then we can conjecture that the (improved) tree level (i.e., mean field) result is *exact* for the BFM in any dimension d. From the fact that $R'''(u) = \sigma$ (for $u > 0$), i.e., all higher derivatives vanish, the results of Sec. II G then show that, for $w > 0$,

$$\overline{e^{L^d \lambda [u(w/2) - w - u(-w/2)]}} = e^{L^d w \hat{Z}(\lambda)}. \tag{178}$$

This should hold in any d for the two-point correlation of the center-of-mass displacements. In (178) $\hat{Z}(\lambda)$ takes the tree expression $\hat{Z}(\lambda) = Z_{\text{tree}}(\lambda) - \lambda$ from (17)

$$\hat{Z}(\lambda) = \frac{1}{2S_m}\left(1 - 2S_m\lambda - \sqrt{1 - 4S_m\lambda}\right), \tag{179}$$

with $S_m = \sigma/m^4 = \sigma t^2$. In Appendix E we show, from our saddle-point method, that the same holds for an *arbitrary number* of ordered points $w_1 < w_2 < \cdots < w_p$,

$$\overline{e^{L^d \sum_{i=1}^p \lambda_i [u(w_i) - w_i]}} = e^{L^d \sum_{i=1}^{p-1} (w_{i+1} - w_i)\hat{Z}(\mu_i)} \tag{180}$$

with $\sum_{i=1}^p \lambda_i = 0$ and $\mu_i = -\sum_{j=1}^i \lambda_j$. It admits a more general formulation

$$\overline{e^{-L^d \int dw \, \mu'(w)[u(w) - w]}} = e^{L^d \int dw \, \hat{Z}(\mu(w))} \tag{181}$$

for any function $\mu(w)$ which vanishes at $w = \pm\infty$, derived in Appendix F. Inserting $\mu(w) = -\sum_i \lambda_i \theta(w - w_i)$ one recovers (180).

Let us now make contact with a remarkable set of results obtained by Carraro-Duchon and Bertoin for the Burgers equation, i.e., the case $d = 0$ [35,36].

We recall the definition of a (homogeneous) *Lévy process*. It is a real random function $X(w)$, continuous on the right with a limit on the left; i.e., it can have jumps. It has homogeneous and independent increments; i.e., $\{X(w_{i+1}) - X(w_i)\}_{i=1,\ldots,p}$ are independent random variables for any ordered set $w_1 < w_2 < \cdots < w_p$ and any p; and for all $w < w'$ the law of $X(w') - X(w)$ is the same as the law of $X(w' - w) - X(0)$. Its characteristic function satisfies, for $w > 0$, and $\omega \in i\mathbb{R}$,

$$\overline{e^{\omega[X(w) - X(0)]}} = e^{w\phi(\omega)}. \tag{182}$$

A Lévy process is thus fully determined by its *Lévy exponent* $\phi(\omega)$, with $\phi(0) = 0$. More generally,

$$\overline{e^{-\int dw \, \omega'(w) X(w)}} = e^{\int dw \, \phi(\omega(w))} \tag{183}$$

for any function $\omega(w)$ which vanishes at $w = \pm\infty$.[8] Equation (182) is recovered using $\omega(v) = \omega\theta(w - v)\theta(v)$. The Lévy-Khintchine theorem [55] then establishes that $X(w)$ is a

[8]In Bertoin [36] ϕ is called ψ and ω is called q.

sum of a Brownian motion (with drift) and an independent jump process, with measure $n(\mathsf{s})d\mathsf{s}$. (We use sans-serif s in order not do confuse with $s = S/S_m$ used earlier.) Here we need the case of (i) only positive or zero jumps (resp. only negative jumps); (ii) finite first and second moments $\int \max(\mathsf{s},\mathsf{s}^2)n(\mathsf{s})d\mathsf{s} < \infty$. In that case

$$\phi(\omega) = b\omega + \int_{\mathsf{s}>0} (e^{-\omega \mathsf{s}} - 1)\, n(\mathsf{s})\, d\mathsf{s}. \tag{184}$$

(The same formula holds with $\mathsf{s} \to -\mathsf{s}$ for only negative jumps.) In (184) ω can be taken in a domain of convergence which includes $\mathrm{Re}(\omega) \geqslant 0$ (but usually is larger).

A remarkable theorem by Carraro and Duchon [35] establishes that if the velocity field $X_t(w) = v_t(w)$ of the inviscid Burgers equation is a Lévy process (with only negative jumps) at initial time [with $\phi'(0) \geqslant 0$], then (i) it remains a Lévy process with only negative jumps for all times; (ii) its associated Lévy exponent $\phi_t(\omega)$ satisfies itself a Burgers equation

$$\partial_t \phi + \phi \partial_\omega \phi = 0. \tag{185}$$

We recall in Appendix F a simple-minded derivation of this formula. Its solution for $\omega > 0$ is obtained by inverting

$$\phi_t(\omega + t\phi_0(\omega)) = \phi_0(\omega), \tag{186}$$

i.e., $\phi_t(\omega) = \phi_0(h_t(\omega))$, where $h_t(\omega)$ is the inverse function of $\omega \to \omega + t\phi(\omega)$. This was applied to the case of the initial Brownian velocity

$$\phi_0(\omega) = \frac{a^2}{2}\omega^2, \tag{187}$$

leading to [35,36]

$$\begin{aligned}\phi_t(\omega) &= \frac{1 + a^2\omega t - \sqrt{1 + 2a^2\omega t}}{a^2 t^2}\\ &= \frac{\omega}{t} + \int_{\mathsf{s}<0} (e^{\omega \mathsf{s}} - 1)\, n(\mathsf{s})\, d\mathsf{s},\end{aligned} \tag{188}$$

$$n(\mathsf{s}) = \frac{1}{a\sqrt{2\pi t^3}|\mathsf{s}|^{3/2}} e^{-|\mathsf{s}|/(2a^2 t)}. \tag{189}$$

This is the same law for the shock-size distribution as the mean-field result (179) for the interface.

We can now identify the results from our present method with those in $d = 0$. Since Eq. (175) gives $v_t(w) = [w - u(w)]/t$, in $d = 0$ the process $u(w) - w$ in the BFM is a Lévy process with only positive jumps. This is consistent with the above, Eq. (179), noting

$$\hat{Z}(\lambda) = \phi_t(\omega = -t\lambda), \tag{190}$$

where we recall $t = 1/m^2$. The result (189) then gives the $P(S)$ of Eqs. (13) and (19) with $\mathsf{s} = -m^2 S$ and $a^2 = 2\sigma$.

To conclude, we conjecture that the BFM model for the interface in any d has center-of-mass displacements given by a Lévy process with positive jumps, i.e., perfectly uncorrelated shocks. In $d = 0$ this was proven in [35,36]. Since we argue that for interfaces for more general disorder (i.e., not restricted to the BFM model but with shorter-ranged correlations) the mean-field theory becomes exact for $d \geqslant d_{\mathrm{uc}}$, we conclude that at (and above) the upper critical dimension the BFM becomes a good description (with $\zeta = 0$) and the center of mass of the interface undergoes a Lévy process. The ϵ expansion then allows us to compute deviations from the independent-avalanche properties.

VI. GENERALIZATION: TREE-LEVEL DIFFERENTIAL EQUATION FOR AN ARBITRARY DISORDER CORRELATOR

We have seen in the previous section that in the case where $|R'''(u)|$ is a constant, (i) the generating functions for the joint probabilities of $u(w) - w$ at an arbitrary number of points is easily computed, from the one at 2 points, and (ii) the (Lévy) exponent of the 2-point generating function itself satisfies the Burgers equation, as shown by Carraro-Duchon [35].

Here we show an even more striking result: We find a generalization of the differential equation, satisfied at tree level, to an *arbitrary* number of points, and for *any* disorder correlator $R(u)$. This equation encodes the complete mean-field results developed in this paper. Here we show how it arises. The question of its solution, and further applications, will be examined elsewhere, but for illustration we discuss an explicit solution for *periodic disorder* at the end of this section.

A. Observable

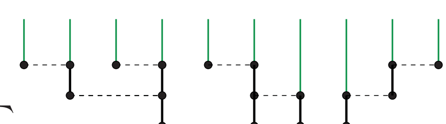
The observable of interest is

$$e^{\hat{\mathbb{Z}}_t[\lambda]} := \overline{e^{\int_w \lambda(w)[u(w)-w]}}, \tag{191}$$

a slight generalization of (75) since it is now a functional of $\lambda(w)$ (hence the notation with a square bracket) and contains information about multiple-point correlations. As in Sec. III we work in dimension $d = 0$ or equivalently (up to the volume factor of L^d) we study the center-of-mass displacement in any d. We compute it in the (improved) tree approximation; i.e., it is the sum of all connected tree graphs (for details of the graphical rules see Sec. VI C below):

$$\hat{\mathbb{Z}}_t[\lambda] := \overline{e^{\int_w \lambda(w)[u(w)-w]}}^{\,c,\mathrm{tree}}$$

$$= \sum_{\mathcal{G}} \;.\tag{192}$$



It is obtained by the expansion of $e^{\int_w \lambda(w)[u(w)-w]}$, where the external lines on the top link to the external $u(w_i) - w_i$ fields at various w_i, following the graphical rules defined in Sec. V C of [20]. It will be convenient to use the notation

$$\Omega(w) = -\lambda(w)t, \quad t = 1/m^2 \tag{193}$$

and loosely denote by the same symbol

$$\hat{\mathbb{Z}}_t[\Omega] := \hat{\mathbb{Z}}_t[\lambda = -\Omega/t]. \tag{194}$$

It contains information about all n-point cumulants $\hat{C}^{(n)}$ of the displacement, or equivalently in $d = 0$ of the Burgers velocity (see Table I for the conventions used). $\hat{\mathbb{Z}}_t[\Omega]$ can be expanded as

$$\hat{\mathbb{Z}}_t[\Omega] = \sum_{p=2}^{\infty} \frac{1}{p!} \int_{w_1,\ldots,w_p} \Omega(w_1)\ldots\Omega(w_p)\hat{C}_t^{(p)}(w_1,\ldots,w_p), \tag{195}$$

TABLE I. Conventions used for the various sources and generating functions.

$t = \frac{1}{m^2}$
$\lambda(w) = -\mu'(w) = -\frac{\Omega(w)}{t} = \frac{\omega'(w)}{t}$
$e^{\hat{\mathbb{Z}}_t[\lambda]} := \overline{e^{\int_w \lambda(w)[u(w)-w]}}$
$\hat{\mathbb{Z}}_t[\Omega] = \hat{\mathbb{Z}}_t[\lambda = -\Omega/t] = \hat{\mathbb{Y}}_t[\Omega,\mathsf{u}]\|_{\mathsf{u}(w)=w}$
$\mathbb{Z}(\lambda,w) = Z(\lambda)w + O(w^2)$
$\hat{\mathbb{Z}}(\lambda,w) = \mathbb{Z}(\lambda,w) - \lambda w$

where the cumulants

$$(-t)^p \hat{C}_t^{(p)}(w_1,\ldots,w_p) = \overline{[u(w_1)-w_1]\ldots[u(w_p)-w_p]}^c \tag{196}$$

were defined in [20]. There we have seen how to calculate them at tree level as a sum over all connected tree graphs $\mathcal{G}$ as in (192) and obtained them explicitly for $p \leqslant 4$.

B. Differential equation

Here we show that $\hat{\mathbb{Z}}_t[\Omega]$ can be obtained very elegantly from *a suitable (functional) generalization of the Carraro-Duchon equation which naturally sums up all tree graphs in the field theory*. The idea is to write an evolution equation in the variable t; hence we have emphasized the dependence on this variable.

To achieve this, one needs to generalize $\hat{\mathbb{Z}}_t[\Omega]$ into a functional $\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}]$ of two variables u and w, defined as

$$\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}] = \sum_{n=2}^{\infty} \frac{1}{n!}\int_{w_1,\ldots,w_n} \Omega(w_1)\ldots\Omega(w_n) \times \hat{C}_t^{(n)}(\mathsf{u}(w_1),\ldots,\mathsf{u}(w_n)). \tag{197}$$

Hence it depends on a background field $\mathsf{u}(w)$, not to be confused with $u(w)$, the center of mass of the manifold in a given disorder realization, the fluctuating field which is averaged over. Then the following property holds:

$\hat{\mathbb{Y}}_t[\Omega,u]$ is the solution of the flow equation

$$\partial_t \hat{\mathbb{Y}}_t[\Omega,\mathsf{u}] = -\int_w \frac{\delta}{\delta \mathsf{u}(w)}\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}]\frac{\delta}{\delta\Omega(w)}\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}] \tag{198}$$

with initial condition

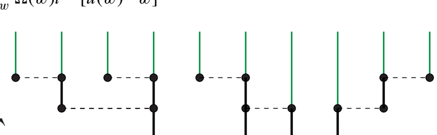
$$\hat{\mathbb{Y}}_{t=0}[\Omega,\mathsf{u}] = \frac{1}{2}\int_{w,w'} \Omega(w)\Omega(w')\Delta(\mathsf{u}(w)-\mathsf{u}(w')). \tag{199}$$

C. Graphical proof

By definition one has

$$\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}] \;\hat{=}\; \overline{e^{-\int_w \Omega(w)t^{-1}[u(w)-w]}}^{c,\text{tree}}$$

$$= \sum_{\mathcal{G}} .$$



(200)

By the notation $\hat{=}$ we mean to use the graphical rules extending the ones defined in [20] Sec. V C as follows:

(1) Draw all connected tree diagrams obtained by the expansion of $e^{-\int_w \Omega(w)t^{-1}[u(w)-w]}$.

(2) Each external point is a contracted variable $-\int_w \Omega(w)t^{-1}[u(w)-w]$, i.e., will contribute a factor of $-\int_w \Omega(w)/t$; hence it does not depend on the background field u.

(3) Each dashed line is a disorder correlator, $R(\mathsf{u}(w_1)-\mathsf{u}(w_2))$, with n_1 derivatives taken with respect to $\mathsf{u}(w_1)$ and n_2 derivatives taken with respect to $\mathsf{u}(w_2)$, where n_1 and n_2 are the number of lines entering the left and right vertex, respectively.

(4) Each solid line is a correlation function at zero momentum, $g_{q=0} = 1/m^2 = t$. All points connected with such a line have the same argument w_i. In the drawing of Eq. (200), we have distinguished external propagators, i.e., lines which end in a $\Omega(w)$ (in green/gray/thin) from internal ones (bold, black). The reason is that the factor of t on an *external* line cancels with the factor of $1/t$ which comes with each $\Omega(w)$. Thus only *internal* lines carry a factor of t.

(5) Once $\hat{\mathbb{Y}}[\Omega,\mathsf{u}]$ has been evaluated, one sets $\mathsf{u}(w) \to w$ to get $\hat{\mathbb{Z}}_t[\Omega]$,

$$\hat{\mathbb{Z}}_t[\Omega] = \hat{\mathbb{Y}}[\Omega,\mathsf{u}]\big|_{\mathsf{u}(w)=w}. \tag{201}$$

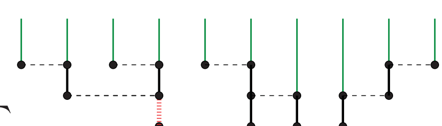
In order to allow for a recursion relation, we perform this last step only at the end.

We now show that there exists a recursion relation for derivatives with respect to *internal lines*:

Consider $\partial_t \hat{\mathbb{Y}}[\Omega,\mathsf{u}]$. Since each *internal* line carries a factor of t, graphically this means a sum over all possibilities $\mathcal{M}$ to mark an internal line (here dotted, red),

$$\partial_t \hat{\mathbb{Y}}[\Omega,\mathsf{u}]$$

$$= \sum_{\mathcal{M}}\sum_{\mathcal{G}} .$$



(202)

One realizes that above and below the marked propagator appear functional derivatives of $\hat{Y}[\Omega,\mathsf{u}]$ itself: $\frac{\delta}{\delta \mathsf{u}(w)}\hat{Y}_t[\Omega,\mathsf{u}]$ at the top, and $-\frac{\delta}{\delta\Omega(w)}\hat{Y}_t[\Omega,\mathsf{u}]$ at the bottom. This implies equation (198). The initial condition (199) is then made to recover the exact second cumulant $\hat{C}^{(2)}$, i.e., the quadratic term in the graphical expansion.

D. Consequences and particular cases

The above mean-field differential equation is remarkable in several respects. First it allows us to compute explicitly the n-point function $\hat{C}^{(n)}$ by integration of (198) in a small-t expansion. One immediately checks that the terms of order Ω^3 and Ω^4 coincide with the expressions (59)–(61) of [20].

We now analyze some special cases, for which the above equations simplify. Suppose we want to compute the p-point expectation

$$e^{\hat{\mathbb{Z}}_t(\{\omega_i,w_i\})} := \overline{e^{-\frac{1}{t}\sum_{i=1}^p \omega_i[u(w_i)-w_i]}} \tag{203}$$

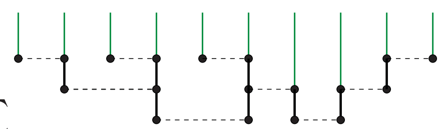
within the tree approximation. We can use the above formalism with the choice

$$\Omega(w) := \sum_{i=1}^{p} \omega_i \delta(w - w_i). \tag{204}$$

The two functions $\Omega(w)$ and $\mathsf{u}(w)$ have been replaced by the two sets of discrete variables ω_i and w_i. The variation with respect to $\mathsf{u}(w)$ gets replaced by the derivative with respect to w_i, so that one can write the recursion relation directly for $\hat{\mathbb{Z}}_t$,

$$\hat{\mathbb{Z}}_t(\{\omega_i, w_i\}) = \overline{e^{-\frac{1}{t}\sum_{i=1}^{p}\omega_i[u(w_i)-w_i]}}^{\text{c, tree}}$$

$$= \sum_{\mathcal{G}} \quad .$$



(205)

Rule (3) now gives $R(w_1 - w_2)$ instead of $R(\mathsf{u}(w_1) - \mathsf{u}(w_2))$. The functional differential Eq. (198) simplifies to an ordinary differential equation,

$$\partial_t \hat{\mathbb{Z}}_t(\{\omega_i, w_i\}) = -\sum_{i=1}^{p} \frac{\partial}{\partial w_i}\hat{\mathbb{Z}}_t(\{\omega_i, w_i\})\frac{\partial}{\partial \omega_i}\hat{\mathbb{Z}}_t(\{\omega_i, w_i\}). \tag{206}$$

The initial condition to this equation is

$$\hat{\mathbb{Z}}_{t=0}(\{\omega_i, w_i\}) = \frac{1}{2}\sum_{i,j=1}^{p} \omega_i \omega_j \Delta(w_i - w_j). \tag{207}$$

Solving Eq. (206) iteratively in powers of t reproduces again the n-point functions of [20], Eqs. (59)–(61). Indeed one also has, from (195) and (204),

$$\hat{\mathbb{Z}}_t(\{\omega_i, w_i\}) = \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n=1}^{p} \omega_{i_1}\ldots\omega_{i_n}\hat{C}_t^{(n)}(w_{i_1},\ldots,w_{i_n}). \tag{208}$$

Formula (206) simplifies even more for the $p = 2$ generating function expressed as a function of the position difference,

$$e^{\hat{\mathbb{Z}}_t(\omega,w)} := \overline{e^{-(\omega/t)[u(w)-u(0)-w]}}. \tag{209}$$

Equation (206) and the initial condition become

$$\partial_t \hat{\mathbb{Z}}_t(\omega,w) = -\frac{\partial}{\partial w}\hat{\mathbb{Z}}_t(\omega,w)\frac{\partial}{\partial \omega}\hat{\mathbb{Z}}_t(\omega,w), \tag{210}$$

$$\hat{\mathbb{Z}}_{t=0}(\omega,w) = \omega^2[\Delta(0) - \Delta(w)]. \tag{211}$$

As an example, we give the solution up to order t^2,

$$\begin{aligned}\hat{\mathbb{Z}}_t(\omega,w) &= \omega^2[\Delta(0) - \Delta(w)] + 2\omega^3 t \Delta'(w)[\Delta(0) - \Delta(w)] \\ &\quad + \omega^4 t^2[5(\Delta(0) - \Delta(w))\Delta'(w)^2 \\ &\quad - 2(\Delta(0) - \Delta(w))^2\Delta''(w)] + O(t^3).\end{aligned} \tag{212}$$

This recursion easily reproduces the 6 first connected Kolmogorov cumulants, explicitly calculated in [20], Eqs. (62)–(66).

There is an interesting property related to the expansion in w: If one writes

$$\hat{\mathbb{Z}}_t(\omega,w) = \sum_{n=1}^{\infty} w^n z_n(\omega,t), \tag{213}$$

then the equations for the $z_p(\omega,t)$, $p \leqslant n$, close; i.e.,

$$\partial_t z_1 + z_1 \partial_\omega z_1 = 0, \tag{214}$$

$$\partial_t z_2 + 2z_2\partial_\omega z_1 + z_1 \partial_\omega z_2 = 0, \tag{215}$$

$$\partial_t z_3 + 3z_3\partial_\omega z_1 + 2z_2\partial_\omega z_2 + z_1\partial_\omega z_3 = 0. \tag{216}$$

More generally

$$\partial_t z_n + \sum_{q=1}^{n} q z_q \partial_\omega z_{n-q+1} = 0, \tag{217}$$

with initial conditions $z_n(\omega, t = 0) = -\frac{1}{n!}\omega^2\Delta^{(n)}(0^+)$.

One particular solution of these equations is $z_n(\omega,t) = 0$ for $n \geqslant 2$. It corresponds to the BFM [where $\Delta^{(n)}(0^+) = 0$ for $n \geqslant 2$], discussed in the previous section, and thus to the equation (185) originally derived by Carraro and Duchon [35]. It describes a Lévy process with exponent $z_1(\omega,t) = \phi_t(\omega)$. The initial condition is

$$z_1(\omega, t = 0) = \phi_{t=0}(\omega) = -\omega^2\Delta'(0^+) = \omega^2\sigma. \tag{218}$$

z_1 is uniquely determined by its initial condition; hence we can use the result (189)

$$z_1(\omega,t) = \frac{1 + 2\sigma\omega t - \sqrt{1 + 4\sigma\omega t}}{2\sigma t^2}. \tag{219}$$

Inserting into (215) we find the general solution for z_2,

$$z_2(\omega,t) = \frac{F\left(\frac{1+\sqrt{1+4\sigma\omega t}}{4\omega\sigma}\right)}{1 + 4\omega\sigma t}. \tag{220}$$

This can be seen by introducing $a = \ln\omega$ and $b = \ln(1 + \sqrt{1 + 4\sigma t\omega})$ in which variables one gets $(\partial_a + \partial_b)z_2 = \frac{2}{e^{-b}-1}z_2$. The function $F(x)$ is determined by the initial condition as

$$F(x) = -\frac{1}{8x^2}\frac{\Delta''(0^+)}{\Delta'(0^+)^2}. \tag{221}$$

Hence

$$z_2(\omega,t) = -\frac{2\Delta''(0^+)\omega^2}{(1 + \sqrt{1 + 4\sigma\omega t})^2(1 + 4\omega\sigma t)}. \tag{222}$$

Now one can check that z_1 and z_2 reproduce the terms $O(w)$ and $O(w^2)$ in (64) obtained there by a completely different method. For the general case one can determine the z_n recursively. Their systematic study is left for the future.

E. Connection with exact RG equations

One easily sees that the generalized Carraro-Duchon equation (198) together with the definition (197) is equivalent to the following RG equation for the cumulants

$$\partial_t \hat{C}_t^{(n)}(w_1,\ldots,w_n) = -\sum_{p,q,p+q=n+1} \frac{n!}{(p-1)!(q-1)!} \times [\hat{C}_t^{(p)}(w_1,w_2,\ldots,w_p)\partial_{w_1}\hat{C}_t^{(q)}(w_1,w_{p+1},\ldots,w_n)]. \tag{223}$$

Here $[\ldots]$ means symmetrization over the n variables $w_1,\ldots,w_n$. The summation over p,q is for $p,q \geqslant 2$ in the case of STS and $p,q \geqslant 1$ in the absence of STS. One can show that if $\hat{C}^{(1)} = 0$ at $t=0$, it remains so. In that case the equation for $\hat{C}^{(n)}$ involves only $\hat{C}^{(n-1)}$. In Appendix H we recall the exact RG (ERG) equations in $d=0$, and show that neglecting one term in these equations (which corresponds to loop corrections) we indeed recover (223) which hence appears as a tree approximation

F. Including loops

It is shown in Appendix G that $\hat{\mathbb{Z}}_t[\Omega]$ satisfies a more general evolution equation

$$\partial_t \hat{\mathbb{Z}}_t = \frac{1}{2}\int_w \Omega'(w)\left[\frac{\delta^2 \hat{\mathbb{Z}}_t}{\delta\Omega(w)^2} + \frac{\delta \hat{\mathbb{Z}}_t}{\delta\Omega(w)}\frac{\delta \hat{\mathbb{Z}}_t}{\delta\Omega(w)}\right]. \tag{224}$$

This equation is exact (i.e., valid beyond the tree approximation) and equivalent to the ERG equations given in Appendix H. Neglecting the first term corresponds to the tree approximation and is equivalent to the recursion of moments (223). As shown in Appendix G2, Eq. (224) can also be obtained by replacing the tree-level equation (198) by the equivalently exact equation

$$\partial_t \hat{\mathbb{Y}}_t[\Omega,\mathsf{u}] = -\int_w \lim_{w'\to w}\left[\frac{\delta}{\delta\Omega(w')}\frac{\delta}{\delta \mathsf{u}(w)}\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}]\right] - \int_w \frac{\delta}{\delta \mathsf{u}(w)}\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}]\frac{\delta}{\delta\Omega(w)}\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}]. \tag{225}$$

The first term generates all loop corrections.

We now consider the BFM, with statistical translation invariance. In Appendix G2 we show that then a solution for $\hat{\mathbb{Z}}_t$ can be obtained from

$$\hat{\mathbb{Z}}_t = f_t(\omega_\infty) + \int_{w_1} \phi_t(\omega(w_1),\omega_\infty), \tag{226}$$

$$\omega(w) = \int_w^\infty dw'\,\Omega(w')\,, \quad \omega_\infty = \int_w \Omega(w), \tag{227}$$

$$\partial_t \phi_t(x,y) = -\phi_t(x,y)\partial_1\phi_t(x,y), \tag{228}$$

$$\partial_t f_t(y) = \frac{1}{2}[\partial_1\phi_t(0,y) + \partial_1\phi_t(y,y)], \tag{229}$$

where ∂_1 denotes the partial derivative with respect to the first argument. The initial condition for the BFM is

$$\phi_{t=0}(x,y) = \sigma x^2 - \sigma x y, \tag{230}$$

$$f_{t=0}(y) = \tfrac{1}{2}\Delta(0)y^2. \tag{231}$$

The solution of the system (228) and (229) with this initial condition is

$$\phi_t(x,y) = \frac{1}{2\sigma t^2}\left[1 + 2\sigma t\left(x - \frac{y}{2}\right) - \sqrt{1 + 4\sigma t\left(x - \frac{y}{2}\right) + \sigma^2 t^2 y^2}\right], \tag{232}$$

$$f_t(y) = \tfrac{1}{2}\ln(1 - t^2 s^2 y^2) + \tfrac{1}{2}\Delta_0(0)y^2. \tag{233}$$

The ln term corresponds to 1-loop corrections, while for this model higher loop contributions identically vanish, as discussed in Appendices G2 and I2.

G. Periodic case

In the periodic case in any dimension it is conjectured that $R''(u) - R''(0) = R'''(0^+)u(1-u)$. Noting $\sigma := R'''(0^+)$, we have $r_4 := R''''(0^+) = -2\sigma$.

Here we compute (in $d=0$ for simplicity but extension is straightforward) the most general 2-point generating function using an arbitrary value for σ. The calculation is performed in Appendix K. The general result for any function $\lambda(w) = -\mu'(w)$ on the interval $[0^-,1^-]$ [this is sufficient since $u(w) - w$ is periodic] with $\mu(0) = \mu(1)$ is

$$\overline{\langle e^{\int_w \lambda(w)[u(w)-w]}\rangle}^{\rm tree} = e^{-\mathcal{S}_\lambda}, \tag{234}$$

$$-\mathcal{S}_\lambda = -\int_0^1 dw\,[\mu(w) + m^2 V(w)] + \sigma A^2. \tag{235}$$

$$V(w) = \frac{m^2 - 2\sigma A}{2\sigma}\left(\sqrt{1 - \frac{4\sigma}{(m^2 - 2\sigma A)^2}\mu(w)} - 1\right). \tag{236}$$

A is given by the self-consistent equation

$$A = \int_0^1 dw\,\frac{m^2 V(w)}{m^2 - 2\sigma A + 2\sigma V(w)}. \tag{237}$$

From this the 2-point function is obtained by taking $\mu(w') = \lambda\theta(0 < w' < w)$:

$$\overline{\langle e^{\lambda[u(w)-w-u(0)]}\rangle}^{\rm tree} = e^{-\mathcal{S}_\lambda}, \tag{238}$$

$$-\mathcal{S}_\lambda = -w(\lambda + m^2 V) + \sigma A^2. \tag{239}$$

V and A both depend on the length of the interval w, and the function $V(w')$ introduced above is $V(w') = \theta(0 < w' < w)V$. The self-consistent equations are

$$V m^2 - 2\sigma A V + \sigma V^2 = -\lambda, \tag{240}$$

$$A(m^2 - 2\sigma A + 2\sigma V) = w V m^2, \tag{241}$$

to be solved for the branch such that $V = \frac{-\lambda}{m^2} + O(\lambda^2)$ and $A = \frac{-\lambda w}{m^2} + O(\lambda^2)$. Up to order t^{10}, or equivalently λ^6 or σ^5, this gives, denoting $t = 1/m^2$,

$$\begin{aligned}-\mathcal{S}_\lambda = &-(\lambda t)^2\sigma(w-1)w + 2(\lambda t)^3\sigma^2 t(w-1)w(2w-1)\\ &-(\lambda t)^4\sigma^3 t^2(w-1)w(24(w-1)w+5)\\ &+2(\lambda t)^5\sigma^4 t^3(w-1)w(2w-1)(44(w-1)w+7)\\ &-2(\lambda t)^6\sigma^5 t^4(w-1)w[52(w-1)w(14(w-1)w+5)\\ &+21] + O(t^{12},\lambda^7).\end{aligned} \tag{242}$$

This is in agreement both with our previous expansions in (64) (up to order w^3) and with (212).

VII. CONCLUSION

In this paper we have presented efficient algebraic tools to study multipoint correlations of the displacement field of an elastic manifold of internal dimension d in a random potential, upon variation of an external parameter. In $d = 0$ these identify with the correlations of the Burgers velocity field with random initial conditions (playing the role of the disorder). Such correlations are of interest in the field of turbulence. In both cases, they yield the statistics of avalanches, i.e., shocks in Burgers.

The first method uses replicas. The saddle-point equations obtained at $T = 0$ resum all tree diagrams and yield among others the avalanche-size distribution in the mean-field limit, i.e., for $d \geqslant d_{\rm uc}$. We have then extended this method to compute the 1-loop corrections. It allowed us to derive the 1-loop avalanche-size distribution more systematically than in our previous work [20], providing an independent check of the latter. This method has a natural extension to the dynamics, which allows us to compute the distribution of velocities in an avalanche near the depinning transition [29,31,32]. Apart from the avalanche-size distribution, other distributions, which were obtained by a resummation of diagrams, such as the width of an interface [57,58] or the distribution of critical forces at depinning [49], should now be obtainable in a purely algebraic way.

The second method arises from the study of the Brownian force model (BFM). That model has the unique property that "its mean-field treatment is exact"; i.e., summation of the tree diagram yields (almost) the exact result. We have argued that this property holds in any d. We also proved the stability, i.e., attractive character, of this model under RG to one loop, but we believe it to be valid more generally. In $d = 0$ this model identifies with the Burgers equation with a (stationary) Brownian initial condition. We recalled results from the mathematical literature: At all times the velocity field remains a Lévy process, implying that the shocks are uncorrelated. Furthermore it was shown that the Lévy exponent of this process obeys itself a Burgers evolution equation in time, the Carraro-Duchon equation. We then pointed out a more general connection between the Carraro-Duchon equation and the mean-field theory of elastic manifolds, not restricted to the BFM, which allowed us to (i) show that avalanches in elastic manifolds at and above their upper critical dimension are described by a Lévy process; and (ii) derive a generalized Carraro-Duchon functional equation, which is in essence the exact RG equation satisfied by the mean-field theory, i.e., the sum of all tree graphs. This allows us in particular to recover very efficiently most of the results of the first method at the level of the mean-field theory. Extensions including loop corrections were presented, but their study was left for the future. They should in principle lead to another, maybe more powerful method to study loop corrections.

Both methods presented here have recently been extended to a manifold with an N-component displacement field, and the results are presented in [31,33].

ACKNOWLEDGMENTS

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APPENDIX A: GENERAL NONUNIFORM w_x

In the general case of the 2-point function (45) at points $w_x = \pm\frac{w}{2} f(x)$ we need to analyze Eq. (29). Expanding for small w, we find

$$\int_{x'} g^{-1}_{xx'}[y(x') - f(x')] = 2R'''(0^+)|y(x)|U(x). \quad \text{(A1)}$$

The other equation to be satisfied is

$$\int_{x'} g^{-1}_{xx'} U(x') - R'''(0^+)\,\mathrm{sgn}(y(x))\,U(x)^2 = \lambda_x. \quad \text{(A2)}$$

These equations can first be studied as an expansion in small λ_x:

$$y(x) = f(x) + 2R'''(0^+)\int_{x'x''} g_{xx'}|f(x')|g_{x'x''}\lambda_{x''} + \dots, \quad \text{(A3)}$$

$$U(x) = g_{xx'}\lambda_{x'} + R'''(0^+)g_{xx'}\mathrm{sgn}(f(x'))(g_{x'x''}\lambda_{x''})^2 + \dots . \quad \text{(A4)}$$

The second-order part of $U(x)$ allows us to retrieve the second moment of local avalanche sizes, from (51),

$$\rho_0^f\langle S_x S_{x'}\rangle = 2R'''(0^+)g_{xx''}g_{x'x''}|f(x'')|. \quad \text{(A5)}$$

These equations were studied in [20] in the case $f(x) = 1$ and $\lambda_x = \lambda\delta(x)$. In that case $y(x) > 0$ does not change sign and the equation (A2) can be studied separately. More generally however, we see from (A3) that if $f(x)$ changes sign, $y(x)$ will also change sign (at least for small enough λ_x), but not necessarily at the same location. A general analysis of these equations demands a more thorough study.

APPENDIX B: DIAGONALIZATION OF REPLICA MATRICES

1. 1-point formulas

Consider a replica matrix M as in (117) specified by the four components M_{11}, $M_{1a} = M_{a1} =: M_1$ for $a \neq 1$, $M_{aa} =: M_c + M$, and $M_{ab} =: M$, where $a \neq b$ are two arbitrary replica indices distinct from 1. The eigenspaces can be split into two groups:

(i) A 2-dimensional subspace of vectors of the form

$$V = \begin{pmatrix} v_1 \\ v\vec{\omega} \end{pmatrix}, \quad \vec{\omega} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \quad \text{(B1)}$$

There the action of M reduces to a simple 2×2 matrix:

$$MV = \begin{pmatrix} v_1' \\ v'\vec{\omega} \end{pmatrix}, \quad \text{(B2)}$$

$$\begin{pmatrix} v_1' \\ v' \end{pmatrix} = \begin{pmatrix} M_{11} & (n-1)M_1 \\ M_1 & M_c + (n-1)M \end{pmatrix} \begin{pmatrix} v_1 \\ v \end{pmatrix}. \tag{B3}$$

(ii) $n-2$ eigenvectors associated to the eigenvalue $\mu = M_c$:

$$V_p = \begin{pmatrix} 0 \\ \vec{\omega}^{(p)} \end{pmatrix}, \quad \vec{\omega}^{(p)} = \begin{pmatrix} \omega_1^{(p)} \\ \ldots \\ \omega_{n-1}^{(p)} \end{pmatrix}, \tag{B4}$$

with $\omega_j^{(p)} = e^{pj\frac{2i\pi}{n-1}}$, ($j = 1,\ldots,n-1$; $p = 1,\ldots,n-2$), the $(n-1)$-vector constructed from the $(n-1)$th root of unity, with $\sum_{j=1}^{n-1} \omega_j^{(p)} = 0$.

To summarize, the eigenvalues and multiplicities are for $n \to 0$

$$\mu = M_c, \quad \text{multiplicity} -2, \tag{B5}$$

$$\mu = \tfrac{1}{2}(\bar{\mu} \pm \sqrt{AB}), \quad \text{multiplicity 1 for each sign}, \tag{B6}$$

with

$$\bar{\mu} = M_{11} + M_c - M, \tag{B7}$$

$$A = M_{11} + M - M_c - 2M_1, \tag{B8}$$

$$B = M_{11} + M - M_c + 2M_1. \tag{B9}$$

2. 2-point formulas

The same analysis can be repeated for the $2n \times 2n$ symmetric matrix $\mathbb{M}$, with

$$\mathbb{M}^\kappa_{ab} = \begin{pmatrix} M^\kappa_{ab} & P^\kappa_{ab} \\ P^\kappa_{ab} & M^\kappa_{ab} \end{pmatrix}. \tag{B10}$$

Since the 2×2 structure is obviously diagonalized by the symmetric $\binom{1}{1}$ and antisymmetric $\binom{1}{-1}$ combination, the task reduces to finding the eigenvalues of $P^\kappa \pm M^\kappa$; hence the first -4 eigenvalues are for $n \to 0$ obtained from (B5),

$$\mu = M_c + P_c, \quad \text{multiplicity} -2, \tag{B11}$$

$$\mu = M_c - P_c, \quad \text{multiplicity} -2. \tag{B12}$$

The remaining four eigenvalues are according to (B3) the eigenvalues of the two following 2×2 matrices

$$\mathcal{U}_+ = \begin{pmatrix} M_{11} + P_{11} & -(M_1 + P_1) \\ M_1 + P_1 & M_c - M + P_c - P \end{pmatrix}, \tag{B13}$$

$$\mathcal{U}_- = \begin{pmatrix} M_{11} - P_{11} & -(M_1 - P_1) \\ M_1 - P_1 & M_c - M - P_c + P \end{pmatrix}. \tag{B14}$$

These four eigenvalues are

$$\mu_{\sigma,\pm} = \tfrac{1}{2}(\bar{\mu}_\sigma \pm \sqrt{A_\sigma B_\sigma}), \tag{B15}$$

with $\sigma = \pm 1$, and

$$\begin{aligned} \bar{\mu}_\sigma &= M_{11} + M_c - M + \sigma(P_{11} + P_c - P), \\ A_\sigma &= M_{11} + M - M_c - 2M_1 + \sigma(P_{11} + P - P_c - 2P_1), \\ B_\sigma &= M_{11} + M - M_c + 2M_1 + \sigma(P_{11} + P - P_c + 2P_1). \end{aligned} \tag{B16}$$

APPENDIX C: $\Gamma_1[u,v]$

The general expression of $\Gamma_1[u,v]$ for 2-point observables is an extension of (105):

$$\begin{aligned} \Gamma_1[u,v] = {} & \frac{1}{2}\mathrm{Tr}\ln\left(g^{-1}\delta_{ab}\mathbb{1} - \mathbb{W}^1_{ab}\right) \\ & - \frac{1}{2}\mathrm{Tr}\ln\left(g^{-1}\delta_{ab}\mathbb{1} - \mathbb{W}^0_{ab}\right) \\ & + \frac{I_2}{4}\mathrm{Tr}[(\mathbb{W}^1)^2 - (\mathbb{W}^0)^2], \end{aligned} \tag{C1}$$

$$\mathbb{W}^\kappa_{ab} = \begin{pmatrix} W^{\kappa,uu}_{ab} & W^{\kappa,uv}_{ab} \\ W^{\kappa,vu}_{ab} & W^{\kappa,vv}_{ab} \end{pmatrix} \tag{C2}$$

with

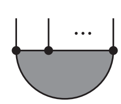
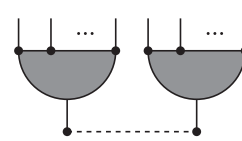
$$\begin{aligned} W^{\kappa,uu}_{ab,xy} = {} & \frac{1}{T}\delta_{xy}\Big[\delta_{ab}\sum_c R''(u_{ac}(x)) - \kappa R''(u_{ab}(x)) \\ & + \delta_{ab}\sum_c R''(u_a(x) - v_c(x))\Big], \end{aligned} \tag{C3}$$

$$\begin{aligned} W^{\kappa,vv}_{ab,xy} = {} & \frac{1}{T}\delta_{xy}\Big[\delta_{ab}\sum_c R''(v_{ac}(x)) - \kappa R''(v_{ab}(x)) \\ & + \delta_{ab}\sum_c R''(v_a(x) - u_c(x))\Big], \end{aligned} \tag{C4}$$

$$W^{\kappa,uv}_{ab,xy} = -\frac{1}{T}\delta_{xy}R''(u_a(x) - v_b(x)) = W^{\kappa,vu}_{ab,xy}. \tag{C5}$$

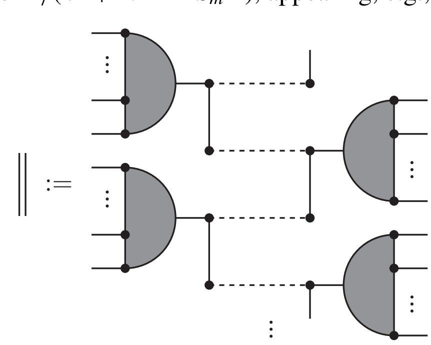
APPENDIX D: DIAGRAMMATIC REPRESENTATION OF 1-LOOP CORRECTIONS

Let us recall the graphical interpretation of the (improved) tree-level self-consistency equation. As already discussed in the text $Z_{\mathrm{MF}} = \lambda + S_m Z_{\mathrm{MF}}^2$ is graphically written as

$Z_{\mathrm{MF}} =$  $= \lambda +$  (D1)

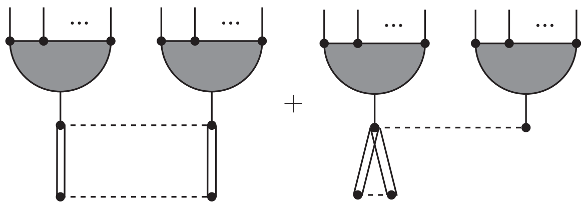
As indicated, the blob denotes Z itself. Note that we work in rescaled variables, where $\Delta'(0^+) = -1$, and in order to lighten the notation, we count the lower vertex as 1 instead of $\Delta'(0^+) = -1$, which explains the change in sign with respect to Eq. (80).

Let us now consider loop corrections. More details can be found in [20]. A graphical interpretation of the dressed propagator $1/(k^2 + m^2 - 2S_m Z)$, appearing, e.g., in (157) is

 (D2)

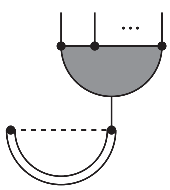
The notation on the right-hand side of the equation is as follows: The left vertex of each disorder is at 0, the right one at w. This is a graphical representation of the *antiferromagnetic* rule described in [20]. The outgoing lines all end in a factor of Z which is explicitly drawn.

There appear two classes of diagrams, corresponding to the first and second line (of the integral) in Eq. (157). The first class, denoted $\mathcal{C}_1$ in [20], can be written as

$\mathcal{C}_1 =$ 

$$= \Delta''(0)[\Delta(w)-\Delta(0)]\int_k \frac{Z^2}{(k^2+m^2-2Z)^2} + \mathcal{O}(w^2)\,. \tag{D3}$$

The diagrams on the second line (of the integral) of Eq. (159) are termed class $\mathcal{C}_2$ in [20]. They look like a correction to the critical force and can be represented as follows:

$\mathcal{C}_2 =$ 

$$= [\Delta'(w)-\Delta'(0^+)]\int_k \frac{Z}{(k^2+m^2-2Z)} - \frac{Z}{k^2+m^2}\,. \tag{D4}$$

Indeed it should be viewed as a loop of Δ', of which exactly one is expanded in w, leading to a loop with one marked vertex. This is the vertex drawn above. Note that the double line needs *at least one* $\Delta'(w)$; otherwise it cannot start at 0 and go to w as indicated. This leads to the last term in (D4) being subtracted.

If one wants the expression in terms of the renormalized disorder, one has to subtract the contribution proportional to $\int_k \frac{1}{(k^2+m^2)^2}$, giving an additional term $-[\Delta'(w)-\Delta'(0^+)]\int_k \frac{2Z}{(k^2+m^2)^2}$.

APPENDIX E: BROWNIAN FORCE MODEL: MANY-POINT CORRELATIONS

In this appendix we derive the correlation function of the center-of-mass displacement for the BFM model for an arbitrary number of points, thereby giving another derivation of its Lévy process character discussed in the text. We provide both a discrete derivation (of the p-point correlation) and a continuum one (functional average). To simplify notations, we set $d=0$, which amounts to omitting the factor of L^d, restored in the main text. Of course this is achieved within tree level, since we have argued that this be exact for the BFM.

1. Discrete calculation

We start with the tree-level equations (56), derived for arbitrary $R(u)$ and specify them to $R'''(u)=\sigma\,\mathrm{sign}(u)$. In this section, for notational simplicity we set $m^2=1$, $R'''(0^+)=\sigma=1$ and denote $\Delta(0):=-R''(0)$. We must solve the following system of equations for U_i and u_{1i}:

$$u_{1i}-w_i+\sum_j |u_{1i}-u_{1j}|U_j = \Delta(0)\sum_j U_j,$$
$$U_i+\sum_{j\neq i}\mathrm{sgn}(u_{1i}-u_{1j})U_iU_j=\lambda_i. \tag{E1}$$

Insert the result in

$$\overline{e^{\sum_i\lambda_i[u(w_i)-w_i]}}^{\text{tree}} = e^{\sum_i(\lambda_i-\frac12 U_i)(u_{1i}-w_i)}. \tag{E2}$$

We choose the w_i ordered as $w_1<w_2<\cdots<w_n$. The second equation above implies $\sum_i U_i=\sum_i\lambda_i$. Hence we can shift $u_{1i}\to u_{1i}+\Delta(0)\sum_i\lambda_i$ and eliminate $\Delta(0)$ without changing the equations. Thus the dependence on $\Delta(0)$ is trivial, and we now compute the rest setting $\Delta(0)=0$:

$$\overline{e^{\sum_i\lambda_i[u(w_i)-w_i]}}^{\text{tree}} = e^{\frac12\Delta(0)(\sum_i\lambda_i)^2}e^{-\frac12\sum_{ij}(\lambda_iU_j+\lambda_jU_i-U_iU_j)|u_{1i}-u_{1j}|}\Big|_{\Delta(0)=0}. \tag{E3}$$

The solution of the first equation of (E1) for n points satisfies

$$u_{i+1}-u_i=\frac{w_{i+1}-w_i}{1+\sum_{j=1}^i U_j-\sum_{j=i+1}^n U_j}\,,\quad 1\leqslant i\leqslant n-1.$$

The second set of equations in (E1) can be rewritten as

$$\lambda_i=U_i\left(1+\sum_{j=1}^{i-1}U_j-\sum_{j=i+1}^{n}U_j\right),\quad 1\leqslant i\leqslant n. \tag{E4}$$

Its solution is

$$2U_1=-1+\sum_{j=1}^n\lambda_j+\sqrt{\Delta_1}, \tag{E5}$$

$$2U_i=\sqrt{\Delta_i}-\sqrt{\Delta_{i-1}},\quad 2\geqslant i\geqslant n-1, \tag{E6}$$

$$2U_n=1+\sum_{j=1}^n\lambda_j-\sqrt{\Delta_{n-1}}, \tag{E7}$$

with

$$\Delta_i=1+\left(\sum_{j=1}^n\lambda_j\right)^2+2\left(\sum_{j=1}^i\lambda_j-\sum_{j=i+1}^n\lambda_j\right). \tag{E8}$$

Hence we find

$$u_{i+1}-u_i=\frac{w_{i+1}-w_i}{\sqrt{\Delta_i}}. \tag{E9}$$

This allows us to rewrite

$$-\frac12\sum_{ij}(\lambda_iU_j+\lambda_jU_i-U_iU_j)|u_{1i}-u_{1j}|$$
$$=-\sum_{i=1}^{n-1}u_{i+1,i}\left[\sum_{j=1}^i\lambda_j\sum_{j=i+1}^n U_j+\sum_{j=1}^i U_j\sum_{j=i+1}^n(\lambda_j-U_j)\right]$$
$$=-\sum_{i=1}^{n-1}\frac{w_{i+1,i}}{4\sqrt{\Delta_i}}\left[\left(\sum_{j=1}^n\lambda_j\right)^2+(1-\sqrt{\Delta_i})^2\right.$$
$$\left.+2(1-\sqrt{\Delta_i})\left(\sum_{j=1}^i\lambda_j-\sum_{j=i+1}^n\lambda_j\right)\right]. \tag{E10}$$

Using now twice the definition of Δ_i it can be rewritten and simplified to

$$\overline{e^{\sum_{i=1}^n \lambda_i [u(w_i)-w_i]}}^{\text{tree}} = e^{\frac{1}{2}\Delta(0)(\sum_i \lambda_i)^2} e^{\frac{1}{2}\sum_{i=1}^{n-1} w_{i+1,i}(1+\sum_{j=1}^{i}\lambda_j - \sum_{j=i+1}^{n}\lambda_j - \sqrt{\Delta_i})}. \tag{E11}$$

This proves formula (180) in the main text.

Consider now $X_i = u(w_i) - w_i$ and choose

$$\lambda_1 = -\mu_1 + \tfrac{1}{2}\mu, \quad \lambda_2 = \mu_1 - \mu_2, \ldots, \lambda_n = \mu_{n-1} + \tfrac{1}{2}\mu. \tag{E12}$$

We then find

$$\overline{e^{\sum_{i=1}^{n-1}\mu_i(X_{i+1}-X_i)+\frac{1}{2}\mu(X_1+X_n)}}^{\text{tree}} = e^{\frac{1}{2}\Delta(0)\mu^2 + \frac{1}{2}\sum_{i=1}^{n-1} w_{i+1,i}(1-2\mu_i - \sqrt{1+\mu^2-4\mu_i})}; \tag{E13}$$

i.e., the variables $X_{i+1} - X_i$ are still independent for fixed μ, but are not independent of $X_1 + X_n$. However, if one considers the rescaled variable $(X_1 + X_n)/\sqrt{\Delta(0)}$, then for large $\Delta(0)$ one recovers statistical independence. A similar result holds with $(X_1 + X_2 + \cdots + X_n)/\sqrt{\Delta(0)}$.

2. Continuous version

Let us consider Eqs. (60), (61) in the main text, and specify to the BFM model, with $\sigma = R'''(0^+)$. One must solve

$$m^2[u_1(w) - w] + \sigma \int_{w'} |u_1(w) - u_1(w')| U(w') = \Delta(0)\int_{w'} U(w'), \tag{E14}$$

$$m^2 U(w) + \sigma \int_{w'} \text{sign}(u_1(w) - u_1(w'))U(w)U(w') = \lambda(w), \tag{E15}$$

and insert into

$$-S_\lambda = \int_w \left[\lambda(w) - \frac{m^2}{2}U(w)\right][u_1(w) - w]. \tag{E16}$$

Here and below $\int_w = \int_{-\infty}^{\infty} dw$. We now restrict to test functions such that $\int_w \lambda(w) = 0$; hence $\int_w U(w) = 0$. We define

$$U(w) = V'(w), \quad \lambda(w) = -\mu'(w). \tag{E17}$$

Let us assume that $\mu(w)$ vanishes sufficiently fast at $w = \pm\infty$; hence the same holds for $V(w)$ and no boundary term arises in any integration by part. The first equation becomes

$$m^2[u_1(w) - w] + \sigma \int_{w'} \text{sign}(u_1(w) - u_1(w')) \times u_1'(w')V(w') = 0.$$

Now assume that

$$\text{sign}(u_1(w) - u_1(w')) = \text{sign}(w - w'), \tag{E19}$$

and take a derivative with respect to w, leading to

$$u_1'(w) = \frac{m^2}{m^2 + 2\sigma V(w)}. \tag{E20}$$

Hence monotonicity holds indeed as long as $-1 < \frac{2\sigma}{m^2}V(w)$, which we now assume, and discuss below. The second equation gives

$$V'(w)\left[m^2 + \sigma \int_{w'} \text{sign}(u_1(w) - u_1(w'))V'(w')\right] = \lambda(w). \tag{E21}$$

Using monotonicity (E19) and integration by part yields

$$V'(w)[m^2 + 2\sigma V(w)] = \lambda(w) = -\mu'(w). \tag{E22}$$

Hence

$$m^2 V(w) + \sigma V(w)^2 = -\mu(w), \tag{E23}$$

which can be solved as

$$V(w) = \frac{m^2}{2\sigma}\left[\sqrt{1 - 4\frac{\sigma}{m^4}\mu(w)} - 1\right]. \tag{E24}$$

Note that $m^2 V(w) = -Z(\mu(w))$ with Z given in (17). Integrating by parts we find

$$\begin{aligned} -S_\lambda &= \int_w \left[-\mu'(w) - \frac{m^2}{2}V'(w)\right][u(w) - w] \\ &= 2\sigma \int_w \left[-\mu(w) - \frac{m^2}{2}V(w)\right]\frac{V(w)}{m^2 + 2\sigma V(w)} \\ &= \sigma \int_w V(w)^2 = \int_w \hat{Z}(\mu(w)). \end{aligned} \tag{E25}$$

Here $\hat{Z}(\mu) = \frac{m^4}{\sigma}\hat{\hat{Z}}(\frac{\sigma}{m^4}\mu)$ and

$$\hat{\hat{Z}}(\mu) = \tfrac{1}{2}(1 - 2\mu - \sqrt{1 - 4\mu}), \tag{E26}$$

which shows formula (181) in the text. Note that the above monotonicity condition for $u_1(w)$ is equivalent to $\frac{4\sigma}{m^4}\mu(w) < 1$, the usual analyticity domain where the generating function is convergent.

3. Cumulants of $u(w) - w$ (Burgers velocity)

It is equivalently interesting to obtain, for the BFM model, the expression for the cumulants of the renormalized pinning force. These were defined in [20], Sec. III B, as

$$m^{2n}\overline{h_1 h_2 \ldots h_n}^c = L^{-(n-1)d}(-1)^n \hat{C}^{(n)}(w_1, \ldots, w_n), \tag{E27}$$

$$h_i := u(w_i) - w_i. \tag{E28}$$

We will choose $w_1 < \cdots < w_n$. For simplicity, we use dimensionless units setting $m = 1$, $R'''(0^+) = 1$, and $d = 0$, all factors being easily recovered. The lowest moments can be computed using the tree-level formula (61) in [20], which should be the exact result for the BFM model according to our conjecture, giving the following simple expressions:

$$\hat{C}^{(2)}(w_1, w_2) = w_1 - w_2 - R''(0), \tag{E29}$$

$$-\hat{C}^{(3)}(w_1, w_2, w_3) = 2(w_1 - 2w_2 + w_3), \tag{E30}$$

$$\hat{C}^{(4)}(w_1, w_2, w_3, w_4) = 3!\,(w_1 - 3w_2 + 3w_3 - w_4). \tag{E31}$$

Higher cumulants have been recalculated and we find that the general result can be written as

$$(-1)^n \hat{C}^{(n)}(w_1,\ldots,w_n) = (n-1)! \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{i+1} w_i. \tag{E32}$$

Equivalently

$$(-1)^n C_n(w_1,\ldots,w_n) = (n-1)!\, a(1-a)^{n-1}|_{a^i \to w_i}, \tag{E33}$$

where the rule $a^i \to w_i$ means to expand in powers of a, and to replace the ith power of a by w_i. This formula has been checked against the Kolmogorov cumulants $K^n(w) := \langle (h_2 - h_1)^n \rangle^c = a_n(w_2 - w_1)$ obtained in (96) of [20].

We have also checked that this result is consistent with the result for the n-point generating function (E11).

APPENDIX F: DERIVATION OF THE CARRARO-DUCHON FORMULA

In this appendix we give a physicist's derivation of Eq. (185) entering (183). For a mathematical derivation see [35]. We use the notation $\int_w = \int_{-\infty}^{\infty} dw$, and recall that our conventions are summarized in Table I.

Consider the Burgers velocity field to be a Lévy process at time t. Then the Lévy-Khintchine theorem [55] implies that

$$\overline{e^{\int_w v_t(w)\Omega(w)}} = e^{\int_w \phi_t(\omega(w))}$$
$$\Omega(w) = -\omega'(w), \quad \omega(w) = \int_w^\infty dw_1\, \Omega(w_1) \tag{F1}$$

for any function $\omega(w)$ such that $\omega(\pm\infty) = 0$. Below we also assume that $\Omega(w)$ vanishes (sufficiently fast) at infinity.

Let us assume that it remains of this form at all times, and check that it is correct provided that ϕ_t satisfies some differential equation. To show this, first take ∂_t on both sides and use the Burgers equation $\partial_t v_t(w) + \frac{1}{2}\partial_w[v_t(w)^2] = 0$. This leads to

$$\begin{aligned}
&\int_w \partial_t \phi_t(\omega(w)) e^{\int_{w'} \phi_t(\omega(w'))} \\
&= \overline{\int_w \Omega(w)\partial_t v_t(w) e^{\int_{w'} v_t(w')\Omega(w')}} \\
&= \frac{-1}{2}\overline{\int_w \Omega(w)\partial_w v_t(w)^2 e^{\int_{w'} v_t(w')\Omega(w')}} \\
&= \frac{1}{2}\overline{\int_w \Omega'(w) v_t(w)^2 e^{\int_{w'} v_t(w')\Omega(w')}} \\
&= \frac{1}{2}\int_w \Omega'(w)\frac{\delta^2}{\delta\Omega(w)^2}\overline{e^{\int_{w'} v_t(w')\Omega(w')}} \\
&= \frac{1}{2}\int_w \Omega'(w)\frac{\delta^2}{\delta\Omega(w)^2} e^{\int_{w'} \phi_t(\omega(w'))} \\
&= \frac{1}{2}\int_w\int_{w_1} \Omega'(w)\frac{\delta}{\delta\Omega(w)}\theta(w-w_1)\phi_t'(\omega(w_1)) e^{\int_{w'} \phi_t(\omega(w'))}.
\end{aligned} \tag{F2}$$

We have used that $\frac{\delta}{\delta\Omega(w)}\omega(w') = \int_{w'}^\infty dw_1 \delta(w - w_1) = \theta(w - w')$; then we obtain

$$\begin{aligned}
\int_w \partial_t\phi_t(\omega(w)) = {}& \frac{1}{2}\int_w\int_{w_1}\int_{w_2} \Omega'(w)\theta(w-w_1)\theta(w-w_2) \\
&\times \phi_t'(\omega(w_1))\phi_t'(\omega(w_2)) + \frac{1}{2}\int_w\int_{w_1} \Omega'(w) \\
&\times \theta(w-w_1)\phi_t''(\omega(w_1)),
\end{aligned} \tag{F3}$$

where we have divided by $e^{\int_{w'}\phi_t(\omega(w'))}$. Integration by parts leads to

$$\begin{aligned}
\int_w \partial_t\phi_t(\omega(w)) &= \int_w\int_{w_1} \omega'(w)\theta(w-w_1)\phi_t'(\omega(w_1))\phi_t'(\omega(w)) \\
&\quad + \frac{1}{2}\int_w \omega'(w)\phi_t''(\omega(w)) \\
&= \int_w\int_{w_1} \theta(w-w_1)\phi_t'(\omega(w_1))\frac{d}{dw}\phi_t(\omega(w)) \\
&\quad + \frac{1}{2}[\phi_t'(\omega(w))]_{-\infty}^{\infty} \\
&= -\int_w \phi_t'(\omega(w))\phi_t(\omega(w)),
\end{aligned} \tag{F4}$$

since $\omega(w)$ vanishes at infinity (and ϕ_t vanishes in zero). Since this is true for any function $\omega(w)$, it implies

$$\partial_t\phi_t(\omega) + \phi_t(\omega)\partial_\omega\phi_t(\omega) = 0. \tag{F5}$$

This is nothing but Eq. (185).

APPENDIX G: BEYOND THE CARRARO-DUCHON FORMULA: INCLUDING LOOPS

1. Evolution equation

We now give the more general evolution equation, valid beyond Lévy processes. We define, as in the text,

$$e^{\hat{\mathbb{Z}}_t} := \overline{e^{\int_w \Omega(w) v_t(w)}}, \tag{G1}$$

where $\hat{\mathbb{Z}}_t$ is *a priori* an arbitrary functional of $\Omega(w)$, and we only assume that $\Omega(w)$ vanishes at infinity [i.e., $\int_w \Omega(w)$ is not necessarily zero]. Similar manipulations as above in (F2), (F3) using the Burgers equation yield the exact evolution equation

$$\partial_t \hat{\mathbb{Z}}_t = \frac{1}{2}\int_w \Omega'(w)\left[\frac{\delta^2\hat{\mathbb{Z}}_t}{\delta\Omega(w)^2} + \frac{\delta\hat{\mathbb{Z}}_t}{\delta\Omega(w)}\frac{\delta\hat{\mathbb{Z}}_t}{\delta\Omega(w)}\right]. \tag{G2}$$

Inserting (195) and expanding in Ω, this equation provides yet another derivation of the exact RG equations (223) for the cumulants $\hat{C}^{(n)}$. (See Appendix H for a derivation using replicas.)

To make contact with our Eq. (198), we note that $\Omega(w)$ and $u(w)$ always appear together, and thus

$$\frac{\delta}{\delta u(w)}\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}] = \frac{\Omega(w)}{u'(w)}\frac{\partial}{\partial w}\frac{\delta}{\delta\Omega(w)}\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}]. \tag{G3}$$

Setting $\hat{\mathbb{Y}}[\Omega,\mathsf{u}]|_{\mathsf{u}(w)=w} \to \hat{\mathbb{Z}}_t[\Omega]$ [possible since we no longer derive with respect to $u(w)$], inserting this into (198), and integrating by part yields the second term in (G2).

The first term in (G2) corresponds to loop corrections and to the first term in the more general equation (225). One can see that these are equivalent as follows:

$$-\int_w \lim_{w'\to w}\left[\frac{\delta}{\delta\Omega(w')}\frac{\delta}{\delta \mathsf{u}(w)}\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}]\right] = -\int_w \lim_{w'\to w}\left[\frac{\delta}{\delta\Omega(w')}\frac{\Omega(w)}{u'(w)}\frac{\partial}{\partial w}\frac{\delta}{\delta\Omega(w)}\hat{\mathbb{Y}}_t[\Omega,\mathsf{u}]\right]. \quad \text{(G4)}$$

Replacing $\mathsf{u}(w)\to w$ [possible since we no longer derive with respect to $u(w)$] yields

$$\begin{aligned}
&-\int_w \lim_{w'\to w}\left[\frac{\delta}{\delta\Omega(w')}\Omega(w)\frac{\partial}{\partial w}\frac{\delta}{\delta\Omega(w)}\hat{\mathbb{Z}}_t[\Omega]\right]\\
&= -\frac{1}{2}\int_w \lim_{w'\to w}\Omega(w)\left[\left(\frac{\partial}{\partial w}+\frac{\partial}{\partial w'}\right)\frac{\delta}{\delta\Omega(w)}\frac{\delta}{\delta\Omega(w')}\hat{\mathbb{Z}}_t[\Omega]\right]\\
&= -\frac{1}{2}\int_w \Omega(w)\frac{\partial}{\partial w}\frac{\delta^2}{\delta\Omega(w)^2}\hat{\mathbb{Z}}_t[\Omega]\\
&= \frac{1}{2}\int_w \Omega'(w)\frac{\delta^2}{\delta\Omega(w)^2}\hat{\mathbb{Z}}_t[\Omega]. \qquad \text{(G5)}
\end{aligned}$$

Thus (G4) is the 1-loop correction (in $d=0$) to be added to (198).

2. Lévy processes and Brownian force model

We now study some particular solutions of the evolution equation (G2). The first one corresponds to the Lévy processes discussed above. Suppose one restricts to the case where $\int_w \Omega(w)=0$, and where $\hat{\mathbb{Z}}_t$ is a function of $\omega(w)=\int_w^\infty dw'\,\Omega(w')$,

$$\hat{\mathbb{Z}}_t = \int_w \phi_t(\omega(w)). \qquad \text{(G6)}$$

Using that $\frac{\delta}{\delta\Omega(w)}=\int dw_1\,\theta(w-w_1)\frac{\delta}{\delta\omega(w)}$, one recovers equation (F5). Equation (G2) could thus be used to study deviations from Lévy processes.

For a Lévy process, we note that the first term in (G2) vanishes when $\int_w \Omega(w)=0$, i.e., there are no loop corrections to averages of velocity differences, and the tree approximation is exact for such observables.

An interesting generalization, within Lévy processes, is to allow for $\int_w \Omega(w)\neq 0$, i.e., study observables which involve the full velocity and not simply velocity differences. In particular, we want to know the full solution for the BFM. In the case of discrete p-point correlations of the BFM this was done in Appendix E. A generalization of the Carraro-Duchon approach allows us to treat that case for continuum observables such as $\hat{\mathbb{Z}}_t[\Omega]$. An interesting output is that we will recover quite simply the full loop corrections for this model obtained via ERG in Appendix H.

We define as before $\omega(w)=\int_w^\infty dw'\,\Omega(w')$, but now $\omega(-\infty)=\omega_\infty := \int_{-\infty}^\infty dw'\,\Omega(w')$ may be nonzero, while $\Omega(w)$ still vanishes at infinity. We show that (178) is replaced by

$$\hat{\mathbb{Z}}_t = f_t(\omega_\infty) + \int_{w_1}\phi_t(\omega(w_1),\omega_\infty). \qquad \text{(G7)}$$

For the integral over w_1 to be convergent at $\pm\infty$ we need the function $\phi_t(x,y)$ to satisfy

$$\phi_t(0,y)=0, \quad \phi_t(y,y)=0, \qquad \text{(G8)}$$

which we assume from now on, and which our solution (G20) given below satisfies. Upon differentiation this also implies $\partial_2\phi_t(0,y)=0$ and $\partial_1\phi_t(y,y)+\partial_2\phi_t(y,y)=0$, which we use below.[9]

Let us prove that (G7) is indeed a solution of (G2):

$$\frac{\delta\hat{\mathbb{Z}}_t}{\delta\Omega(w)} = f_t'(\omega_\infty) + \int_{w_1}[\theta(w-w_1)\partial_1\phi_t(\omega(w_1),\omega_\infty) + \partial_2\phi_t(\omega(w_1),\omega_\infty)], \qquad \text{(G10)}$$

$$\frac{\delta^2\hat{\mathbb{Z}}_t}{\delta\Omega(w)^2} = f_t''(\omega_\infty) + \int_{w_1}\big[\theta(w-w_1)\big(\partial_1^2\phi_t(\omega(w_1),\omega_\infty) + 2\partial_1\partial_2\phi_t(\omega(w_1),\omega_\infty)\big) + \partial_2^2\phi_t(\omega(w_1),\omega_\infty)\big]. \qquad \text{(G11)}$$

Let us first compute the loop contributions, using that $\int_w \Omega'(w)=0$, and $\frac{d}{dw}[A(\omega(w),\omega_\infty)] = -\Omega(w)\partial_1 A(\omega(w),\omega_\infty)$:

$$\begin{aligned}
&\frac{1}{2}\int_w \Omega'(w)\frac{\delta^2\hat{\mathbb{Z}}_t}{\delta\Omega(w)^2}\\
&= -\frac{1}{2}\int_w \Omega(w)\big[\partial_1^2\phi_t(\omega(w),\omega_\infty) + 2\partial_1\partial_2\phi_t(\omega(w),\omega_\infty)\big]\\
&= \frac{1}{2}[\partial_1\phi_t(0,\omega_\infty) + 2\partial_2\phi_t(0,\omega_\infty)\\
&\quad -\partial_1\phi_t(\omega_\infty,\omega_\infty) - 2\partial_2\phi_t(\omega_\infty,\omega_\infty)]\\
&= \frac{1}{2}[\partial_1\phi_t(0,\omega_\infty) + \partial_1\phi_t(\omega_\infty,\omega_\infty)]. \qquad \text{(G12)}
\end{aligned}$$

We now compute the tree contribution $\frac{1}{2}\int_w \Omega'(w)\frac{\delta\hat{\mathbb{Z}}_t}{\delta\Omega(w)}\frac{\delta\hat{\mathbb{Z}}_t}{\delta\Omega(w)} = A+B$:

$$\begin{aligned}
A &= \left[f_t'(\omega_\infty) + \int_{w_1}\partial_2\phi_t(\omega(w_1),\omega_\infty)\right]\\
&\quad\times\int_{w,w_2}\Omega'(w)\theta(w-w_2)\partial_1\phi_t(\omega(w_2),\omega_\infty)\\
&= \left[f_t'(\omega_\infty) + \int_{w_1}\partial_2\phi_t(\omega(w_1),\omega_\infty)\right]\\
&\quad\times[\phi_t(0,\omega_\infty) - \phi_t(\omega_\infty,\omega_\infty)] = 0, \qquad \text{(G13)}
\end{aligned}$$

[9]Note that assuming the disorder to be statistically translational invariant (STS) can be expressed as a Ward identity

$$\int_w \Omega'(w)\frac{\delta\hat{\mathbb{Z}}_t}{\delta\Omega(w)} = 0 \qquad \text{(G9)}$$

obtained by performing the change of variables $w\to w+a$ in all w integrals appearing in $\hat{\mathbb{Z}}_t$, e.g., in its definition (195). It implies that if $\hat{\mathbb{Z}}_t$ is a solution of (G2) then $\hat{\mathbb{Z}}_t + F(\int_{-\infty}^\infty \Omega(w))$, where $F(y)$ is an arbitrary function, is also a solution, with a different initial condition. This invariance corresponds to adding the so-called Larkin random force in the language of interface pinning. One easily checks that STS is satisfied by our ansatz. Inserting (G7) into (G9) one finds that it vanishes after integration by part because of (G8).

$$\begin{aligned} B &= \frac{1}{2}\int_{w,w_1,w_2} \Omega'(w)\theta(w-w_1)\theta(w-w_2) \\ &\quad \times \partial_1\phi_t(\omega(w_1),\omega_\infty)\partial_1\phi_t(\omega(w_2),\omega_\infty) \\ &= -\int_{w,w_1} \Omega(w)\theta(w-w_1)\partial_1\phi_t(\omega(w_1),\omega_\infty) \\ &\quad \times \partial_1\phi_t(\omega(w),\omega_\infty) \\ &= -\int_w \phi_t(\omega(w),\omega_\infty)\partial_1\phi_t(\omega(w),\omega_\infty). \end{aligned} \tag{G14}$$

Hence we find that if the unknown functions $\phi_t(x,y)$ and $f_t(y)$ satisfy the following two equations, then (G2) is satisfied:

$$\partial_t\phi_t(x,y) = -\phi_t(x,y)\partial_1\phi_t(x,y), \tag{G15}$$

$$\partial_t f_t(y) = \tfrac{1}{2}[\partial_1\phi_t(0,y) + \partial_1\phi_t(y,y)]. \tag{G16}$$

Consider now the BFM. The initial condition is

$$\phi_{t=0}(x,y) = \sigma x^2 - \sigma x y, \tag{G17}$$

$$f_{t=0}(y) = \tfrac{1}{2}\Delta(0)y^2. \tag{G18}$$

This can be seen by rewriting

$$\begin{aligned} Z_0[\Omega] &= \frac{1}{2}\int_{w_1,w_2} \Omega(w_1)\Omega(w_2)\Delta(w_1-w_2) \\ &= \frac{1}{2}\Delta(0)\left(\int\Omega\right)^2 - \sigma\int_{w_1>w_2}\Omega(w_1)\Omega(w_2)(w_1-w_2) \\ &= \frac{1}{2}\Delta(0)\left(\int\Omega\right)^2 - \sigma\int_{w_1>w_2}\omega(w_1)\Omega(w_2) \\ &\quad + \sigma[\omega(w_1)\Omega(w_2)(w_1-w_2)]_{w_1=w_2}^{w_1=\infty}. \end{aligned} \tag{G19}$$

Using that $\omega(\infty)=0$, the boundary term is zero and one obtains (G17), (G18).

The solution of the system (G15)–(G16) with this initial condition is

$$\begin{aligned} \phi_t(x,y) &= \frac{1}{2\sigma t^2}\left[1 + 2\sigma t\left(x - \frac{y}{2}\right)\right. \\ &\quad \left. - \sqrt{1 + 4\sigma t\left(x-\frac{y}{2}\right) + \sigma^2 t^2 y^2}\,\right], \end{aligned} \tag{G20}$$

$$f_t(y) = \tfrac{1}{2}\ln(1 - t^2 s^2 y^2) + \tfrac{1}{2}\Delta_0(0)y^2. \tag{G21}$$

One checks that $\phi_t(x,y)$ satisfies the conditions (G8), and that this recovers the loop corrections obtained by the ERG method in (H8)–(H13).

APPENDIX H: ERG EQUATIONS AND THEIR MEAN-FIELD VERSION

It is instructive to compare the equations (223) with the known exact RG equations; we restrict here to $d=0$. In Ref. [25] Sec. IV A2 (arXiv version) the ERG equations for the cumulant of the renormalized potential $\hat S^{(n)}(w_1,\dots w_n) = (-1)^n \overline{\hat V(w_1)\dots\hat V(w_n)}^c$ were obtained. It was shown that the function

$$\hat U(w_a) = \sum_n \frac{1}{n!T^n}\sum_{a_1,\dots a_n}\hat S^{(n)}(w_{a_1},\dots w_{a_n}) \tag{H1}$$

satisfies

$$2\partial_t\hat U = T\sum_a \partial_{w_a}^2\hat U + \sum_a\left(\partial_{w_a}\hat U\right)^2. \tag{H2}$$

Hence the $\hat S^{(n)}$ satisfy at $T=0$

$$\begin{aligned} &2\partial_t\hat S^{(n)}(w_1,\dots w_n) \\ &\quad = n\left[\hat S^{(n+1)}_{110\dots0}(w_1,w_1,\dots w_n)\right] + \sum_{p,q,p+q=n+1}\frac{n!}{(p-1)!(q-1)!} \\ &\qquad \times\left[\hat S^{(p)}_{10\dots0}(w_1,\dots w_p)\hat S^{(q)}_{10\dots0}(w_1,w_{p+1}\dots w_n)\right], \end{aligned} \tag{H3}$$

where we have used that

$$\begin{aligned} &\sum_{a_1,\dots,a_p}\partial_{w_a}^2\hat S^{(p)}(w_{a_1},\dots,w_{a_p}) \\ &\quad = p\sum_{a_1,\dots,a_p}\delta_{aa_1}\hat S^{(p)}_{20\dots0}(w_a,w_{a_2},\dots,w_{a_p}) + p(p-1) \\ &\qquad \times\sum_{a_1,\dots,a_p}\delta_{aa_1}\delta_{aa_2}\hat S^{(p)}_{110\dots0}(w_a,w_a,w_{a_3},\dots,w_{a_p}). \end{aligned} \tag{H4}$$

We now use that $\hat C^{(n)}(w_1,\dots,w_n) = (-1)^n\partial_{w_1}\dots\partial_{w_n}\hat S^{(n)}(w_1,\dots,w_n)$ to obtain

$$\begin{aligned} \partial_t\hat C^{(n)}(w_1,\dots,w_n) &= -\frac{n}{2}[\partial_{w_1}\hat C^{(n+1)}(w_1,w_1,\dots,w_n)] \\ &\quad - \frac{1}{2}\sum_{p,q,p+q=n+1}\frac{n!}{(p-1)!(q-1)!} \\ &\quad \times[\partial_{w_1}(\hat C^{(p)}(w_1,\dots,w_p) \\ &\quad \times\hat C^{(q)}(w_1,w_{p+1},\dots,w_n))]. \end{aligned} \tag{H5}$$

This formula works when the $\hat C$ are continuous functions of their arguments.[10] Hence we see here that if one takes out the first term one recovers exactly the mean-field RG equation (223) of the main text. Hence this provides a further derivation of this equation.

In the STS case the lowest-order ERG equations (including loops) are

$$\begin{aligned} \partial_t\hat C^{(2)}(w_1,w_2) &= -\tfrac{1}{2}\partial_{w_1}\hat C^{(3)}(w_1,w_1,w_2) \\ &\quad - \tfrac{1}{2}\partial_{w_2}\hat C^{(3)}(w_1,w_2,w_2), \end{aligned} \tag{H6}$$

$$\begin{aligned} \partial_t\hat C^{(3)}(w_1,w_2,w_3) &= -\tfrac{1}{2}\partial_{w_1}\hat C^{(4)}(w_1,w_1,w_2,w_3) \\ &\quad - \tfrac{1}{2}\partial_{w_2}\hat C^{(4)}(w_1,w_2,w_2,w_3) \\ &\quad - \tfrac{1}{2}\partial_{w_3}\hat C^{(4)}(w_1,w_2,w_3,w_3) \\ &\quad - \partial_{w_1}\hat C^{(2)}(w_1,w_2)\hat C^{(2)}(w_1,w_3) \\ &\quad - \partial_{w_3}\hat C^{(2)}(w_2,w_3)\hat C^{(2)}(w_1,w_3) \\ &\quad - \partial_{w_2}\hat C^{(2)}(w_1,w_2)\hat C^{(2)}(w_2,w_3). \end{aligned} \tag{H7}$$

An exact solution *including loop corrections* exists for all $\hat C^{(n)}$ in the STS-Brownian case (stationary BFM discussed in

[10] Here we study only the case with STS (translation invariance). The non-STS case, e.g., a two-sided Brownian force landscape starting at zero, violates this condition at the origin and requires a special treatment, which is left for the future.

Sec. V). It reads

$$\hat{C}^{(1)}(w_1) = 0, \tag{H8}$$

$$\hat{C}^{(2)}(w_1,w_2) = \sigma(w_1 - w_2) - \sigma^2 t^2, \tag{H9}$$

$$\hat{C}^{(3)}(w_1,w_2,w_3) = -2t\sigma^2(w_1 - 2w_2 + w_3), \tag{H10}$$

$$\hat{C}^{(4)} = 3!t^2\sigma^3(w_1 - 3w_2 + 3w_3 - w_4) - 6\sigma^4 t^4, \tag{H11}$$

$$\hat{C}^{(5)} = -4!t^3\sigma^4(w_1 - 4w_2 + 6w_3 - 4w_4 + w_5), \tag{H12}$$

$$\hat{C}^{(6)} = 120t^4\sigma^5(w_1 - 5w_2 + 10w_3 - 10w_4 + 5w_5 - w_6) - c_6\sigma^6 t^6. \tag{H13}$$

One has thus a constant part $-c_n\sigma^n t^n$ with $c_n = (n-1)!$ for n even and $c_n = 0$ for n odd. If one sets c_n to zero one recovers the expressions in (E33). The fact that this simple exact solution exists in that case is a consequence of the property that $\Gamma = \mathcal{S}$ in the sense discussed in Sec. I2.

APPENDIX I: FRG PROPERTIES OF THE BFM MODEL

1. Stability of the BFM fixed point

Let us express the FRG equation using the rescaled force correlator $\tilde{\Delta}(u) = -\tilde{R}''(u)$ defined in (66). To one loop (first 2 lines) and 2 loops (third line) the FRG flow for $\tilde{\Delta}'(u)$, derived in [15–17], is

$$\begin{aligned}
-\frac{m\partial}{\partial m}\tilde{\Delta}'(u) &= (\epsilon - \zeta)\tilde{\Delta}'(u) + \zeta u\tilde{\Delta}''(u) \\
&\quad -3\tilde{\Delta}'(u)\tilde{\Delta}''(u) - \tilde{\Delta}'''(u)[\tilde{\Delta}(u) - \tilde{\Delta}(0)] \\
&\quad +\frac{1}{2}\partial_u^3[(\tilde{\Delta}(u) - \tilde{\Delta}(0))(\tilde{\Delta}'(u)^2 - \tilde{\lambda}\Delta'(0)^2)]
\end{aligned} \tag{I1}$$

with $\tilde{\lambda} = 1$ for the statics (the BFM model studied here) and $\tilde{\lambda} = -1$ for depinning (the ABBM model generalized to an interface). In both cases, there is a fixed point corresponding to $\zeta = \epsilon$ with

$$\tilde{\Delta}'(u) = -\tilde{\sigma}\,\mathrm{sign}(u), \quad \tilde{\Delta}(0) - \tilde{\Delta}(u) = \tilde{\sigma}|u|. \tag{I2}$$

We note that 1- and 2-loop corrections identically vanish for the flow of $\tilde{\Delta}'(u)$ at this fixed point—even for its generalization to an N-component field [59]. This is in fact more general, and we claim it to be true to all loop orders. Indeed, it is easy to see that higher loops bring more derivatives, hence vanishing contributions. For the statics, it can be checked to four loops in $d = 0$ [25]. Hence we conjecture that this property holds for any d.

Note that some parts of the effective action are flowing. For instance, one has, in the BFM model,

$$-m\partial_m\tilde{\Delta}(0) = -\epsilon\tilde{\Delta}(0) - \tilde{\sigma}^2. \tag{I3}$$

Hence $\tilde{\Delta}(0) = Cm^\epsilon - \frac{\tilde{\sigma}^2}{\epsilon}$, i.e., $\Delta(0) = C - \frac{\tilde{\sigma}^2 m^{-\epsilon}}{\epsilon}$, and the constant $C = \Delta_0(0)$ has to be chosen sufficiently large. This is consistent with the fact that the model, defined with statistical translational invariance, must be defined with a regularization, e.g., a periodic box of size much larger than any other scale, as discussed in the text. Another regularization would be to choose a Brownian force with origin at $u = 0$, i.e., $V'(0) = 0$, but this leads to a different FRG equation which we leave for future investigations.

Let us now show that the above fixed point is *attractive*, at least for a class of perturbations (defined precisely below) which are at most of the same range as the BFM model. It thus defines a universality class in any d. The stability analysis is performed to first order in $\epsilon = 4 - d$, within the 1-loop FRG equation. We look for perturbations of the form

$$\tilde{\Delta}'(u) = -\tilde{\sigma} + g(u). \tag{I4}$$

Physically acceptable solutions for $g(u)$ must vanish at infinity and have a regular Taylor expansion in powers of $|u|$ at $u = 0$. One then obtains to linear order in $g(u)$

$$-m\partial_m g(u) = (\epsilon - \zeta)g(u) + \zeta u g'(u) + 3\tilde{\sigma}g'(u) + g''(u)\tilde{\sigma}|u|. \tag{I5}$$

Using $\zeta = \epsilon$ we can rescale $a \to \epsilon a$, $\tilde{\sigma} \to \epsilon\tilde{\sigma}$, and $u \to \tilde{\sigma}u$. Thus one must solve for $u > 0$

$$-m\partial_m g(u) = (u+3)g'(u) + ug''(u) = -ag(u). \tag{I6}$$

As indicated, we search for eigenvalues a, where $a > 0$ are stable and $a < 0$ unstable modes. We find a general solution, noting $L_n^a(x)$ the generalized Laguerre-L polynomial and $U(a,b,z)$ the confluent hypergeometric function

$$g(u) = C_1 e^{-u}U(3-a,3,u) + C_2 e^{-u}L_{a-3}^2(u). \tag{I7}$$

The first solution $e^{-u}U(3-a,3,u)$ behaves as u^{-2} at small u (not physically acceptable), except when $a - 3 = 0,1,\ldots$ is a positive integer, in which case the two solutions are linearly dependent. Hence we set $C_1 = 0$. The remaining solution is the second independent function, indexed by a,

$$g_a(u) := e^{-u}L_{a-3}^2(u) \equiv L_{-a}^2(-u), \tag{I8}$$

which has a regular Taylor expansion at $u = 0$ for all a. It is thus a physically acceptable solution. For noninteger a,

$$g_a(u) \sim u^{-a} \quad \text{for} \quad u \to \infty. \tag{I9}$$

Thus for $a > 0$ this is a long-ranged (attractive) perturbation of (I2). The cases of integer $a > 0$ must be treated separately, because then $L_{-a}^2(-u)$ either vanishes identically, or is a short-ranged solution (see below). Taking the derivative of Eq. (I6) with respect to a with the solution (I8) in mind yields

$$-m\partial_m[\partial_a g_a(u)] = -a[\partial_a g_a(u)] - g_a(u). \tag{I10}$$

For $a = 1,2$, $g_a(u)$ vanishes, and $\partial_a g_a(u)$ is a long-range correlated eigenfunction of the RG flow. For $a = 3,4,5,\ldots$ we can restrict our analysis to the 2-dimensional space spanned by $g_a(u)$ and $\partial_a g_a(u)$. It has a Jordan block structure and the general solution of the flow equation is

$$\tilde{g}_a(u) = c_1(m)[\partial_a g(u)]|_{m=m_0} + c_2(m)g(u)|_{m=m_0} \tag{I11}$$

with

$$c_1(m) = \left(\frac{m}{m_0}\right)^a c_1(m_0), \tag{I12}$$

$$c_2(m) = \left(\frac{m}{m_0}\right)^a\left[c_2(m_0) + \ln\left(\frac{m}{m_0}\right)c_1(m_0)\right]. \tag{I13}$$

Note that for $a = 3,4,5,\ldots \partial_a g_a(u) \sim u^{-a}$ for $u \to \infty$, whereas $g_a(u)$ is short-ranged, as we discuss now:

$$g_{a=3}(u) = e^{-u}, \tag{I14}$$

$$g_{a=4}(u) = e^{-u}(3-u), \tag{I15}$$

$$g_{a=5}(u) = e^{-u}(6-u)(2-u). \tag{I16}$$

The functions $g_a(u)$ for $a = 1,2$ vanish. For negative a, there are polynomial solutions, consistent with the asymptotic behavior (I9),

$$g_{a=-1}(u) = 3 + u, \tag{I17}$$

$$g_{a=-2}(u) = \tfrac{1}{2}(2+u)(6+u), \tag{I18}$$

$$g_{a=-3}(u) = \frac{u^3}{6} + \frac{5u^2}{2} + 10u + 10. \tag{I19}$$

These solutions are unstable and physically unacceptable, since they grow stronger than $|u|$ at large u. They correspond to models with even longer-ranged correlations than the BFM.

For instance the leading short-ranged eigenmode $a = 3$ reads

$$\Delta(0) - \Delta(u) = \epsilon|u| + b(1 - e^{-|u|}) = \int \frac{dq}{2\pi}\left(\frac{2\epsilon}{q^2} + \frac{2b}{q^2+1}\right)[1 - \cos(qu)]; \tag{I20}$$

hence it has a positive Fourier transform (as long as $b > -\epsilon$), and thus corresponds to a physical disorder direction.

The question remains whether we have found *the complete spectrum for all physically allowed perturbations*. We argue that this is indeed the case: First of all, we have found a complete basis for short-ranged perturbations, the functions $g_a(u)$ for integer a. Functions decaying as a *power law* can be expanded in the basis $g_a(u)$ for noninteger a, or $\partial_a g_a(u)$ for integer a. As perturbations depending on m, they decay to 0, either as a power law (noninteger a) or with additional logarithmic corrections (integer a). In conclusion, the BFM fixed point is stable with respect to perturbations of $R'''(u)$ which decay at least as a power law at infinity.

2. More on loop expansion

To obtain a deeper understanding of the properties of the BFM we must look at its replicated effective action functional $\Gamma[u]$, with $u \equiv \{u_a(x)\}_{a=1,\ldots,n;x\in\mathbb{R}^d}$. The statement that the (improved) tree level is exact means that for any replica field $u_a(x)$,

$$\Gamma[u] = \mathcal{S}_R[u]. \tag{I21}$$

The flow of the *exact* $R(u)$ was discussed above, with the claim that $R'''(u) = \sigma\,\mathrm{sign}(u)$ does not flow (i.e., is independent of m), while $R''(0)$ flows, but its flow is unimportant. (It is part of the regularization required to define the model.)

Let us examine the meaning and validity of the property (I21). The *exact* RG flow (as a function of m in any d) of the p-replica part of $\Gamma[u]$, $S^{(p)}(u_{12,\ldots,1p})$ was written in [25]; see Eqs. (384), (385) of the arXiv version. We want to study the force correlator (corresponding in $d = 0$ to the Burgers velocity), hence look at $\partial_{u_1}\ldots\partial_{u_p}S^{(p)}(u_{12,\ldots,1p}) \equiv \partial^p S^{(p)}$. By analogy with the second moment, where we found that $\partial^3 S^{(2)} = R'''$ does not flow (away from coinciding arguments), we want to examine $\partial^{p+1}S^{(p)}$. If we assume that all $\partial^4 R$ and higher derivatives vanish, it is clear from these equations that the feeding term for this quantity *vanishes*. The natural conclusion is thus that $\partial^{p+1}S^{(p)}$ is zero for all $p \geqslant 3$ for the BFM model. Hence (I21) holds *in the sense of derivatives*, i.e., up to terms $O(u^p)$ in the p-replica terms. These terms are called random force terms since they can be set to zero by a STS transformation $u^a(x) \to u^a(x) + g_{xx'}f(x')$ where $f(x)$ is a (Larkin) random force, coupling linearly to the displacement field. In $d = 0$ this is sufficient to ensure that the tree calculation is exact for the BFM model. Recently we also showed this property for the dynamics in $d = 0$ [29].

In $d > 0$ one should worry about x (i.e., space) dependent fields, i.e., nonlocal parts of the functionals $S^{(p)}$. From Eq. (104) (see also (461) of [25], arXiv version) where the complete local+nonlocal two-replica functional $p = 2$ is obtained from the exact RG equations to order R^2, one finds for the BFM model

$$R''_{xy}[u_{ab}] = \sigma^2 g_{xy}^2\,\mathrm{sign}(u_{ab}(x))\mathrm{sign}(u_{ab}(y)). \tag{I22}$$

If we take a third derivative it vanishes away from the singular points. Thus naively there is no nonlocal part for $\partial^3 S^{(2)}$ and the same will be true for higher p. This leaves open the question of how the derivatives act on the singular points (where two replica fields coincide for some values of their argument x). We will not attempt to answer this question here, but leave it for future research. However, we emphasize that the fact that the tree level is exact for “sufficiently reasonable,” i.e., uniform or nearly uniform, field configurations is sufficient for computing, e.g., center of mass observables using tree-level formulas. This can be seen from (I22) since if $u_{ab}(x)$ does not change sign, i.e., the replicas are in partially ordered configurations, taking a third derivative again gives zero. Hence we can safely assume that (I21) holds in any d for (i) the needed derivatives of the p-replica part and (ii) partially ordered configurations. This is sufficient to argue that tree calculations are exact for the BFM model in most applications.

APPENDIX J: TOY MODEL: MARKOVIAN AND POISSON PROCESS FOR AVALANCHES

In this appendix we describe two simple toy models: (i) avalanche positions being a Markov process and (ii) avalanche positions and sizes being a totally uncorrelated process (Poisson process).

First consider a Markovian model where the location w_n of avalanche n depends only on the previous one, with the “waiting time” or interval between avalanche $\ell = w_{n+1,n} = w_{n+1} - w_n$ distributed according to a distribution $Q(\ell)$ with $\int_0^\infty d\ell\, Q(\ell) = 1$. Given that a first avalanche occurs in w_1, the probability that the n subsequent ones occur in $\prod_{i=2}^n [w_i, w_i + dw_i]$ is thus $Q(w_{n,n-1})\ldots Q(w_{2,1})dw_2\ldots dw_n$, also normalized to unity. Also assume statistical translation invariance with a uniform density of avalanches, noted ρ_0; hence a given avalanche can occur anywhere with the same probability.

For this model one shows that the probability that the interval $[0,w]$ with $w > 0$ contains n avalanches and that their positions are $0 < w_1 < w_2 < \cdots < w_n < w$ is given for $n \geqslant 1$ by

$$p_w^{(n)}(w_1,\ldots,w_n) = \frac{1}{\langle \ell \rangle}\int dw_0 dw_{n+1}\theta(-w_0) \times \theta(w_{n+1}-w)\prod_{i=0}^{n} Q(w_{i+1,i}) = \frac{1}{\langle \ell \rangle}\tilde{Q}(w_1)\prod_{i=1}^{n-1} Q(w_{i+1,i})\tilde{Q}(w-w_n), \quad \text{(J1)}$$

with $\tilde{Q}(w) = \int_w^\infty d\ell Q(\ell)$. To prove this one notes that the probability that the origin belongs to an interval of size $w_{1,0}$ is $\frac{w_{1,0}Q(w_{1,0})}{\langle \ell \rangle}$; hence the probability that the first positive shock occurs in $[w_1, w_1 + dw_1]$, $w_1 > 0$, is $\frac{1}{\langle \ell \rangle}\int dw_0 \frac{\theta(w_{1,0}-w_1)}{w_{1,0}}\frac{w_{1,0}Q(w_{1,0})}{\langle \ell \rangle} = \frac{1}{\langle \ell \rangle}\tilde{Q}(w_1)$. Inserting the factors $1 = \prod_{i=1}^\infty [\theta(w-w_i) + \theta(w_i - w)]$ and $\prod_{j=1}^\infty Q(w_{j+1,j})$ and expanding one gets (J1). The probabilities that there are n avalanches in $[0,w]$ are thus

$$p^{(n)} = \int dw_1 \ldots dw_n \, p_w^{(n)}(w_1,\ldots,w_n), \quad \text{(J2)}$$

$$p_w^{(0)} = \frac{1}{\langle \ell \rangle}\int_w^\infty dw_1 \int_{w_1}^\infty dw_{10} Q(w_{10}) = \frac{1}{\langle \ell \rangle}\int_w^\infty dw_1 (w-w_1) Q(w_1). \quad \text{(J3)}$$

These expression are easier written Laplace transformed, and one finds for $p_n(s) = \int_0^\infty dw\, e^{-ws} p_w^{(n)}$

$$p_n(s) = \frac{1}{\langle \ell \rangle s^2}(1-Q(s))^2 Q(s)^{n-1} \quad \text{for } n \geqslant 1, \quad \text{(J4)}$$

$$p_0(s) = \frac{1}{\langle \ell \rangle s^2}(Q(s) - 1 + s\langle \ell \rangle), \quad \text{(J5)}$$

with $Q(s) = \int_0^\infty dw\, e^{-sw} Q(w)$ and $\langle \ell \rangle = -Q'(0)$, which satisfy the normalization $\sum_{n=0}^\infty p_n(s) = 1/s$; i.e., $\sum_{n=0}^\infty p^{(n)} = 1$.

The case where avalanches are fully independent events, i.e., a Poisson process of density ρ_0, corresponds to $Q(w) = \rho_0 e^{-\rho_0 w}$ and $\langle \ell \rangle = 1/\rho_0$. Then $Q(s) = \frac{\rho_0}{s+\rho_0}$ and one finds $p_n(s) = \frac{\rho_0^n}{(s+\rho_0)^{n+1}}$ and, not surprisingly,

$$p^{(n)} = \frac{1}{n!}(\rho_0 w)^n e^{-\rho_0 w} \quad \text{(J6)}$$

for $n \geqslant 0$, i.e., the Poisson distribution for the number of shocks in the interval. More precisely one finds

$$p_w^{(n)}(w_1,\ldots,w_n) = \rho_0^n e^{-\rho_0 w}\theta(0 < w_1 < \cdots < w_n < w), \quad \text{(J7)}$$

i.e., a uniform distribution for the positions of the shocks.

Let us now add information about avalanche sizes. Assume that the process contains only positive jumps $u(w) - u(0) = \sum_\alpha S_\alpha \theta(0 < w_\alpha < w)$; then

$$\overline{e^{\lambda[u(w)-u(0)]}} = \sum_{n=0}^\infty \int_{w_i,S_i} p_w^{(n)}(w_1,\ldots,w_n;S_1,\ldots,S_n) e^{\lambda(S_1+\cdots+S_n)}. \quad \text{(J8)}$$

If we further assume that the avalanche sizes are independent events uncorrelated with their location,

$$p_w^{(n)}(w_1,\ldots,w_n;S_1,\ldots,S_n) = p_w^{(n)}(w_1,\ldots,w_n)\prod_{i=1}^n P(S_i), \quad \text{(J9)}$$

where $P(S)$ is a normalized probability distribution, one finds

$$\overline{e^{\lambda[u(w)-u(0)]}} = \sum_{n=0}^\infty p_w^{(n)} \langle e^{\lambda S}\rangle^n, \quad \text{(J10)}$$

$$\langle e^{\lambda S}\rangle := \int dS\, P(S) e^{\lambda S}. \quad \text{(J11)}$$

The case of a general $Q(w)$ can be solved in Laplace,

$$\int_0^\infty dw\, e^{-sw}\overline{e^{\lambda[u(w)-u(0)]}} = \frac{1}{s} + \frac{1}{\langle \ell \rangle s^2}\frac{[1-Q(s)][\langle e^{\lambda S}\rangle - 1]}{1 - \langle e^{\lambda S}\rangle Q(s)}. \quad \text{(J12)}$$

For the Poisson process, we find that (J12) simplifies into $[s + \rho_0(1 - \langle e^{\lambda S}\rangle)]^{-1}$; hence

$$\overline{e^{\lambda[u(w)-u(0)]}} = e^{\rho_0 w \int dS\, P(S)(e^{\lambda S}-1)} = e^{w\int dS\, \rho(S)(e^{\lambda S}-1)} = e^{wZ(\lambda)} \quad \text{(J13)}$$

in agreement with our general result (we have set $d = 0$). Although here we have assumed ρ_0 finite (which is the case, e.g., in numerical simulations [28]) the above formula remains valid when the total density of shocks is infinite as long as the density for a given size $\rho(S)$ [also noted $n(s)$ in the text] is finite and $\int dS\, S\rho(S)$ is finite. One then recovers the Lévy process formula for the case of only positive jumps. For a proper mathematical formulation see [35,36].

We observe that in the Poisson case (for $d = 0$) $Z(\lambda) \to -\rho_0 L^{-d}$ at large negative λ. This is dominated by the probability that there is no avalanche in the interval w. This limit can also be written as $Z(-\infty) = -1/\langle S\rangle$. Hence the mean-field formula (17) is valid only for $\lambda > -1/S_{\min}$, where $S_{\min}$ is a typical small-scale cutoff for the avalanche size, as discussed in Sec. V E of [20].

APPENDIX K: PERIODIC CASE

We now solve Eqs. (60) and (61) for the case where $R(u)$ is periodic. It is sufficient to choose $\lambda(w) = -\mu'(w)$ on the interval $[0,1]$ with $\mu(0) = \mu(1)$. Indeed since in the periodic case we know that $u_1(w) - w$ is periodic of period 1, we can write

$$\overline{\langle e^{\int_w \lambda(w)[u_1(w)-w)]}\rangle} = \overline{\langle e^{\int_0^1 dw\,[u_1(w)-w)]\tilde{\lambda}(w)}\rangle}, \quad \text{(K1)}$$

$$\sum_n \lambda(w+n) = \tilde{\lambda}(w), \quad \text{(K2)}$$

with $\int_0^1 dw\, \tilde{\lambda}(w) = 0$. Using $u_1(w+n) = u_1(w) + n$ one finds for $w \in [0,1]$ that Eqs. (60) and (61) hold where all integrals are over $w' \in [0,1]$ and $U(w) \to \tilde{U}(w)$. We define again for $w \in [0,1]$

$$\tilde{\lambda}(w) = -\mu'(w), \quad \tilde{U}(w) = V'(w). \tag{K3}$$

Note that since $\tilde{\lambda}(0) = \tilde{\lambda}(1)$ and $\int_0^1 dw\, \tilde{\lambda}(w) = 0$ one has $\mu(0) = \mu(1)$ and similarly $V(0) = V(1)$. The second equation in (60) gives after integration by part

$$V'(w)\left[m^2 + \int_0^1 dw' R''''(u_1(w) - u_1(w'))u_1'(w')V(w')\right] = -\mu'(w). \tag{K4}$$

The boundary term $[R'''(u_1(w) - u_1(w'))V(w')]_0^1$ vanishes only if the integral goes from 0^- to 1^-, i.e., contains the delta function at 0, a convention which we use here. Noting that for the periodic case

$$R''''(u) = R''''(0) + \sum_n 2R'''(0^+)\delta(u-n), \tag{K5}$$

we obtain for $w \in [0,1]$

$$\left[m^2 + R''''(0)\int_0^1 dw' u_1'(w')V(w') + 2R'''(0^+)V(w)\right] \times V'(w) = -\mu'(w). \tag{K6}$$

Integration yields

$$V(w)\left[m^2 + R''''(0)\int_0^1 dw' u_1'(w')V(w')\right] + R'''(0^+)V(w)^2 = -\mu(w). \tag{K7}$$

(Note that a possible integration constant has been dropped since it does not enter any physical observable.) The first equation in (60) gives after integration by part

$$m^2[u_1(w) - w] + \int_0^1 dw' R'''(u_1(w) - u_1(w'))u_1'(w')V(w') = 0. \tag{K8}$$

Taking a derivative and using again Eq. (K5), we finally arrive at the system of two equations:

$$u_1'(w) = \frac{m^2}{m^2 + R''''(0)\int_0^1 dw' u_1'(w')V(w') + 2R'''(0^+)V(w)}, \tag{K9}$$

$$V(w)\left[m^2 + R''''(0)\int_0^1 dw' u_1'(w')V(w')\right] + R'''(0^+)V(w)^2 = -\mu(w). \tag{K10}$$

Noting $\sigma = R'''(0)$ and $r_4 = R''''(0)$ we obtain, multiplying $u_1'(w)$ from (K9) with $V(w)$ given by (K10),

$$u_1'(w)V(w) = \frac{-m^2\mu(w)}{[m^2 + r_4 A + 2\sigma V(w)][m^2 + r_4 A + \sigma V(w)]}. \tag{K11}$$

We have defined $A = \int_0^1 dw'\, u'(w')V(w')$. We can now close the system of equations. This gives the result quoted in the text with $r_4 = -2\sigma$. The action becomes

$$\begin{aligned} -S_\lambda &= \int_0^1 dw \left[-\mu'(w) - \frac{m^2}{2}V'(w)\right][u_1(w) - w] \\ &= \int_0^1 dw \left[\mu(w) + \frac{m^2}{2}V(w)\right][u_1'(w) - 1] \\ &= -\int_0^1 dw\,[\mu(w) + m^2 V(w)] - \frac{1}{2}r_4 A^2 \end{aligned} \tag{K12}$$

as quoted in the text.

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