

Recognition of stable distribution with Lévy index α close to 2

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We address the problem of recognizing α -stable Lévy distribution with Lévy index close to 2 from experimental data. We are interested in the case when the sample size of available data is not large, thus the power law asymptotics of the distribution is not clearly detectable, and the shape of the empirical probability density function is close to a Gaussian. We propose a testing procedure combining a simple visual test based on empirical fourth moment with the Anderson-Darling and Jarque-Bera statistical tests and we check the efficiency of the method on simulated data. Furthermore, we apply our method to the analysis of turbulent plasma density and potential fluctuations measured in the stellarator-type fusion device and demonstrate that the phenomenon of the L-H transition from low confinement, L mode, to a high confinement, H mode, which occurs in this device is accompanied by the transition from Lévy to Gaussian fluctuation statistics.

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I. INTRODUCTION

Stable distributions, also called α -stable, or Lévy stable [1], are ubiquitous in nature due to the generalized central limit theorem. It says that the stable distributions, like the Gaussian one, attract distributions of sums of independent identically distributed random variables [2]. For this reason, Lévy stable distributions naturally appear when the evolution of a system or result of an experiment is determined by a sum of many random factors. An important parameter characterizing stable distribution is its index of stability, or the Lévy index α , $0 < \alpha \leq 2$. If α is strictly less than 2, the probability density function (PDF) of the stable law exhibits slowly decaying power law asymptotic behavior of the form $|x|^{-1-\alpha}$ (“heavy tail”), hence, the variance diverges. For this reason, the stable PDFs appear naturally in the description of random processes with large outliers, far from equilibrium. The limiting case $\alpha = 2$ gives Gaussian distribution possessing fast decaying tails and finite variance. In general, each stable distribution is characterized by four parameters: Lévy index α , skewness parameter β , $-1 \leq \beta \leq 1$, scale parameter γ , and location parameter δ . For the Gaussian distribution β is irrelevant, and it is characterized by standard deviation and mean. The comprehensive theory of α -stable Lévy distributions is presented in monographs [3–5].

Some measurable quantities obey Lévy statistics exactly, such as, for example, the gravitational field of masses distributed randomly in space, hitting times for one-dimensional Brownian motion, and points of arrival in the two-dimensional one [6]; interesting examples can also be found in [3,7]. Lévy statistics may also appear asymptotically due to the generalized central limit theorem like, for example, in non-Brownian random walks with jumps and/or waiting times obeying heavy-tailed distributions (see the reviews [8–10]). We mention here such illuminating examples as circulation of dollar bills [11], Lévy flights for light in fractal medium, called Lévy glass [12], and Lévy-like behavior of the marine vertebrates in response to patchy distribution of food resources [13]. Stably distributed random noises are observed in such diverse applications as plasma physics (density and electric field fluctuations [14,15]), stochastic climate dynamics [16], physiology (heartbeats [17]), electrical engineering [18], biology [19], and economics [20]. These and a lot of other observations of stably (or stablelike) distributed quantities require reliable methods of random data analysis which allow one to detect stable distributions (or distributions which belong to their domains of attraction) and estimate their parameters.

There are at least three procedures for estimating Lévy stable law parameters: (i) the maximum likelihood method based on numerical approximation of the Lévy stable likelihood function [21]; (ii) the quantile method using tabulated quantiles of Lévy stable laws [22]; and (iii) the method using regression on the sample characteristic function [23]. All presented methods work well assuming that the sample under consideration is indeed Lévy stable. The regression method is both fast and accurate, thus we use it in our analysis below.

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However, if the data come from a different distribution, these procedures may mislead more than the Hill [24] and direct log-log scale tail estimation method [25,26], which focus only on the tail behavior without assuming the parametric form for the whole distribution function.

The present paper, motivated by the analysis of plasma data (see Sec. V below), addresses the issue of recognizing stable distributions with the Lévy index close to 2. In this case, if the data sample is not large enough, the shape of empirical PDF is close to a Gaussian, and both the log-log scale analysis and the Hill estimator give an overestimated value of the Lévy index for the number of observation less than 10^6 [27] (see also examples in Figs. 3–7 of Ref. [25]). In applications, especially in physics and biology, the number of observations is often less. In particular, we deal with plasma data containing a few thousand data points. Thus, one has to be cautious in that domain of the Lévy indices. Here we propose a certain testing algorithm which, in our belief, could be helpful for the data analysis in that case. It combines “visual inspection” of Lévy-type behavior with a few statistical tests, namely, the Anderson-Darling and Jarque-Bera. Being direct and relatively simple, they, however, are rarely used in physical and biological applications, to our knowledge. We demonstrate effectiveness of the proposed testing procedure by using simulated data with known Lévy stable distribution. As an application, we analyze the PDFs of plasma fluctuations measured in the fusion plasma device and detect an interesting change of plasma fluctuation statistics from Lévy stable to a Gaussian one in the so-called L-H transition which occurs in the plasma device during operation.

The structure of the paper is as follows. In Sec. II we first plot empirical PDFs for simulated data with the Lévy index close to two and 5000 observations, and make sure that the difference from the Gaussian distribution is almost not visible. Then we plot the cumulative fourth moment of the simulated data as a function of the observation number and demonstrate that this simple visual test may help to distinguish between the Lévy stable and Gaussian PDFs. In Sec. III we outline the algorithm used to distinguish between the Lévy stable and Gaussian distributions and demonstrate how this algorithm works for simulated data. This is accompanied by a short sketch of the statistical tests which we employ for the analysis. In Sec. IV we apply the testing algorithm for the analysis of the experimental data from plasma physics. Finally, in Sec. V we add a few remarks on the two issues: the proposed testing procedure and change of statistics observed in Sec. IV.

II. SIMULATIONS

In order to demonstrate behavior of the Lévy stable distribution with the Lévy index close to 2 we first turn to the analysis of simulated samples. The procedures for simulating stable random variables one can find in [5,27]. In Fig. 1 we present in the log-log scale the right tail of the empirical PDF of the symmetric stable distribution with parameters $\alpha = 1.98$, $\beta = 0$, $\sigma = 1$, and $\mu = 0$ together with the empirical PDF of the Gaussian distribution with mean equal to zero and variance equal to 2. Let us mention that the Gaussian distribution with zero mean and variance equal to 2 corresponds to the Lévy stable one with $\alpha = 2$, $\sigma = 1$, and $\mu = 0$ (the β parameter is

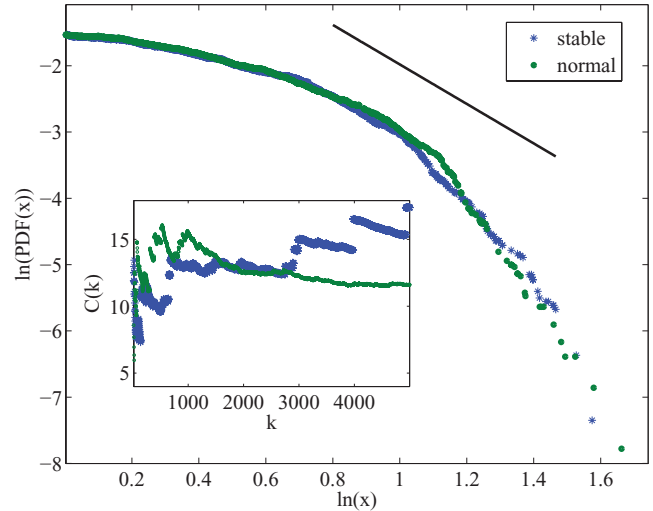


FIG. 1. (Color online) The two empirical PDFs calculated for simulated samples from Lévy stable distribution with parameters $\alpha = 1.98$, $\beta = 0$, $\sigma = 1$, and $\mu = 0$, and from Gaussian distribution with mean equal zero and variance equal 2, are shown in the log-log scale. Number of observations in both samples is 5000. The straight line shows the asymptotic slope of the Lévy stable PDF with $\alpha = 1.98$. (Inset) Empirical cumulative fourth moment for the samples considered.

irrelevant in this case). In both examined samples the number of observations is set to 5000. Moreover, in Fig. 1 we also show the straight line with the slope parameter -2.98 corresponding to the asymptotics of the Lévy stable PDF with $\alpha = 1.98$. As we observe, the difference between the two empirical PDFs is hardly visible. Therefore, we propose to use a simple visual test to distinguish between the distributions. Namely, we calculate the empirical cumulative fourth moment (ECFM) of the simulated data sets, which for a sample of observations $\{x_1, \dots, x_n\}$ is defined as follows:

$$C(k) = \frac{1}{k} \sum_{i=1}^k (x_i - \bar{x})^4, k = 1, 2, \dots, n, \quad (1)$$

where \bar{x} is the mean of the random sample.

The formula (1) can be calculated for any sample obtained from an arbitrary probability distribution. For a fixed k it forms a random variable. For distributions with finite fourth moment (e.g., Gaussian), the ECFM, as a function of k , converges to a constant, whereas for distributions with infinite fourth moment (e.g., Lévy stable with $\alpha < 2$) it diverges to infinity. The latter, for a finite sample, can be observed as an irregular chaotic behavior. Such reasoning stands for the idea of the first step of our algorithm. This simple test, which is presented in Fig. 1 (inset), clearly indicates that there is a noticeable difference between Lévy and Gaussian distributions. For the former the ECFM does not tend to a constant value with number k of observations increasing, and behaves chaotically, while for the latter one for large numbers of observations it goes to the theoretical fourth moment that in this case is equal to 12. Now, if we take stably distributed variables with smaller Lévy index, say, $\alpha = 1.9$, and the same total number of observations n , then the difference between the empirical Lévy stable PDF and Gaussian PDF becomes more visible; this is shown in

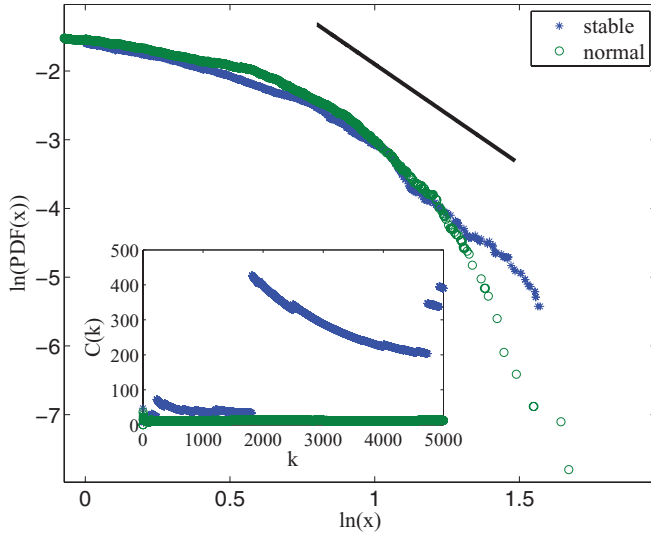


FIG. 2. (Color online) Empirical PDF calculated for a simulated sample from the Lévy stable distribution with parameters $\alpha = 1.9$, $\beta = 0$, $\sigma = 1$, and $\mu = 0$ and for the Gaussian distribution (with mean equal zero and variance equal 2) in the log-log scale. The number of observations in both samples is 5000. The straight line shows the asymptotic slope of the Lévy stable PDF with $\alpha = 1.9$. (Inset) Empirical cumulative fourth moment for the samples considered.

Fig. 2. Of course, in this case we also observe the chaotic behavior of the ECFM for stably distributed variables (see the inset in Fig. 2).

In Fig. 3 we demonstrate the right tail of the empirical PDF (in the log-log scale) for totally skewed Lévy distribution with $\alpha = 1.98$, $\beta = 1$, $\sigma = 1$, and $\mu = 0$. The number of observations is the same as in Figs. 1 and 2 (i.e., $n = 5000$). Similarly to the symmetric case $\beta = 0$, in Fig. 3 we also plot

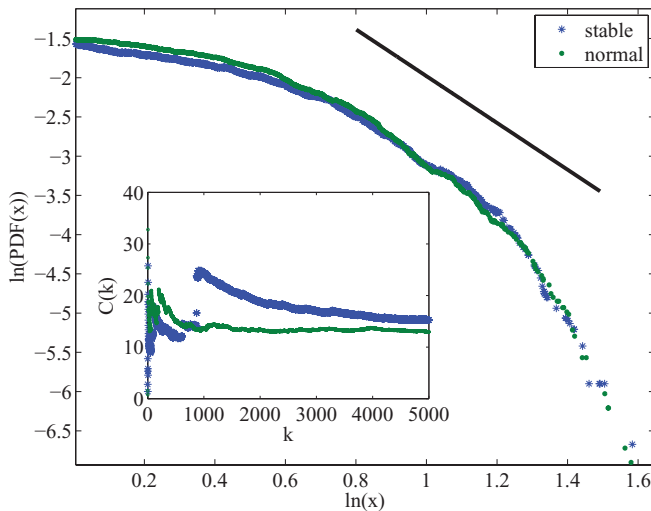


FIG. 3. (Color online) Empirical PDF calculated for a simulated sample from the Lévy stable distribution with parameters $\alpha = 1.98$, $\beta = 1$, $\sigma = 1$, and $\mu = 0$ and for normal distribution (with mean equal zero and variance equal 2) in the log-log scale. The number of observations in both samples is 5000. The straight line shows the asymptotic slope of the Lévy stable PDF with $\alpha = 1.9$. (Inset) Empirical cumulative fourth moment for the samples considered.

the PDF of the Gaussian distribution with mean equal zero and variance equal to 2, and the slope of the asymptotics of the Lévy stable PDF. The difference between the considered PDFs is almost not visible, similar to Fig. 1. However, in contrast to Fig. 1, the visual test based on ECFM does not indicate clearly a difference between the analyzed distributions, since for both empirical PDFs the ECFM tends to a constant (see the inset in Fig. 3). This example demonstrates that obviously there is a need to use a more advanced technique for recognition of the Lévy stable distribution with α close to 2.

III. PROCEDURE OF TESTING

In this section we present an algorithm on how to recognize the difference between the Gaussian distribution and the Lévy stable distribution with the index α close to 2. It is based on the empirical fourth moment studied in detail in Sec. II and Anderson-Darling and Jarque-Bera statistical tests.

Goodness-of-fit statistical tests, in general, are constructed to check whether a hypothetical distribution can be rejected for the given data or not. Their idea basically relies on checking how probable is reaching the value of some statistic calculated for the data if the underlying distribution was the hypothetical one.

Statistics usually measure the distance between the empirical and the fitted analytical cumulative distribution functions. For a sample of observations $\{x_1, \dots, x_n\}$ the empirical cumulative distribution function $F_n(x)$ is a piecewise constant function starting from zero with jumps of size $1/n$ at points x_i . The distance is usually measured either by a supremum or a quadratic norm [28].

The most well-known supremum statistic is the Kolmogorov-Smirnov (KS) statistic. It is just the supremum of the set of distances:

$$D = \sup_x |F_n(x) - F(x)|, \quad (2)$$

where $F(x)$ is the analytical cumulative distribution function.

The Cramer-von Mises family of statistics incorporates the idea of the quadratic norm. It is defined by

$$Q = n \int_{-\infty}^{\infty} \{F_n(x) - F(x)\}^2 \psi(x) dF(x), \quad (3)$$

where $\psi(x)$ is a suitable function which gives weights to the squared difference $\{F_n(x) - F(x)\}^2$. When $\psi(x) = [F(x)\{1 - F(x)\}]^{-1}$, Eq. (3) yields the Anderson-Darling (AD) statistic.

The tests derived from the KS and AD statistics are called Kolmogorov-Smirnov and Anderson-Darling, respectively. It is well known that the KS test exhibits poor sensitivity to deviations from the hypothetical distribution that occurs in the tails, whereas the AD test is considered one of the most powerful when the fitted distribution departs from the true distribution in this area [28]. Since the fit in the tails is of crucial importance in the stable case, AD will be our choice when testing for the non-Gaussian stable law.

For testing the Gaussianity we propose to use the standard Jarque-Bera (JB) test [29], which is different from the KS and AD. It is a goodness of fit test of whether sample data have the skewness and kurtosis matching a normal distribution. The JB

statistic is defined as

$$J = \frac{n}{6} \left(S^2 + \frac{(K - 3)^2}{4} \right), \quad (4)$$

where S and K are the sample skewness and kurtosis, respectively, namely

$$S = \frac{1/n \sum_{i=1}^n (x_i - \bar{x})^3}{\left(\sqrt{1/n \sum_{i=1}^n (x_i - \bar{x})^2} \right)^3}, \quad (5)$$

$$K = \frac{1/n \sum_{i=1}^n (x_i - \bar{x})^4}{\left(\sqrt{1/n \sum_{i=1}^n (x_i - \bar{x})^2} \right)^2}. \quad (6)$$

Equations (4)–(6), similarly to Eq. (1), can be used for any sample from an arbitrary probability distribution. The value of the JB statistic given by Eq. (4) forms a random variable which converges to zero if the underlying distribution has skewness zero and kurtosis 3 (e.g., Gaussian). Any deviation from zero skewness and kurtosis equal to 3 increases the JB statistic. For distributions with infinite kurtosis (e.g., Lévy stable with $\alpha < 2$) it diverges to infinity. The test is quite standard and implemented in various numerical packages, like, for example, R or MATLAB.

To employ any of the tests, first, we need to estimate the parameters of the hypothetical distribution. In the Gaussian case the standard method is the maximum likelihood, whereas in the non-Gaussian stable case we suggest the use of the regression method estimates [23,30].

The general test of fit is structured as follows. The hypothesis is that a specific distribution is acceptable, whereas the alternative is that it is not. Small values of its test statistic T are evidence in favor of the hypothesis; large ones indicate its falsity. To see how unlikely such a large outcome would be if the hypothesis is true, we calculate the p value by: $p \text{ value} = P(T \geq t)$, where t is the statistic value for a given sample. It is typical to reject the hypothesis when a small p value is obtained, like, for example, below 3% or 5%. To calculate p values for KS and AD tests we apply the procedure proposed in [31] and described in detail in [32]. The p values for the JB test were evaluated in MATLAB using its standard procedure “jbstest.”

Now, we can combine the AD and JB tests with the test based on ECFM, to introduce an algorithm for recognizing the α -stable Lévy distribution with the Lévy index close to 2.

In the first step of the algorithm, we propose to observe the ECFM of the sample. If the ECFM tends to a constant, we check for the Gaussian distribution by using the JB test. If its p value exceeds the confidence level (usually 5%), then we can assume the underlying distribution of the time series is Gaussian. In this case we estimate its parameters by using the standard maximum likelihood estimation method. If the JB test shows the data cannot be modeled by a Gaussian distribution, then we test them for the Lévy stable distribution. If the AD test gives the p value that exceeds the confidence level, then we can assume the time series can be described by the α -stable distribution. In this case we estimate its parameters via the regression approach [30].

If the ECFM exhibits chaoticlike behavior, then we test for the Lévy stable distribution. If the p value is greater than the

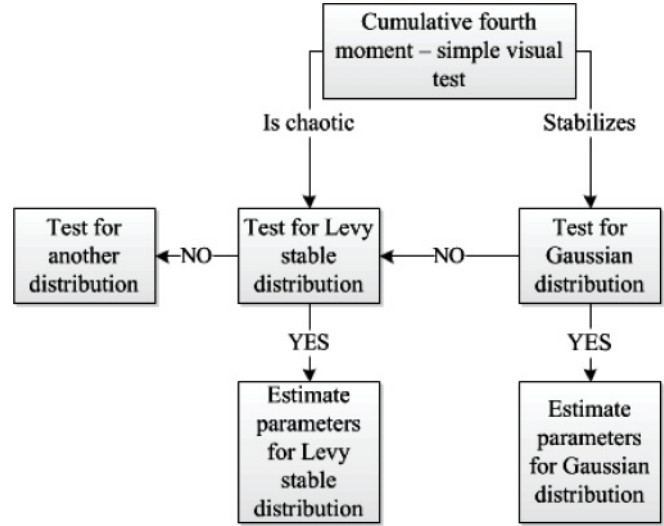


FIG. 4. Schematic algorithm for recognition of Lévy stable distribution with Lévy index close to 2.

confidence level, then we can assume the data follow the Lévy stable law. Its parameters are regression method estimates. A scheme of the whole procedure is depicted in Fig. 4.

In order to illustrate the procedure we apply it to the simulated random samples from Sec. II. The obtained results are presented in Table I. In the last two columns we demonstrate the values of the JB and AD statistics, respectively, together with the corresponding p values (in parenthesis).

For the sample from the Lévy stable distribution with parameters $\alpha = 1.98$, $\beta = 0$, $\sigma = 1$, and $\mu = 0$ we can observe chaotic behavior of the ECFM (see Fig. 1). Therefore, according to the procedure, we test its stability using the AD test statistic. The obtained p value indicates the time series constitutes independent identically distributed (i.i.d.) Lévy stable random variables.

As we observe in Fig. 2, the ECFM for the sample from symmetric Lévy stable distribution with $\alpha = 1.9$ and $\sigma = 1$ behaves chaotically, therefore we use the AD statistic here.

TABLE I. Illustration of the introduced procedure for simulated samples from Sec. II. The last two columns contain values of the appropriate statistics, calculated by Eqs. (2)–(6), and the corresponding p values (in parentheses).

Sample distribution	ECFM tends to const?	JB stat. (p value)	AD stat. (p value)
Lévy $\alpha = 1.98, \beta = 0, \sigma = 1, \mu = 0$	No	—	0.24 (0.64)
Lévy $\alpha = 1.90, \beta = 0, \sigma = 1, \mu = 0$	No	—	0.88 (0.42)
Lévy $\alpha = 1.98, \beta = 1, \sigma = 1, \mu = 0$	Yes	61.33 (<0.01)	0.18 (0.86)
Gaussian $\mu = 0, \sigma = \sqrt{2}$	Yes	0.55 (0.5)	—

The obtained p value clearly indicates the underlying stable distribution.

The ECFM for simulated data from the totally skewed stable distribution with $\alpha = 1.98$ for a large number of observations tends to a constant (see Fig. 3); therefore first we test the Gaussian law hypothesis. The JB test indicates the time series cannot be modeled by the normal distribution, therefore we check for the Lévy stable by using the AD test that justifies the stable distribution.

For the last simulated sample from the normal distribution we observe that the ECFM tends to a constant. The obtained p value clearly indicates that the underlying distribution of the analyzed time series is Gaussian.

The algorithm proposed in the paper provides a procedure to distinguish between the case $\alpha \approx 2$ and $\alpha = 2$. In order to distinguish the case of $\alpha \approx 2$ and $\beta = 1$ with $\alpha \approx 2$ and $\beta = 0$ one has to analyze the estimated skewness parameter β . However, we note that the confidence interval constructed for this parameter can be quite wide if the number of observations is not large enough. This circumstance makes the difference not definite.

IV. APPLICATIONS

In this section we investigate the data obtained in experiment on the controlled thermonuclear fusion device. Namely, we analyze the statistical properties of plasma fluctuations before and after the so-called L-H transition phenomenon, that is, sudden transition from the low confinement mode (L mode) to a high confinement mode (H mode) accompanied by suppression of turbulence and a rapid drop of turbulent transport at the edge of the thermonuclear device [33]. The implementation of the H-mode regime, which is chosen as the operating mode for the future ITER device [34], requires detailed investigation of the transition physics. Electric Langmuir probes represent an effective diagnostic tool for this purpose [35]. The important characteristics of edge plasma turbulence, such as fluctuation amplitudes, spectra, and turbulence-induced transport are investigated in the Uragan-3M (U-3M) stellarator torsatron by the use of high resolution measurements of density (ion saturation current) and potential (floating potential) fluctuations with the help of movable Langmuir probe arrays [36]. Changes in fluctuation behavior are good indicators of the transition to improved confinement modes. In particular, investigations of spontaneous LH-like transition (hereinafter, transition) in the U-3M torsatron reveal a distinct decrease of the fluctuation level and radial turbulent particle flux [37]. In this section we analyze the three data sets. They present typical floating potential fluctuations (in volts) in turbulent plasma, registered in the U-3M torsatron for the small torus radial positions $r = 9.9$ cm (data set S1), $r = 10$ cm (S2), and $r = 11.25$ cm (S3). For the detailed description of the experimental setup, see Refs. [14,36]. One can observe an obvious change of the fluctuation amplitude in the middle position of every data source, which indicates the transition [37]. Therefore, we split each data set into two parts: before (data1) and after the transition point (data2). Moreover, we examine only such parts of each data which constitute the stationary process. Precisely, each data1 contains observations between 6000 and 12 000, whereas data2 between 14 000 and

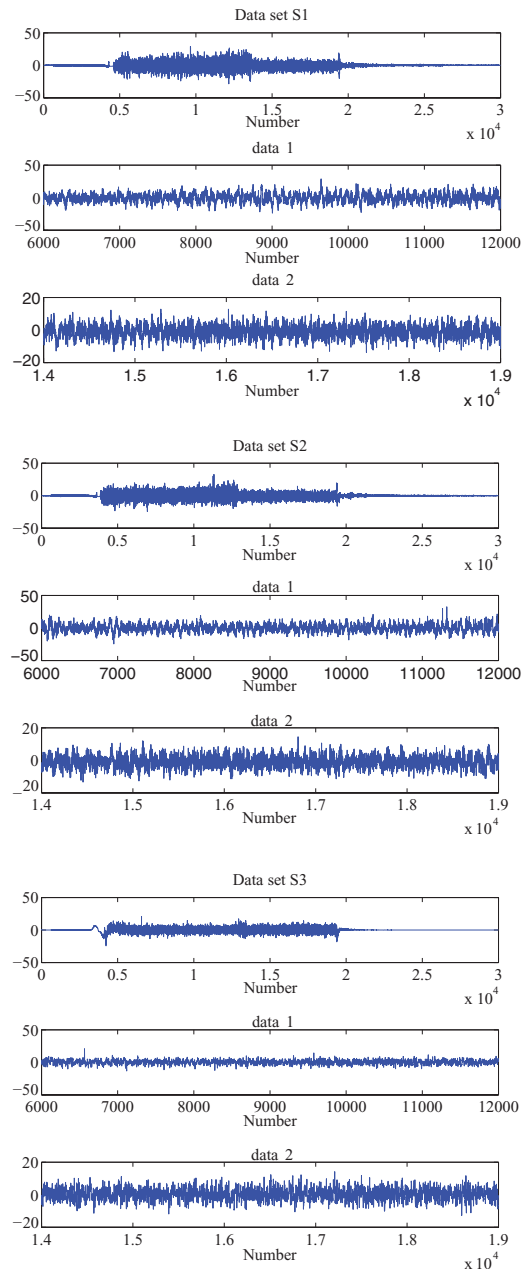


FIG. 5. (Color online) The empirical time series from plasma physics described in Sec. IV.

19 000. The ADC sampling rate is 625 kHz that means the time resolution between two consecutive measurement points is $1.6 \mu\text{s}$. In Fig. 5 we depict the examined data sets and the selected parts before and after the L-H point.

According to the procedure of testing between Lévy stable and Gaussian distributions presented in Sec. III, we start with examining the ECFMs for all data sets. The result of this simple test together with right tails of the empirical PDFs are presented in Fig. 6.

This visual test indicates that the following data sets cannot be considered as a Gaussian time series: S1-data1, S2-data1, and S3-data1, whereas for the rest of the examined samples (i.e., S1-data2, S2-data2, and S3-data2), the ECFM stabilizes. In order to confirm this fact, we test all examined data sets

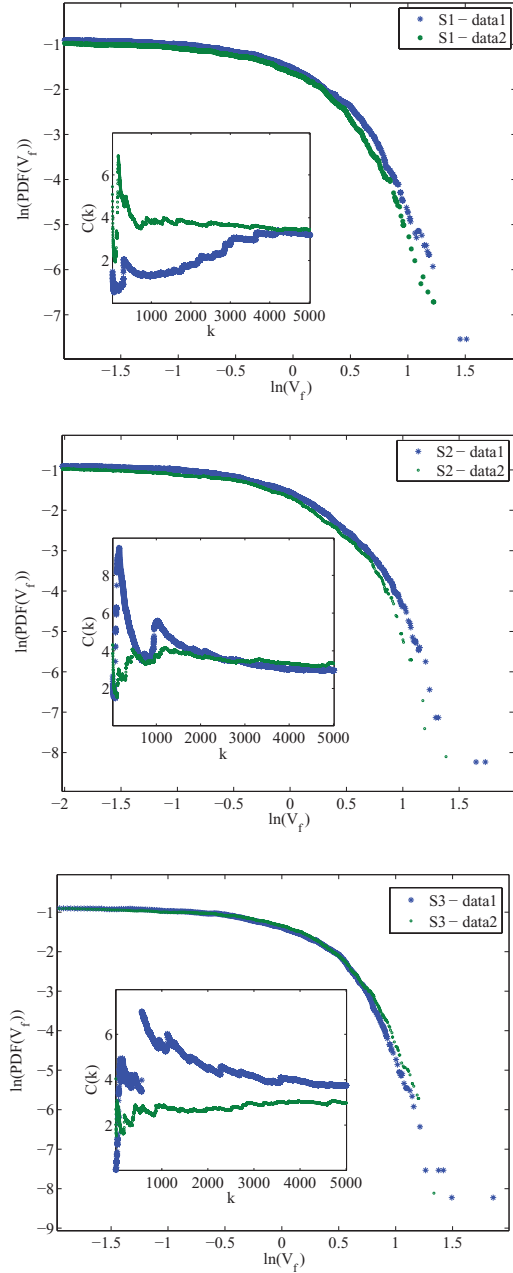


FIG. 6. (Color online) The right tail of the empirical PDFs and ECFMs for six examined data sets.

for the Gaussian distribution by using the JB test presented in Sec. III. The p values and values of the JB statistics are depicted in Table II.

The JB test clearly indicates that S1-data1, S2-data1, and S3-data1 follow a non-Gaussian distribution because the p values are smaller than the confidence level 5% (in fact they are even less than 1%). Let us now consider the sample S1-data2. In this case the ECFM tends to a constant whereas the JB test rejects the hypothesis of Gaussianity with p value close to the confidence level. This is why we use another test in order to confirm one of the hypotheses (Gaussian or non-Gaussian distribution). We propose to use the AD test presented in Sec III, for testing for Gaussianity. The value of

TABLE II. Value of the test statistic and corresponding p values (in parentheses) for the Gaussian law hypothesis (the JB test).

Data set	JB stat. (p value)
S1-data1	12.35 (<0.01)
S1-data2	7.39 (0.025)
S2-data1	71.59 (<0.01)
S2-data2	1.67 (0.43)
S3-data1	62.25 (<0.01)
S3-data2	0.96 (0.5)

the AD statistic is 0.5125, while the corresponding p value is equal to 0.16, which indicates the Gaussian behavior.

In the next step of our analysis we test the samples with chaotic ECFMs for the Lévy distribution, namely S1-data1, S2-data1, and S3-data1. In Table III we present p values and values of the AD statistics for the hypothesis of the stable distribution that indicate stable behavior of examined data sets.

Finally, we can present the index of stability and the skewness parameter for samples S1-data1, S2-data1, and S3-data1. They were obtained by the regression method [23]. The estimated β parameter for S1-data1 is equal to 0.2 while for S2-data1 it is on the level of 1, and for S3-data1 it is -1 . The results of the estimated parameter α and the corresponding confidence intervals are presented in Table IV.

V. DISCUSSION

Let us first add a word of caution: Of course, as with any other family of distributions it is not possible to prove that a given data set is or is not stable [38], and even testing for normality is still an active field of research [39]. In the empirical data analysis like the one presented here the best that can be done is to determine whether or not the data are consistent with the hypothesis of stability.

However, with the testing procedure developed in the present paper we were able to distinguish between Gaussian and stable distribution with the Lévy index close to 2. In other words, we showed that the difference between such laws is essential and one can identify the α -stable distribution with α

TABLE III. Values of the test statistics and corresponding p values (in parentheses) for the Lévy stable law hypothesis (AD test).

Data set	AD stat.
S1-data1	0.39 (0.24)
S2-data1	0.48 (0.72)
S3-data1	0.62 (0.91)

TABLE IV. Estimated parameter α of the Lévy stable distribution with the corresponding confidence intervals.

Data set	α
S1-data1	1.98 [1.97; 1.99]
S2-data1	1.97 [1.96; 1.99]
S3-data1	1.98 [1.96; 1.99]

close to 2, even if the number of available data points is not too large. We illustrated the procedure on simulated samples from Gaussian and non-Gaussian stable distributions. The results show that the algorithm is reliable and works well. It might be useful for detecting Lévy stable distributions in different experiments.

As an example, we applied the developed procedure to assess the stable properties of the fluctuations measured in the edge plasma of stellarator torsatron “Uragan 3M” and demonstrated that the statistics of fluctuations changes from stable to normal at the LH transition.

Another remark concerns the observed phenomenon of changing statistics. We first note that the Lévy statistics of

plasma fluctuations has been detected in the measurements on stellarators “Uragan 3M” and Heliotron J by using the method of quantiles [14] and in the tokamak ADITYA by measuring probability to stay at the origin [15]. The Lévy indexes of the detected stable distributions were sufficiently less than 2. Non-Gaussian heavy-tailed PDFs for low frequency turbulence have also been observed in other toroidal plasma confinement systems such as T-10 tokamak, L-2M, TJ II, and LHD stellarators [40] as well as in astrophysical plasmas [41,42]. As to the physics of the change of the statistics, it in turn implies qualitative changes of the basic properties of plasma turbulence, and should be taken into account in the theoretical models of the LH transition. This important issue is out of the scope of the present paper. On the other hand, we believe that the change of statistics that we observed in the data from “Uragan 3M” may inspire other authors to check whether LH transition is accompanied by a change of statistics in other plasma confinement devices as well.

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