Explicit solutions from eigenfunction symmetry of the Korteweg–de Vries equation

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In nonlinear science, it is very difficult to find exact interaction solutions among solitons and other kinds of complicated waves such as cnoidal waves and Painlevé waves. Actually, even if for the most well-known prototypical models such as the Kortewet–de Vries (KdV) equation and the Kadomtsev-Petviashvili (KP) equation, this kind of problem has not yet been solved. In this paper, the explicit analytic interaction solutions between solitary waves and cnoidal waves are obtained through the localization procedure of nonlocal symmetries which are related to Darboux transformation for the well-known KdV equation. The same approach also yields some other types of interaction solutions among different types of solutions such as solitary waves, rational solutions, Bessel function solutions, and/or general Painlevé II solutions.

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I. INTRODUCTION

With the development of science and technology in modern society, people have come to realize that nonlinear science plays a more and more important role both in the advancement of science advancement and in our life. As one of the main parts of nonlinear science, the theory of solitons [1] has been applied to many areas of mathematics, biology, and even almost all of the branches of physics such as nonlinear optics, solid state physics, hydrodynamics, condensed matter physics, the theory of relativity, plasma physics, and fluid dynamics. Since solitons or their weak analogies, the solitary waves, embody many properties of particles and waves, they have strong universality in nature.

It is known that the solitary wave phenomenon was first found in a river by Russell in 1844 and 60 years later, Korteweg and de Vries derived the very famous shallow water equation, the so-called Korteweg-de Vries (KdV) equation,

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1}$$

from which they solved the bell soltion solution described by Russell earlier and proved the existence of solitary waves theoretically. In fact, the KdV equation is found valid in a very large variety of physical situations when a few general assumptions about the structure of nonlinearity and dispersion are made. In particular, the KdV equation arises quite naturally in plasma physics, solid state physics, biology, and many other areas [2]. A considerable number of explicit solutions of the KdV equation, especially the multisoliton solutions and periodic solutions, have been obtained by various nonlinear mathematical physics methods including inverse scattering transformation, the symmetry method, the Hirota bilinear method, the Darboux transformation (DT), the Bäcklund transformation, the homogeneous balance method, etc.

Clearly, in the real physics world, solitary waves must interact with other waves, say, the cnoidal waves, which are periodic and maybe described by Jacobi elliptic functions. This would give more freedom in controlling solitons in a given environment. The solutions (soliton + cnoidal wave) can be used to analyze the behavior of the soliton in optically induced lattices and to describe localized states in optically induced refractive index gratings [3–5]. However, due to mathematical difficulty, there are few works in the literature that study such kinds of solutions except for Ref. [6], which focuses on using the DT method to construct solitons on a cnoidal wave background for Schrödinger equations and Sine-Gordon equations. We know that when the DT is applied once on a given starting solution, a new solution of "a soliton + starting solution" type will be obtained. However, the original DT problem with the seed solutions being taken as the cnoidal or Painlevé waves is very difficult to solve. For the KdV equation, this soliton + cnoidal wave interaction solution has not yet been found.

In this paper, the nonlocal symmetry [7–9] method is developed to reveal the exact interaction solutions among solitons and other complicated waves including periodic cnoidal waves and Painlevé waves of the KdV equation. The same approach also yields some other different types of solutions such as solitary waves, rational solutions, Bessel function solutions, and cnoidal waves solutions. In this effective method, we no longer concentrate on the traditional DT itself but switch to study its corresponding infinitesimal transformation based on which rich solutions of the KdV equation are calculated.

The paper is organized as follows. In Sec. II, the new nonlocal symmetry method is introduced to derive the eigenfunction symmetries from the DT of the KdV equation and to localize the nonlocal symmetry by extending the KdV equation to a prolonged system. Section III is devoted to finding new explicit solutions of the KdV equation through the use of the Lie-Bäcklund transformation and similar reductions. The last section contains a summary and discussion.

II. EIGENFUNCTION SYMMETRIES FOR THE KDV EQUATION

A. Nonlocal symmetries from the DT

The Lax pair of Eq. (1) reads

$$\psi_{xx} - u\psi + \lambda\psi = 0, \tag{2}$$

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$$\psi_t + 4\psi_{xxx} - 6u\psi_x - 3u_x\psi = 0, \tag{3}$$

where ψ is the spectral function, λ is the spectral parameter, and the subscripts x and t represent derivatives.

It is known that Eq. (1) has the following DT:

Proposition 1 [10–12]. Let u be a solution of the KdV equation (1), where ψ satisfies Eqs. (2) and (3). Then $\bar{u} = u - 2\partial^2 \ln\psi/\partial x^2$ is a solution of Eq. (1).

The DT presented in Proposition 1 is very famous and thriving. It shows a finite transformation from one solution to another solution, which hints at some symmetry of the KdV equation. Nevertheless, for a long time almost no one has bothered to study the corresponding symmetry of the DT. Recently, one of our authors, Lou, and his collaborator, Hu [13], investigated this kind of nonlocal symmetry from the DT given in Proposition 1, resulting in the following proposition.

Proposition 2 [13]. $\sigma = (\tilde{\phi}/\phi)_{xx}$ is a symmetry of the KdV equation, Eq. (1), where $\tilde{\phi}$ and ϕ satisfy the following equations:

$$\begin{aligned}
&\phi_{xx} - [u + 2\ln(\phi)_{xx}]\phi = 0,\\
&\phi_t + [u_x + 2\ln(\phi)_{xxx}]\phi - 2[u + 2\ln(\phi)_{xx}]\phi_x = 0,\\
&-\tilde{\phi}_{xx} + [u + 2\ln(\phi)_{xx}]\tilde{\phi} = \phi,\\
&\tilde{\phi}_t + [u_x + 2\ln(\phi)_{xxx}]\tilde{\phi} - 2[u + 2\ln(\phi)_{xx}]\tilde{\phi}_x = 4\phi_x.
\end{aligned} \tag{4}$$

Regarding the proof of Proposition 2, one can refer to Ref. [13] in detail. In particular, the symmetries proposed in Proposition 2 involve a well-known seed symmetry of the KdV equation, which can be written as the following.

Proposition 3 [13]. This proposition states that

$$\sigma_u = (\psi^2)_x \tag{5}$$

is a nonlocal symmetry of the KdV equation, Eq. (1), where ψ satisfies the Lax pair, Eqs. (2) and (3).

The above nonlocal symmetries are all obtained from the DT and related to the Lax pair of Eq. (1); therefore they are also called eigenfunction symmetries. Eigenfunction symmetries have played an important role in the topics of symmetry constraints, soliton equations with sources, positive and negative hierarchies, etc. The nonlocal symmetry (5) can also be obtained by virtue of inverse recursion operators [14–16] and the Möbious transformation [17] of the KdV equation, respectively.

B. Localization of eigenfunction symmetry

As we know, nonlocal symmetries cannot be directly employed to construct explicit solutions for differential equations (DEs). Hence, nonlocal symmetries need to be transformed into local ones [18,19]. In general, the prolongation of nonlocal Lie-Bäcklund symmetries does not close, but sometimes the inclusion of pseudopotentials may lead to new nonlocal symmetries [9,20,21], which are equivalent to Lie point symmetries of the prolonged systems of DEs. Therefore the nonlocal symmetries with closed prolongation are anticipated. Here, we show that the eigenfunction symmetry (5) also possesses a closed prolongation; in other words, one can find a related system which possesses a Lie point symmetry that is equivalent to the eigenfunction symmetry (5).

First, we rewrite eigenfunction symmetry (5) as

$$\sigma_u = 2\psi\psi_x, \tag{6}$$

and we introduce $\psi_1 \equiv \psi_1(x,t)$ by

$$\psi_x = \psi_1, \tag{7}$$

which leads symmetry (6) to

$$\sigma_u = 2\psi\psi_1. \tag{8}$$

In order to compute local symmetries for the variables ψ and ψ_1, we have to introduce another prolonged potential $p \equiv p(x,t)$ through

$$p_x = \psi^2, \quad p_t = 4\psi_x^2 + (8\lambda - 2u)\psi^2. \tag{9}$$

Then the inclusion of p and ψ_1 yields

$$\sigma_\psi = \tfrac{1}{2}p\psi, \quad \sigma_{\psi_1} = \tfrac{1}{2}(\psi^3 + p\psi_1), \quad \sigma_p = \tfrac{1}{2}p^2, \tag{10}$$

where $\sigma_\psi, \sigma_{\psi_1}$, and σ_p denote the symmetries of ψ, ψ_1, and p, respectively.

Now the prolongation is closed after covering four dependent variables u, ψ, ψ_1, and p for the symmetry (5) with the vector form

$$V = 2\psi\psi_1\frac{\partial}{\partial u} + \frac{1}{2}p\psi\frac{\partial}{\partial\psi} + \frac{1}{2}(\psi^3 + p\psi_1)\frac{\partial}{\partial\psi_1} + \frac{1}{2}p^2\frac{\partial}{\partial p}. \tag{11}$$

Clearly, V shown by Eq. (11) is a Lie point symmetry of the prolonged system Eqs. (1)–(3), (7), and (9).

Another interesting point one can see is that the introduced auxiliary dependent variable p just satisfies the Schwartzian form of the KdV equation (SKdV equation):

$$p_t = -\{p;x\}p_x + 6\lambda p_x, \tag{12}$$

where $\{p;x\} = (p_{xxx}/p_x) - \frac{3}{2}(p_{xx}/p_x)^2$ is the Schwartzian derivative. The reason is that $\sigma_p = p^2$ embodies the Möbious (conformal) invariance property

$$p \longrightarrow \frac{a + bp}{c + dp}(ad \neq cb).$$

This may provide us with a way to seek for the Schwartzian form of DEs without using singularity analysis and this will be more meaningful for discrete models. Since this topic is not the object of our concern in this paper, we do not discuss it in detail here but it is worth pursuing in our future research.

III. EXPLICIT SOLUTIONS FROM EIGENFUNCTION SYMMETRY

After we succeed in making the eigenfunction symmetry (5) or (6) be equivalent to Lie point symmetry (11) of the related prolonged system, the explicit solutions can be constructed naturally by Lie group theory in two aspects.

A. Finite symmetry transformation

Thanks to Lie point symmetry (11), by solving the following initial value problem,

$$\begin{aligned}\frac{d\bar{u}}{d\epsilon}&=2\bar{\psi}\bar{\psi}_1,\quad \bar{u}|_{\epsilon=0}=u,\\ \frac{d\bar{\psi}}{d\epsilon}&=\frac{1}{2}\bar{p}\bar{\psi},\quad \bar{\psi}|_{\epsilon=0}=\psi,\\ \frac{d\bar{\psi}_1}{d\epsilon}&=\frac{1}{2}(\bar{\psi}^3+\bar{p}\bar{\psi}_1),\quad \bar{\psi}_1|_{\epsilon=0}=\psi_1,\\ \frac{d\bar{p}}{d\epsilon}&=\frac{1}{2}\bar{p}^2,\quad \bar{p}|_{\epsilon=0}=p,\end{aligned} \tag{13}$$

the finite symmetry transformation can be calculated as

$$\begin{aligned}\bar{u}&=u-\frac{4\epsilon\psi\psi_1}{\epsilon p-2}+\frac{2\epsilon^2\psi^4}{(\epsilon p-2)^2},\quad \bar{\psi}=\frac{2\psi}{2-\epsilon p},\\ \bar{\psi}_1&=\frac{2\psi_1}{2-\epsilon p}+\frac{2\epsilon\psi^3}{(\epsilon p-2)^2},\quad \bar{p}=\frac{2p}{2-\epsilon p}.\end{aligned} \tag{14}$$

Remark 1. For a given solution u of Eq. (1), above finite symmetry transformation will arrive at another solution $\bar{u}$. What is more interesting is that the eigenfunction symmetry (5) results from the DT in Proposition 1, but its corresponding finite symmetry transformation around u presented in Eq. (14) is not back to the original DT. In fact, the transformation (14) is just the so-called Levi transformation or the second type of DT. The results of this paper show that two kinds of finite transformation possess the same infinitesimal form (5).

As an example, we take the trivial solution $u=0$ of Eq. (1). From Eqs. (2), (3), (7), and (9) with $\lambda=-k_1^2$, we obtain the corresponding special solutions for the introduced dependent variables:

$$\begin{aligned}\psi&=\cosh\left[-k_1(x-x_0)+4k_1^3t\right],\\ \psi_1&=-k_1\sinh\left[-k_1(x-x_0)+4k_1^3t\right],\\ p&=\frac{x}{2}-6k_1^2t+c_0-\frac{\sinh\left[-2k_1(x-x_0)+8k_1^3t\right]}{4k_1}.\end{aligned} \tag{15}$$

Substituting Eq. (15) into Eq. (14) leads to the transformed solution of Eq. (1):

$$\bar{u}=16\epsilon k_1^2\frac{\epsilon\cosh\zeta+[k_1\epsilon\zeta_1-4k_1]\sinh\zeta+\epsilon}{[\epsilon\sinh\zeta-2k_1\epsilon\zeta_1+8k_1]^2}, \tag{16}$$

with $\zeta=-2k_1(x-x_0)+8k_1^3t$ and $\zeta_1=x-12k_1^2t+2c_0$.

B. Similarity reductions

One can also construct special solutions which are invariant under the symmetry transformations by reducing dimensions of a partial differential equation. To search for more similarity reductions of Eq. (1), we study Lie point symmetries of the whole prolonged equation system instead of the single Eq. (1).

Supposing Eqs. (1)–(3), (7), and (9) are invariant under the infinitesimal transformations

$$\begin{aligned}u&\to u+\epsilon\sigma,\quad \psi\to\psi+\epsilon\sigma_1,\\ \psi_1&\to\psi_1+\epsilon\sigma_2,\quad p\to p+\epsilon\sigma_3,\end{aligned} \tag{17}$$

with

$$\begin{aligned}\sigma&=X(x,t,u,\psi,\psi_1,p)u_x+T(x,t,u,\psi,\psi_1,p)u_t\\ &\quad-U(x,t,u,\psi,\psi_1,p),\\ \sigma_1&=X(x,t,u,\psi,\psi_1,p)\psi_x+T(x,t,u,\psi,\psi_1,p)\psi_t\\ &\quad-\Psi(x,t,u,\psi,\psi_1,p),\\ \sigma_2&=X(x,t,u,\psi,\psi_1,p)\psi_{1,x}+T(x,t,u,\psi,\psi_1,p)\psi_{1,t}\\ &\quad-\Psi_1(x,t,u,\psi,\psi_1,p),\\ \sigma_3&=X(x,t,u,\psi,\psi_1,p)p_x+T(x,t,u,\psi,\psi_1,p)p_t\\ &\quad-P(x,t,u,\psi,\psi_1,p),\end{aligned} \tag{18}$$

then substituting the expressions (18) into the following symmetry equations,

$$\begin{aligned}&\sigma_t+\sigma_{xxx}-6u\sigma_x-6u_x\sigma=0,\\ &\sigma_{1xx}-\psi\sigma+(\lambda-u)\sigma_1=0,\\ &\sigma_{1t}+4\sigma_{1xxx}-6u\sigma_{1x}-6\psi_x\sigma-3u_x\sigma_1-3\sigma_x\psi=0,\\ &\sigma_{1x}-\sigma_2=0,\quad \sigma_{3x}-2\psi\sigma_1=0,\\ &\sigma_{3t}-8\psi_x\sigma_{1x}-(16\lambda-4u)\psi\sigma_1+2\psi^2\sigma=0,\end{aligned} \tag{19}$$

and collecting together coefficients of the independent partial derivatives of dependent variables, we arrive at a system of overdetermined linear equations for the infinitesimals X, T, U, Ψ, Ψ_1, and P, which can be easily solved by using computer programs, such as *maple*, to give

$$\begin{aligned}X(x,t,u,\psi,\psi_1,p)&=c_1(x-12\lambda t)+c_4,\\ T(x,t,u,\psi,\psi_1,p)&=3c_1t+c_2,\\ U(x,t,u,\psi,\psi_1,p)&=-2c_1(u-\lambda)+4c_3\psi\psi_1,\\ \Psi(x,t,u,\psi,\psi_1,p)&=c_3p\psi+c_5\psi,\\ \Psi_1(x,t,u,\psi,\psi_1,p)&=c_3(\psi^3+p\psi_1)+(c_5-c_1)\psi_1,\\ P(x,t,u,\psi,\psi_1,p)&=c_3p^2+(c_1+2c_5)p+c_6,\end{aligned} \tag{20}$$

where c_i $(i=1,\ldots,6)$ are six arbitrary constants. When $c_1=c_2=c_4=c_5=c_6=0$, the obtained symmetry is just Eq. (11). Four classical Lie point symmetries [8] of Eq. (1), known as ∂_x-space translation, ∂_t-time translation, $(t\partial_x+\partial_u)$-Galilean boost, and $(x\partial_x+3t\partial_t-2u\partial_u)$ scaling, are also included in Eq. (20).

To give more group invariant solutions, we would like to solve the symmetry constraint conditions $\sigma=0$ and $\sigma_i=0$ $(i=1,2,3)$ defined by Eqs. (18) with Eqs. (20), which is equivalent to solving the following characteristic equation:

$$\begin{aligned}\frac{dx}{c_1(x-12\lambda t)+c_4}&=\frac{dt}{3c_1t+c_2}=\frac{du}{-2c_1(u-\lambda)+4c_3\psi\psi_1}\\ &=\frac{d\psi}{c_3p\psi+c_5\psi}=\frac{\psi_1}{c_3(\psi^3+p\psi_1)+(c_5-c_1)\psi_1}\\ &=\frac{dp}{c_3p^2+(c_1+2c_5)p+c_6}.\end{aligned} \tag{21}$$

In the following part of the paper, two nontrivial cases under the consideration of $c_3\neq 0$ in Eq. (21) are listed.

Case 1. $c_1\neq 0$ and $c_2=c_4=c_5=0$.

First, we redefine the parameter c instead of c_6 by $c^2=(c_1^2-4c_3c_6)/(36c_1^2)$ for facilitating the later computation. Two situations with $c\neq 0$ and $c=0$ are given, respectively.

When $c \neq 0$, by solving Eq. (21), we have

$$\begin{aligned} u &= \lambda + t^{-\frac{2}{3}} U(z) + t^{-\frac{2}{3}} \exp\left(-\frac{2}{3}P(z)\right) \\ &\quad \times \left[\frac{4c_3}{3c_1c} Q(z)Q_1(z)\tanh\{c[\ln t + P(z)]\} \right. \\ &\quad \left. - \frac{2c_3^2}{9c_1^2c^2} Q(z)^4 \text{sech}^2\{c[\ln t + P(z)]\}\right], \\ \psi &= t^{-\frac{1}{6}} \exp\left(-\frac{1}{6}P(z)\right) Q(z)\text{sech}\{c[\ln t + P(z)]\}, \qquad (22) \\ \psi_1 &= \frac{1}{3c_1c} t^{-\frac{1}{2}} \exp\left(-\frac{1}{2}P(z)\right)\text{sech}\{c[\ln t + P(z)]\} \\ &\quad \times [3c_1cQ_1(z) + c_3Q^3(z)\tanh\{c[\ln t + P(z)]\}], \\ p &= -\frac{c_1}{2c_3}[1 + 6c\tanh\{c[\ln t + P(z)]\}], \end{aligned}$$

with $z = (x + 6\lambda t)/t^{\frac{1}{3}}$.

Here, $U(z)$, $Q(z)$, $Q_1(z)$, $P(z)$, and z in Eq. (22) represent five group invariants and substituting Eq. (22) into the prolonged system yields

$$\begin{aligned} U(z) &= \frac{z}{6} + 2c^2P_z^2(z) + \frac{P_{zz}^2(z)}{2P_z^2(z)} - \frac{1}{2P_z(z)}, \\ Q(z) &= \sqrt{-3\frac{c_1c^2}{c_3}P_z(z)\exp\left(\frac{1}{6}P(z)\right)}, \qquad (23) \\ Q_1(z) &= Q_z(z)\exp\left(\frac{1}{3}P(z)\right) + \frac{c_3}{18c_1c^2}Q^3(z), \end{aligned}$$

where $P(z)$ satisfies a three-order ordinary differential equation:

$$\begin{aligned} &6P_{zzz}(z)P_z(z) - 9P_{zz}^2(z) - 12k^2P_z^4(z) \\ &\quad - 2zP_z^2(z) + 6P_z(z) = 0. \end{aligned} \qquad (24)$$

To deal with Eq. (24), we introduce $P_1(z_1)$ by

$$P_z(z) = \frac{3^{\frac{2}{3}}}{2P_{1z_1}(z_1) + 2P_1^2(z_1) + z_1} \quad \left(z_1 = 3^{-\frac{1}{3}}z\right), \qquad (25)$$

which converts Eq. (24) into the second Painlevé equation (PII):

$$P_{1z_1z_1}(z_1) = 2P_1^3(z_1) + z_1P_1(z_1) + \alpha, \qquad (26)$$

with $\alpha = -3c - \frac{1}{2}$.

It appears naturally that when $P_1(z_1)$ is solved from Eq. (26), the explicit solutions of Eq. (1) would be immediately obtained through Eqs. (22) with Eqs. (25) and (23).

To see more clearly, after the substitution of Eq. (23) we rewrite u in Eq. (22) as

$$\begin{aligned} u = \lambda + t^{-\frac{2}{3}}&\left[\frac{z}{6} + 2c^2P_z^2(z) - \frac{1}{2P_z(z)} + \frac{P_{zz}^2(z)}{2P_z^2(z)}\right. \\ &-2cP_{zz}(z)\tanh\{c[\ln t + P(z)\}) \\ &\left. -2c^2P_z^2(z)\text{sech}^2\{c[\ln t + P(z)]\}\right], \end{aligned} \qquad (27)$$

where $P(z)$ is closely related to PII (26) in terms of the transformation (25).

Remark 2. It is obvious that Eq. (27) can be considered as an interaction solution of an explicit solitary wave with the general Painlevé II wave.

We know that generic solutions of PII [22] are transcendental, but for special values of the parameters PII has two families of rational solutions and Bessel and Airy function solutions, which can be used to construct rich explicit solutions of Eqs. (1) through (27).

Rational solutions of the KdV equation. For every $\alpha = N \in Z$ integers, there exists a unique rational solution of PII (26), on the basis of which we can find a sequence of rational solutions of Eq. (1). The first two of them are listed as follows.

For $\alpha = 1$ $(c = -\frac{1}{2})$, PII (26) has a simple solution:

$$P_1(z_1) = -\frac{1}{z_1}; \qquad (28)$$

then by solving Eq. (25) we obtain

$$P(z) = \ln(z^3 + 12) \qquad (29)$$

which further results in a rational solution of Eq. (1):

$$u = \lambda + \frac{6Z(Z^3 - 24t - 2)}{(Z^3 + 12t + 1)^2}, \quad Z = x + 6\lambda t. \qquad (30)$$

For $\alpha = 2$ $(c = -\frac{5}{6})$, the rational solution of PII (26) reads

$$P_1(z_1) = \frac{1}{z_1} - \frac{3z_1^2}{z_1^3 + 4}, \qquad (31)$$

which immediately yields one corresponding solution of Eq. (1):

$$u = \lambda + \frac{2(6Z^{10} - 18Z^5 + 32400t^2Z^4 + 259200t^3Z + 1)}{(Z^6 + 60tZ^3 + Z - 720t^2)^2}, \qquad (32)$$

with $Z = x + 6\lambda t$.

Bessel or Airy function solutions of KdV equation. Since for every $\alpha = N + \frac{1}{2}$, with $N \in Z$, there exists a unique one-parameter family of classical solutions of PII (26) which are written in terms of Bessel and Airy functions, a series of corresponding Bessel and Airy function solutions of Eq. (1) can be obtained with the help of Eqs. (27) and (25).

For example, for $\alpha = \frac{1}{2}$ $(c = -\frac{1}{3})$, Eq. (26) has a solution in the form of a Bessel function, written as

$$P_1(z_1) = -\frac{\sqrt{2}}{2}\frac{\sqrt{z_1}J\left(-\frac{2}{3}, \frac{\sqrt{2}}{3}z_1^{\frac{3}{2}}\right)}{J\left(\frac{1}{3}, \frac{\sqrt{2}}{3}z_1^{\frac{3}{2}}\right)}, \qquad (33)$$

where $J(n,\eta)$ is the first kind of Bessel function. By solving Eq. (25), we obtain

$$\begin{aligned} P(z) = \frac{5}{2}\ln\frac{2}{3} + \frac{3}{2}\ln& \\ \times\left[J^2\left(-\frac{2}{3}, \frac{\sqrt{6}}{9}z^{\frac{3}{2}}\right) - z^2J^2\left(\frac{1}{3}, \frac{\sqrt{6}}{9}z^{\frac{3}{2}}\right)\right]&. \end{aligned} \qquad (34)$$

The corresponding solution for Eq. (1) is presented as follows:

$$u=\lambda-\frac{Z}{6t}+\frac{32Z^2J_1^4}{\left[4Z^2\left(J_1^2+J_2^2\right)-3\times 18^{1/3}\right]^2}-\frac{8\sqrt{6}Z^{3/2}J_1J_2}{3\sqrt{t}\left[4Z^2\left(J_1^2+J_2^2\right)-3\times 18^{1/3}\right]}. \quad (35)$$

For simplicity, we denote $J_1\equiv J(\frac{1}{3},\frac{\sqrt{6}}{9}\frac{Z^{\frac{3}{2}}}{\sqrt{t}})$ and $J_2\equiv J(-\frac{2}{3},\frac{\sqrt{6}}{9}\frac{Z^{\frac{3}{2}}}{\sqrt{t}})$, with $Z=x+6\lambda t$.

When $c=0$ in Eq. (21), following the similar steps of the above case $c\neq 0$ and omitting the tedious calculations, the explicit solution of KdV equation is provided as

$$u=2\lambda+\frac{x}{6t}+t^{-\frac{2}{3}}\left[\frac{2H_z^2(z)}{H^2(z)}-\frac{3c_1}{2c_3}H^2(z)+\frac{4c_3}{3c_1}\frac{H_z(z)}{H^3(z)}\frac{1}{\ln t+\frac{c_3}{3c_1}\int\frac{1}{H^2(z)}\mathrm{d}z}+\frac{2c_3^2}{9c_1^2H^4(z)}\frac{1}{\left(\ln t+\frac{c_3}{3c_1}\int\frac{1}{H^2(z)}\mathrm{d}z\right)^2}\right], \quad (36)$$

where we have $z=(x+6\lambda t)/t^{\frac{1}{3}}$ and $H(z)$ satisfies a special PII equation,

$$H_{zz}(z)=\frac{3c_1}{2c_3}H^3(z)-\frac{1}{6}zH(z). \quad (37)$$

Remark 3. One can see that the solution (36) of the KdV equation represents the interactions between a Painlevé II wave and a logarithm of singularity [$\ln(t)$].

Case 2. $c_1=0$.

Without loss of generality, we let $c_2\equiv 1$ and $c_4\equiv k$. For simplicity, we redefine the parameter $d\equiv\sqrt{c_5^2-c_3c_6}$. Next, the two cases $d\neq 0$ and $d=0$ are both taken into account.

When $d\neq 0$, by solving Eq. (21), we have

$$u=U(\xi)-\frac{2c_3^2}{d^2}Q^4(\xi)\mathrm{sech}^2\{d[t+P(\xi)]\}+\frac{4c_3}{d}Q(\xi)Q_1(\xi)\tanh\{d[t+P(\xi)]\},$$

$$p=-\frac{c_5+d\tanh\{d[t+P(\xi)]\}}{c_3},$$

$$\psi=Q(\xi)\mathrm{sech}\{d[t+P(\xi)]\},$$

$$\psi_1=\frac{Q_1(\xi)}{\cosh\{d[t+P(\xi)]\}}+\frac{c_3}{d}\frac{Q^3(\xi)\sinh\{d[t+P(\xi)]\}}{\cosh^2\{d[t+P(\xi)]\}}, \quad (38)$$

with $\xi=x-kt$.

Substituting Eq. (38) into the prolonged system yields

$$P_\xi(\xi)=-\frac{c_3}{d^2}W(\xi),\quad Q(\xi)=\sqrt{W(\xi)},\quad Q_1(\xi)=\frac{W_\xi(\xi)}{2\sqrt{W(\xi)}},$$

$$U(\xi)=4\lambda+\frac{k}{2}+\frac{d^2}{2c_3W(\xi)}+\frac{W_\xi^2(\xi)}{2W^2(\xi)}+\frac{2c_3^2}{d^2}W^2(\xi), \quad (39)$$

where $W(\xi)$ satisfies

$$W_\xi^2(\xi)=4k_4^2W^4(\xi)+4k_3W^3(\xi)+k_2W^2(\xi)+k_1W(\xi), \quad (40)$$

with

$$k_1=-\frac{d^2}{c_3},\quad k_4=-\frac{c_3}{d},\quad k_2=-12\lambda-2k. \quad (41)$$

After summarizing the above formulas, we get the explicit solution of the KdV equation:

$$u=\lambda+\frac{k_2}{4}+2k_3W(\xi)+2k_4^2W^2(\xi)+2k_4^2\tanh^2\left(k_1k_4t+k_4\int W(\xi)\mathrm{d}\xi\right)-2k_4W_\xi(\xi)\tanh\left(k_1k_4t+k_4\int W(\xi)\mathrm{d}\xi\right), \quad (42)$$

where $W(\xi)$ satisfies Eq. (40) and we have $\xi=x+(\frac{k_2}{2}+6\lambda)t$.

Remark 4. We know that the general solution of Eq. (40) can be written out in terms of Jacobi elliptic functions. Hence, the solution expressed by Eq. (42) is just the explicit exact interaction between the soliton and the cnoidal periodic wave. As Shin [6] has mentioned, these soliton + cnoidal wave solutions, though they can be easily applicable to the analysis of physically interesting processes, seem rather rare in the literature of physics.

To show more clearly of this kind of solution, we offer two special cases of Eq. (42) by solving Eq. (40).

Type 1. A simple solution of Eq. (40) is given as

$$W(\xi)=a_1+a_1\mathrm{sn}^2(c\xi,n), \quad (43)$$

which leads (42) to a KdV soliton residing on a cnoidal wave background where

$$u=\lambda-\frac{c^2}{4}-\frac{n^2c^2}{4}-c^2n^2\mathrm{sn}(c\xi,n)+\frac{n^2c^2}{2}\mathrm{sn}^2(c\xi,n)+\frac{n^2c^2}{2}[1+\mathrm{sn}(c\xi,n)]^2\tanh^2\left[\frac{nc}{2}\left(x+\frac{3c^2+n^2c^2+12\lambda}{2}t+\int_{\xi_0}^{\xi}\mathrm{sn}(c\xi_1,n)\mathrm{d}\xi_1\right)\right]-nc^2\mathrm{cn}(c\xi,n)\mathrm{dn}(c\xi,n)\tanh\left[\frac{nc}{2}\left(x+\frac{3c^2+n^2c^2+12\lambda}{2}t+\int_{\xi_0}^{\xi}\mathrm{sn}(c\xi_1,n)\mathrm{d}\xi_1\right)\right]\times\left[\xi=x+\left(6\lambda+\frac{5}{2}n^2c^2-\frac{1}{2}c^2\right)t\right], \quad (44)$$

with n, c, λ, and ξ_0 being four arbitrary constants. Here sn, cn, and dn are usual Jacobian elliptic functions with modulus n.

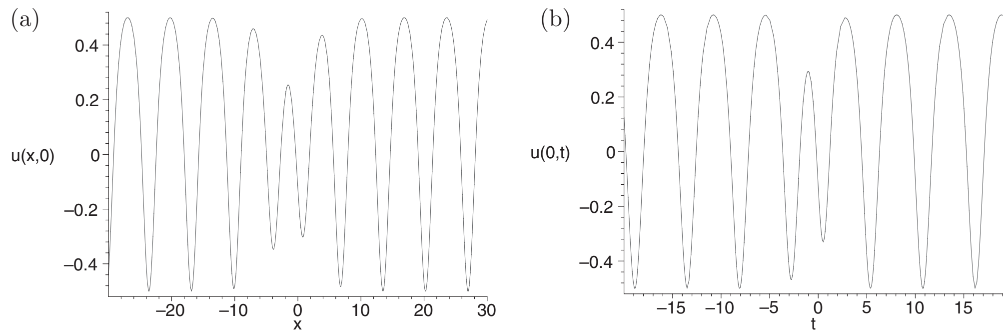

FIG. 1. The solution (44) showing a KdV soliton residing on a cnoidal wave background with the parameter selection listed in Eq. (45). (a) Time-sliced view at $t=0$; (b) space-sliced view at $x=0$.

Figures 1(a) and 1(b) plot the solution (44) which can be seen from a time-sliced view at $t=0$ and a space-sliced view at $x=0$, respectively. The parameters used are

$$n=\tfrac{1}{2},\quad c=1,\quad \lambda=\tfrac{3}{16},\quad \xi_0=0. \tag{45}$$

Type 2. Another special solution of Eq. (40) reads

$$W(\xi)=\frac{1}{a_0+a_2\mathrm{sn}^2(m\xi,n)}, \tag{46}$$

which yields the solution (42) of the KdV equation to a dark soliton on a cnoidal wave background where

$$\begin{aligned} u=&\frac{l_1\mathrm{sn}^4(m\xi,n)+l_2\mathrm{sn}^2(m\xi,n)+l_3}{a_0(a_0+a_2)(a_0n^2+a_2)[(a_0+a_2\mathrm{sn}^2(m\xi,n)]^2}\\ &-\frac{4k_4a_2m\mathrm{sn}(m\xi,n)\mathrm{cn}(m\xi,n)\mathrm{dn}(m\xi,n)}{[a_0+a_2\mathrm{sn}^2(m\xi,n)]^2}\tanh(\Xi)\\ &-\frac{2k_4^2\mathrm{sech}^2(\Xi)}{[a_0+a_2\mathrm{sn}^2(m\xi,n)]^2},\\ \Bigg(\Xi=&\frac{4k_4^3n^2}{a_0(a_0+a_2)(a_0n^2+a_2)}t\\ &-k_4\int_{d_0}^{\xi}\frac{1}{a_0+a_2\mathrm{sn}^2(m\xi',n)^2}d\xi',\\ \xi=&x+\left(6\lambda+\frac{2k_4^2(3a_0n^2+a_2n^2+a_2)}{a_0(a_0+a_2)(a_2+a_0n^2)}\right)t\Bigg), \end{aligned} \tag{47}$$

with

$$\begin{aligned} m&=\sqrt{-\frac{a_2k_4^2}{a_0(a_0+a_2)(a_0n^2+a_2)}},\\ l_1&=a_2^2\big[a_0(a_0+a_2)(a_0n^2+a_2)\lambda+k_4^2(3a_0n^2+n^2a_2+a_2)\big],\\ l_2&=2a_2\big[a_0^2(a_0+a_2)(a_0n^2+a_2)\lambda-k_4^2a_2(a_0+n^2a_0+a_2)\big],\\ l_3&=a_0\big[a_0^2(a_0+a_2)(a_0n^2+a_2)\lambda\\ &\quad+k_4^2\big(a_0a_2+2a_2^2+n^2a_0^2+n^2a_0a_2\big)\big], \end{aligned} \tag{48}$$

where a_0, a_2, k_4, d_0, and λ are five independent constants.

For $|a_0|>|a_2|>0$, the solution given in Eq. (47) also describes the analytic interaction solution between a soliton and a cnoidal periodic wave. Here a dark soliton residing on a cnoidal wave instead of on a constant background for the KdV equation is first found. In fact, it is of interest to study these types of solutions, for example, in describing localized states in optically refractive index gratings. In the ocean, there are some typical nonlinear waves such as the solitary waves (which can also be used to describe a tsunami) and the cnoidal periodic waves (which can be described by elliptic cosine function solution of the KdV and/or the Kadomtsev-Petviashvili (KP) equation). Solution (47) opens a good beginning to study the interactions among theses types of ocean waves.

In Fig. 2, we plot a dark soliton coupled to a cnoidal wave background expressed by Eq. (47) with

$$\begin{aligned} n&=\tfrac{1}{2},\quad \lambda=-1,\ d_0=0,\quad a_0=2,\\ a_2&=-\tfrac{1}{3},\quad k_4=\tfrac{10\sqrt{31}}{31}. \end{aligned} \tag{49}$$

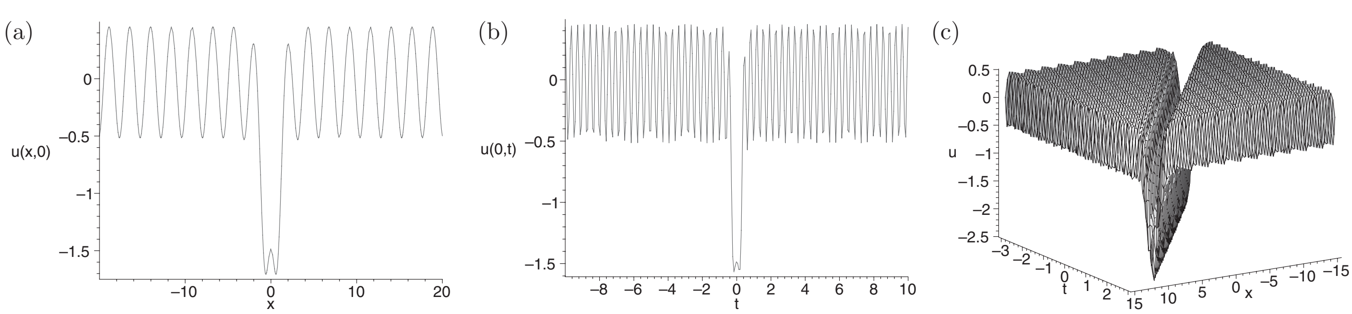

FIG. 2. Plot of a dark soliton on a cnoidal wave background expressed by Eq. (47) with Eq. (49) of the KdV equation in one and two dimensions, respectively. (a) One-dimensional image at $t=0$; (b) one-dimensional image at $x=0$; (c) the corresponding two-dimensional image.

Figures 2(a) and 2(b) plot solution (47) at $t = 0$ and $x = 0$, respectively. Figure 2(c) is the corresponding two-dimensional image.

When $d = 0$ in Eq. (21), in accordance with the above process, the explicit solution of the KdV equation can be written as

$$u = -2\lambda - \frac{k}{2} - 2b_1 V^2(\xi) + \frac{2c_3^2 V^4(\xi)}{\left(t + c_3 \int V^2(\xi)\mathrm{d}\xi\right)^2} - \frac{4c_3 V(\xi)V_\xi(\xi)}{t + c_3 \int V^2(\xi)\mathrm{d}\xi}, (\xi = x - kt), \tag{50}$$

where $V(\xi)$ satisfies

$$V_\xi^2(\xi) = \frac{1}{4c_3} - \left(3\lambda + \frac{k}{2}\right)V^2(\xi) - b_1 V^4(\xi), \tag{51}$$

and λ, k, b_1, and c_3 are four arbitrary constants.

Remark 5. From Eq. (51), we know that since $V(\xi)$ can be expressed as an elliptic integration, solution (50) denotes the interactions among cnoidal periodic waves and rational waves for the KdV equation.

IV. SUMMARY AND DISCUSSIONS

In this paper, a variety of exact explicit interaction solutions for the KdV equation are obtained based on the eigenfunction symmetry coming from the DT.

On the one hand, besides covering the results created by the classical Lie point symmetries, different classes of explicit solutions are provided in the present paper, for example, the rational solution hierarchy, the Bessel function solution hierarchy, soliton + Painlevé II wave solutions, soliton + cnoidal wave solutions, and other combinational solutions of elliptic functions. The most interesting and meaningful solution among them is the analytic soliton + cnoidal wave solution. This kind of solution describing solitons moving on a cnoidal wave background instead of on the plane continuous wave background can be easily applicable to the analysis of physically interesting processes, but seems rather rare in the literature of physics. More general applications of the solutions obtained in this paper both in atmospheric dynamics and in other physical fields deserve more investigation.

One the other hand, the eigenfunction symmetry is applied to construct explicit solutions of DEs. With regard to Lie point symmetries, a large collection of explicit solutions have been constructed by the classical symmetry method; however, only a small amount of the literature deals with the nonlocal symmetry and its related nonlocal group. Here, the DT provides an effective way of seeking a type of nonlocal symmetry known as eigenfunction symmetry, which is concerned with the Lax pair of integrable equation(s). Moreover, we explore how to use the eigenfunction symmetry to build similarity solutions. The idea is to incorporate the original equation(s) in an extended related system by introducing other auxiliary dependent variables. In this case, the primary nonlocal symmetry is equivalent to a Lie point symmetry of a prolonged system, on the basis of which one can find nonlocal groups as well as the explicit similarity solutions.

Using the DT to search for nonlocal symmetries of integrable DEs and then applying them to construct explicit solutions are both of considerable interest. It would be possible to extend the approach presented here to many other interesting integrable models. However, since in general the prolongation does not close, neither for the local nor for the nonlocal variables, there is not a universal way to estimate what kind of nonlocal symmetries can be localized to the Lie point symmetries of some related prolonged system. More details on the results of this paper, especially on the applications of soliton + cnoidal wave solutions and the method to obtain these types of solutions for nonlinear systems, are worthy of further study.

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