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Interaction forces among two-dimensional bright solitons and many-soliton molecules

U. Al Khawaja
Department of Physics, United Arab Emirates University, P.O. Box 17551, Al-Ain, United Arab Emirates

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We consider two-dimensional bright matter-wave solitons in two-dimensional Bose-Einstein condensates. From the asymptotic form of their wave function, we derive an analytic expression for the force of interaction between solitons in the large separation limit, which turns out to decay with solitons separation Δ as $F(\Delta) \propto \exp(-\Delta)/\sqrt{\Delta}$. Simulating the dynamics of two solitons using the relevant Gross-Pitaevskii equation, we obtain the force of the interaction for the full range of Δ, which turns out to be of molecular type. We show that many-soliton molecules can exist as a result of such a molecular-type of interaction. These include string-shaped, ring-shaped, or regular-lattice-shaped soliton molecules. By calculating their binding energy, we investigate the stability of these structures. Contrary to one-dimensional soliton molecules, which have no binding energy, two-dimensional molecules of a lattice of solitons with alternating phases are robust and have a negative binding energy. Lattices of size larger than 2×2 solitons have many discrete equilibrium values of the separation between two neighboring solitons.

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I. INTRODUCTION

It is well established that bright solitons are not stable against collapse or decay in two- and three-dimensional homogeneous spaces [1]. Different mechanisms to stabilize these solitons have been considered, including dissipation and gain effects [2], higher-order contributions such as the cubic-quintic nonlinearity [3], saturable nonlinearity, and composite solitons in multicomponent systems [4–9]. Two-dimensional lump solitons were also shown to exist [10]. Vortex solitons (or doughnut-shaped solitons) were also found to have stability windows for some of their parameter regimes [11]. The force of the interaction between one-dimensional solitons is well known and has been well studied for decades in optical solitons in fibers [12–14] and in quasi-one-dimensional matter-wave soliton trains [15,16]. Interactions within N-soliton trains have also been studied both in the homogeneous case [17] and in the presence of external potentials [18,19]. The interaction between solitons in two and three dimensions has also been a subject of interest for many researchers. In Ref. [3] the force of the interaction between two- and three-dimensional solitons was derived in the large separation limit in the presence of cubic-quintic nonlinearity. In Ref. [4] it was shown that a state of spiraling two-dimensional spacial solitons is possible as a stable dynamical regime, in agreement with the experimental observation. In Ref. [6] a vortex mode vector soliton was suggested and in Ref. [5] a robust optical vector soliton (dipole mode vector soliton) was predicted, which was experimentally realized afterward [7]. In Ref. [20] it was shown that even in a homogeneous medium with only cubic self-focusing nonlinearity a necklace-shaped ring soliton exhibits stable propagation for distances much larger than the dispersion length.

The above-mentioned mechanisms will have their effect on the nature of the interaction between solitons. Ideally, one would be interested in finding the force between solitons that is intrinsic to their mutual interaction, i.e., in a homogeneous background and with the presence of only the cubic nonlinearity. It is therefore more appealing to consider a confining potential as a stabilizing mechanism. In a confining potential, such as the harmonic trapping potential of Bose-Einstein condensates, two- and three-dimensional solitons can be stabilized [21]. The effect of the harmonic potential on the interaction between solitons can be removed by setting solitons in an orbital motion around the center of the trapping potential with an angular speed that equals the characteristic angular frequency of the trapping potential. The centripetal force cancels that of the trapping potential and the relative dynamics of the two solitons is governed only by their mutual interaction. This work is therefore distinguished from previous studies by accounting for the intrinsic interaction between solitons in terms of the full range of their separation, which will show clearly the molecular shape of the potential. The molecular potential is then exploited to construct different stable molecular structures. We refer to these structures as soliton molecules, which correspond to two- and many-soliton solutions of the nonlinear Schrödinger equation. The main goals of this paper are thus to (i) obtain the force of the interaction between two-dimensional bright solitons for all separations and (ii) explore the implications on the existence and stability of two-dimensional soliton molecules. This work is also distinguished from the above-mentioned previous works first by the alternative approach and setup we follow to extract the force of interaction between two two-dimensional solitons in a harmonic trapping potential. Second we find that, unlike one-dimensional soliton molecules, two-dimensional soliton molecules may be robust and stable for some discrete values of the bond length.

We approach this problem by first considering two bright Bose-Einstein solitons confined by a cylindrically symmetric trapping potential that is much tighter in the axial direction than in the radial one [21]. The force of the interaction between solitons in this case will be the resultant of that of the trapping potential and the mutual interaction between the solitons. The latter is typically defined to be proportional to the acceleration of the solitons separation [13]. This requires the center of mass of the solitons to be well defined, which is the case when the solitons are sufficiently separated from each other such that interference effects are not large enough. Alternatively,

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a more rigorous definition can be found in Ref. [22]. In this manner we obtain the force of the interaction between the solitons for the full range of their separation. As expected, the force turns out to be composed of a negative part for large separations and a positive part for short separations. Such a molecular type of interaction indicates the possibility of forming two-dimensional soliton molecules, which is another subject of interest in this paper.

It is known for one-dimensional solitons that a soliton molecule forms only when two solitons are located close enough to each other with zero relative speed [23]. It is expected that similar conditions are required to create a soliton molecule in two dimensions. In the presence of the harmonic confining potential, the condition of zero relative speed can be achieved only by rotating both solitons around the axis of the trap with the same angular speed such that the centripetal force cancels the force of the trap. In the rotating frame the solitons will be stationary with respect to each other and the molecule forms only due to their mutual interaction. A similar setup is proposed in Ref. [24] in a homogeneous medium. Interesting structures, which can also be considered as many-soliton molecules, such as a string of solitons, a soliton ring, or a regular lattice of solitons turn out to be stable in such a setup. Considering the initial relative phases of the solitons, two distinct situations turn out to be particularly interesting, namely, the case with zero relative phase for the neighboring solitons and the case of alternating phases between 0 and π. The dynamics and stability of these two cases will be drastically different. For instance, a regular square lattice of size $n \times n$ of solitons with vanishing relative phases will be oscillating between its initial lattice size and a reduced square lattice of size $(n-1) \times (n-1)$. For a similar lattice with relative phases alternating between 0 and π, such transitions do not occur and the initial structure of the lattice is maintained with small soliton vibrations around their equilibrium positions. Another interesting example is the soliton ring that results from the out-of-phase superposition of two or more solitons. In Refs. [25,26] it was shown that such a structure corresponds to a vortex soliton.

The binding energy of a soliton molecule is an important indication of its stability. In addition to its dependence on the initial relative phase between neighboring solitons, as mentioned above, the binding energy of a soliton lattice oscillates with the size of the primitive cell. As an example, the binding energy of a molecule composed of a string of solitons has no local minima, indicating that a finite equilibrium soliton separation does not exist. This is contrary to the case of, for instance, an $n \times n$ square lattice, where a number of local minima (depending on n) in the binding energy exists. Furthermore, the value of the binding energy at these minima is relatively large and negative, which means that such structures are predicted to be stable and robust.

The rest of the paper is organized as follows. In Sec. II we derive an analytic form of the interaction potential between two-dimensional solitons in the large separation limit and obtain the force of interaction for the complete range of separations numerically. In Sec. III we present two-dimensional molecules. In Sec. IV we discuss their stability. We conclude in Sec. V with discussion of our results and outlook.

II. FORCE OF THE INTERACTION BETWEEN TWO-DIMENSIONAL SOLITONS

The dynamics of solitons in axially symmetric traps, with tight binding in the axial direction, is described by the dimensionless effective Gross-Pitaevskii equation

$$i\frac{\partial\psi}{\partial t} = \left[-\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\frac{1}{2}\lambda^2(x^2+y^2)-g|\psi|^2\right]\psi. \tag{1}$$

Here the strength of the trapping potential is given in terms of the anisotropy ratio $\lambda = \omega_z/\omega_\perp \gg 1$, where ω_z and $\omega_\perp$ are the characteristic frequencies of the trapping potential in the axial and radial directions, respectively. The effective two-dimensional interaction strength is given by $g = \sqrt{8\pi}Na/a_z$, where a is the magnitude of the s-wave scattering length, $a_z = \sqrt{\hbar/m\omega_z}$, and N is the number of atoms in the soliton. In Eq. (1) the length is scaled to $a_\perp = \sqrt{\hbar/m\omega_\perp}$, time is scaled to $1/\omega_\perp$, and the order parameter ψ is scaled to $1/a_\perp$.

Solitonic solutions to the homogeneous version of this equation exist in either a numerical form [27] or an analytic (algebraic) form, which are known as two-dimensional lumps [28]. However, as mentioned above, these solutions are unstable against collapse or decay. While the analytic form of localized solitonic solutions of Eq. (1) is not known, the asymptotic behavior of the solution at a large distance can be derived.

At large distances from the center of the soliton the density is low and the nonlinear term in Eq. (1) can be ignored. The trapping potential can also be ignored in the local density approximation, i.e., when the size of the soliton is much less than the characteristic length of the trapping potential. Under these assumptions, the asymptotic form of the interaction potential is shown in the Appendix to take the form

$$V(\Delta) \propto \frac{e^{-\Delta}}{\sqrt{\Delta}}. \tag{2}$$

This is to be compared with the one-dimensional result $V(\Delta) \propto e^{-\Delta}$. A similar calculation can be performed for the three-dimensional case and the result will be $V(\Delta) \propto e^{-\Delta}/\Delta$. It should be mentioned that in the proportionality constant the phase factor $\cos\varphi$ is present, where φ is the phase difference between solitons. A similar result was derived in Ref. [3] for a Gross-Pitaevskii equation with qubic-quentic nonlinearity.

To account for the interaction potential between solitons for all separations we simulate, with the Gross-Pitaevskii equation (1), the dynamics of two solitons. The dynamics is started with two solitons of equal number of atoms at opposite sides from the center of the trap and with zero initial center-of-mass speeds. The normalized initial wave function is taken as

$$\psi_0 = A(e^{-2g(r-x_1)^2} + e^{-2g(r-x_2)^2+i\varphi_0}), \tag{3}$$

where $r = \sqrt{x^2+y^2}$, x_1 and x_2 are the initial positions of the center of mass of the two solitons, φ_0 is an initial phase difference, and $A = (4g/\pi)^{1/4}\exp(2gx_1^2)/\sqrt{\exp(4gx_1^2)+\cos\varphi_0}$ is a factor that guarantees normalization to 2. Two distinct behaviors are obtained for $\varphi_0 = 0$ and π. While in the former case the two solitons coalesce, the π-phase difference in the latter case prevents them from doing so. The center-of-mass

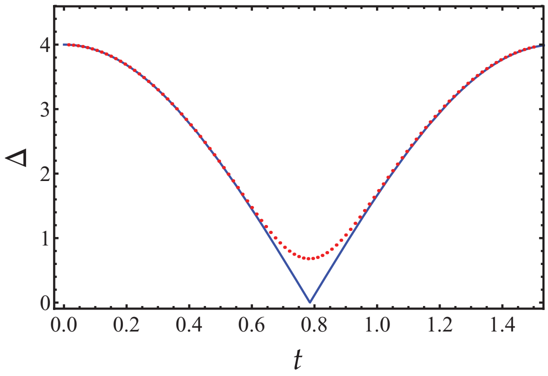

FIG. 1. (Color online) Separation between solitons $\Delta(t)$. Dots correspond to the numerical solution and the curve corresponds to the noninteracting case. The parameters are $x_1 = -x_2 = -2$, $\lambda = 2$, and $g = 0.5$.

separation $\Delta(t)$ is plotted in Fig. 1. The deviation of the separation $\Delta(t)$ from that of the noninteracting solitons increases when the separation is small as a result of the interaction between solitons. Obtaining the force between the solitons by differentiating numerically $\Delta(t)$ twice with respect to time and then eliminating time between $\Delta(t)$ and $\ddot{\Delta}(t)$, we obtain the force in terms of separation, as shown in Fig. 2. The curve in this figure corresponds to the resultant force $F_R = F_{\text{trap}} + F_{\text{interaction}}$, where F_{trap} is the force due to the trapping potential and $F_{\text{interaction}}$ is the force of the interaction between solitons. For large separations the curve asymptotically approaches that of two noninteracting solitons in a trap. This is of course expected since the interaction force between solitons decays at large separations. Subtracting the force due to the trapping potential $\lambda^2\Delta$, we obtain the force between the two solitons that is due to their mutual interaction. The result is shown in Fig. 3, where $F(\Delta)$ has a negative part acting at large separations and a repulsive part that acts at small separations.

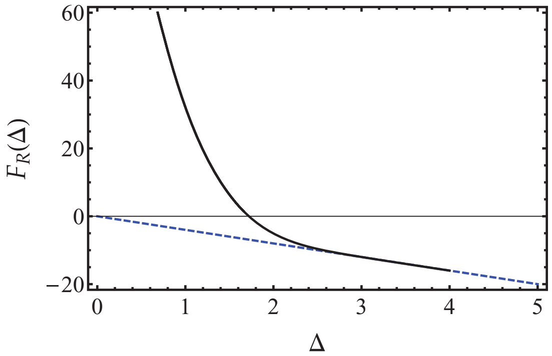

FIG. 2. (Color online) Resultant force between solitons $F_R(\Delta) \equiv \ddot{\Delta} = F_{\text{trap}}(\Delta) + F_{\text{interaction}}(\Delta)$. The dashed line corresponds to the force between the solitons due to the trapping potential. The parameters of Fig. 1 are used.

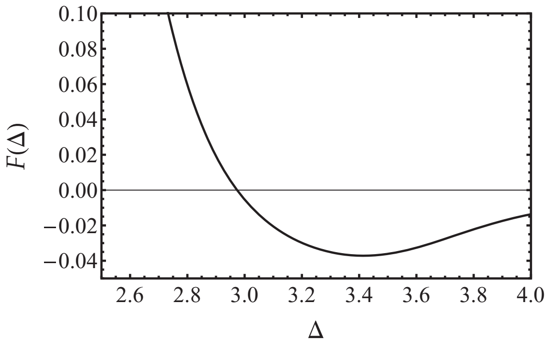

FIG. 3. Force of interaction between solitons $F_{\text{interaction}}(\Delta) \equiv F(\Delta)$. The parameters of Fig. 1 are used.

III. TWO-DIMENSIONAL SOLITON MOLECULES

Due to the molecular nature of the interaction potential between two solitons, it is expected that stable molecular structures exist. In this section we investigate the existence of such structures. While the trapping potential is needed for the internal stability of the solitons, its effect on their dynamics hides the interesting dynamics of the solitons due to their mutual interaction. Setting the solitons in orbital motion around the axis of the trap with an angular speed equal to the angular frequency of the trapping potential will result in a centripetal force that cancels the force of the trap. In such a rotating frame the dynamics of the solitons will be driven only by their mutual interaction. Therefore, all molecules considered in this section will be given such an initial angular speed. The initial wave function of an n-soliton molecule can be written as

$$\psi_0 = \sqrt{\frac{4g}{\pi}} \sum_{i=1}^{n} e^{-2g[(x-x_i)^2+(y-y_i)^2]+i[v_{ix}(x-x_i)+v_{iy}(y-y_i)+\varphi_i]}, \tag{4}$$

where x_i, y_i, v_{ix}, v_{iy}, and φ_i denote the initial center-of-mass positions, speeds, and phase of the ith soliton.

The simplest structure that can be considered is a two-soliton molecule. We set the two solitons initially on the x axis, symmetrically distributed off the origin, with coordinates $(x_1,y_1) = (-1,0)$ and $(x_2,y_2) = (1,0)$. The initial speeds needed to cancel the trapping force are given by $(v_{1x},v_{1y}) = (0,-\lambda)$ and $(v_{2x},v_{2y}) = (0,\lambda)$. The initial wave function thus takes the form

$$\psi_0 = \sqrt{\frac{4g}{\pi}}[e^{-2g[y^2+(x-1)^2]+i\lambda y} + e^{-2g[y^2+(x+1)^2]-i\lambda y+i\varphi_0}]. \tag{5}$$

For a zero initial phase difference $\varphi_0 = 0$, the two solitons coalesce, as shown by the top panels in Fig. 4. The lower panels show that a $\varphi_0 = \pi$ phase difference prevents the two solitons from coalescing.

A generalization of this type of molecule is a string of finite number of solitons, as shown in Fig. 5. The initial state is a string of seven solitons that are in phase, as in the top panels, or out of phase with respect to each other, as in the bottom panels. The initial wave function for the latter molecule is given by

$$\begin{aligned}\psi_0 = \sqrt{\frac{4g}{\pi}}[&-e^{-2g[y^2+(x-3)^2]+3i\lambda y} + e^{-2g[y^2+(x-2)^2]+2i\lambda y}\\ &- e^{-2g[y^2+(x-1)^2]+i\lambda y} + e^{-2g(y^2+x^2)} - e^{-2g[y^2+(x+1)^2]-i\lambda y}\\ &+ e^{-2g[y^2+(x+2)^2]-2i\lambda y} - e^{-2g[y^2+(x+3)^2]-3i\lambda y}].\end{aligned} \tag{6}$$

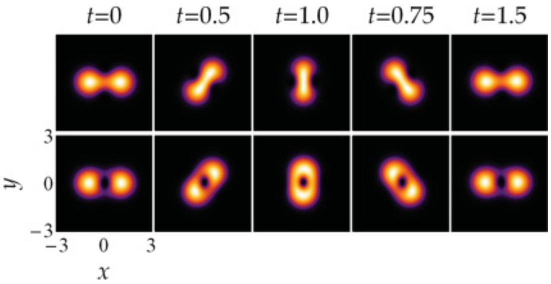

FIG. 4. (Color online) Density plots of the amplitude of the wave function $|\psi(x,y,t)|^2$ of a soliton molecule in a harmonic trap. The molecule is rotating counterclockwise with a frequency equal to $\lambda = 2$. The initial wave function is given by Eq. (5) with the parameters $x_1 = -x_2 = -1$, $y_1 = y_2 = 0$, and $g = 0.5$. The initial phase difference equals zero in the top panels and equals π in the bottom panels.

For the in-phase solitons a flat-top-like soliton forms, while for the out-of-phase case the initial soliton string periodically transforms to a deformed soliton ring. This again shows the importance of the relative phase in giving rise to different dynamics of soliton molecules.

Another interesting generalization is a lattice of solitons. As an example we take a square lattice of 6×6 solitons. The initial wave function of this lattice takes a form similar to that of the previous one [Eq. (6)] but with 36 terms. The center-of-mass positions of the soliton at the bottom left corner are $(x_i,y_i) = (-40/9,-40/9)$ and the distance between two consecutive solitons is 16/9. The speeds are given by $(v_{ix},v_{iy}) = \lambda(-y_i,x_i)$. In Fig. 6 we show in the top panels the case with in-phase solitons and in the bottom panels the case with alternating phases. For the case of in-phase solitons the lattice preserves its structure and number of solitons. Strikingly, the lattice of out-of-phase solitons oscillates between the initial 6×6 and a 5×5 square lattice. Similar results are obtained for a hexagonal lattice.

Decreasing the solitons separation in a lattice with out-of-phase solitons results in a soliton ring. For the in-phase solitons the same procedure results in a high-intensity soliton that collapses later on. For small lattice sizes the resulting ring will be deformed from the circular shape, as shown previously in the bottom panels of Fig. 4. In addition, the intensity will be oscillating considerably along the circumference of the ring. The shape of the soliton ring becomes more circular and with more homogeneous intensity for larger lattice sizes such as the case of size 3×3 shown in Fig. 7. The initial wave function corresponding to this soliton ring is similar to that of the previous case of the square lattice molecule but with considerably smaller distance between nearest neighbors. The left soliton ring at the initial time is formed by a square lattice with center-of-mass positions of the soliton at the bottom left corner equal to $(x_i,y_i) = (-3.5,2.7)$ and those of the right ring are $(x_i,y_i) = (3.5,2.7)$. The distance between two consecutive solitons in both cases is 0.2. The speeds of the solitons in the left ring, which is rotating clockwise, are given by $(v_{ix},v_{iy}) = \lambda(y_i,-x_i)$ while those for the right ring rotating counterclockwise are $(v_{ix},v_{iy}) = \lambda(-y_i,x_i)$. We show also in this figure a collision between two soliton rings initially set with opposite speeds on the same circular orbit around the axis of the trap. It is clear that after the collision the two soliton rings preserved their shapes, confirming their solitonic nature.

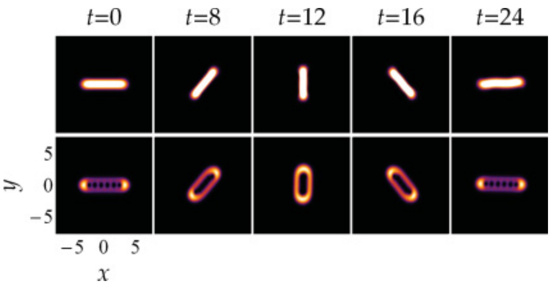

FIG. 5. (Color online) String of seven solitons molecule periodically transforming to a ring soliton. The molecule is rotating counterclockwise with a frequency equal to $\lambda = 2$. The initial wave function is given by Eq. (6) with the parameters $x_i = -3,-2,-1,0,1,2,3$; $y_i = 0$, and $g = 0.5$. The initial phase difference equals zero in the top panels and alternates between 0 and π in the bottom panels.

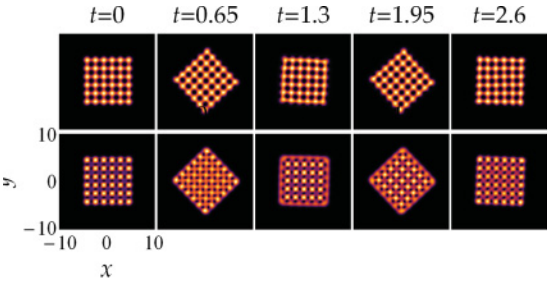

FIG. 6. (Color online) Square lattice of the initially 6×6 soliton molecule. The molecule is rotating counterclockwise with a frequency equal to $\lambda = 2$. The parameters are as follows: the coordinate of the soliton at the lower left corner $(x_1,y_1) = (-40/9,-40/9)$, the initial soliton separation equal to 16/9, and $g = 1.5$. The initial phase difference is equal to zero in the top panels and alternates between 0 and π in the bottom panels.

IV. STABILITY ANALYSIS

To investigate the stability of the soliton molecules descried in the preceding section we calculate their binding energy. The binding energy is defined as the difference between the energy of the molecule and that of its constituent solitons. Energy is calculated in terms of the wave function of the n-soliton

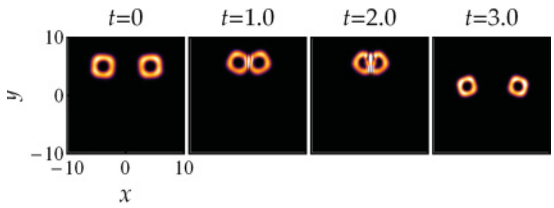

FIG. 7. (Color online) Collision of two ring solitons. The parameters are $g = 1$ and $\lambda = 2$. The ring soliton is a superposition of nine out-of-phase solitons on a 3×3 square lattice of lattice size equal to 1.

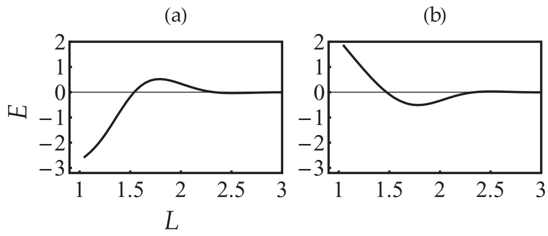

FIG. 8. Binding energy of a 2 × 2 square lattice of solitons versus the soliton separation. The parameters are the same as in Fig. 6 for (a) in-phase solitons and (b) out-of-phase solitons.

molecule

$$\psi_m(x,y)=\sum_{j=1}^{n}\psi_{sj}(x,y), \tag{7}$$

where $\psi_{sj}(x,y)$ is the wave function of the jth single soliton. The binding energy is thus written as

$$E_b=E[\psi_m]-\sum_{j=1}^{n}E[\psi_{sj}], \tag{8}$$

where the energy functional reads

$$E[\psi]=\int_{-\infty}^{\infty}dx\,dy\left[\frac{1}{2}\left(\left|\frac{\partial\psi}{\partial x}\right|^2+\left|\frac{\partial\psi}{\partial y}\right|^2\right)\right.$$
$$\left.+\frac{1}{2}\lambda^2(x^2+y^2)|\psi|^2-\frac{1}{2}g|\psi|^4\right]. \tag{9}$$

As an example, we calculate the binding energy of a 2 × 2 square lattice. The result is shown in Fig. 8(a) for in-phase solitons and in Fig. 8(b) for out-of-phase solitons. The curves show that the binding energy of the first case is the negative of the second case. The curves show that no stable equilibrium exists for the in-phase case while a stable equilibrium exists for the out-of-phase case at $L=1.75$.

Increasing the size of the lattice to 3 × 3, the binding energy acquires more structure, as shown in Fig. 9. Similarly, the binding energy of the in-phase case is the negative of the binding energy of the out-of-phase case. Here a stable equilibrium exists for both cases. More than one equilibrium point exists if we increase the size of the lattice, as shown in Fig. 10 for a 4 × 4 lattice where two main equilibrium points exist for both the in-phase and out-of-phase cases. However, the greater depth of the energy at the equilibrium point for the out-of-phase case indicates that soliton molecules are more stable than for the in-phase case.

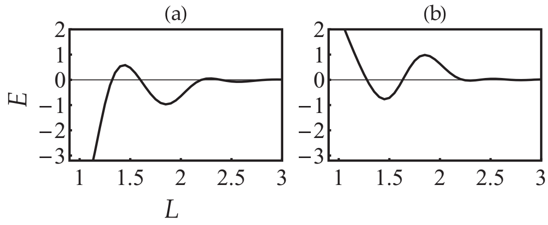

FIG. 9. Binding energy of a 3 × 3 square lattice of solitons versus the soliton separation. The parameters as the same as in Fig. 6 for (a) in-phase solitons and (b) out-of-phase solitons.

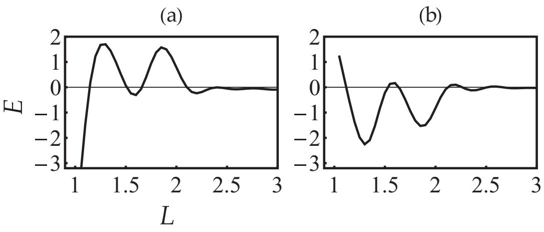

FIG. 10. Binding energy of a 4 × 4 square lattice of solitons versus the soliton separation. The parameters are the same as in Fig. 6 for (a) in-phase solitons and (b) out-of-phase solitons.

In contrast to the above cases of lattice molecules, no stable equilibrium exists for string molecules, where we found the energy to decrease (increase) monotonically to zero for the out-of-phase (in-phase) case. This is consistent with the known exact result that the binding energy vanishes for n-soliton molecules in the homogeneous one-dimensional case [29]. The above results show that this is not the case for lattice soliton molecules. To understand this behavior and to confirm the role of dimensionality in stabilizing the molecule, we calculate analytically the binding energy of a string of two solitons and a lattice of 2 × 2 solitons for the out-of-phase case. For the string molecule, we use the wave function

$$\psi_{\text{string}}=A(\Delta)[e^{-(x-\Delta)^2-(y-\Delta)^2}e^{i[-\Delta(x-\Delta)+\Delta(y-\Delta)]}$$
$$+e^{-x^2-(y-\Delta)^2}e^{-i\Delta(x-\Delta)}e^{i\pi}], \tag{10}$$

where $A(\Delta)$ is a normalizing factor. This wave function corresponds to two solitons located at $(x,y)=(\Delta,\Delta)$ and $(x,y)=(0,\Delta)$ with tangential speeds $(v_x,v_y)=(-\Delta,\Delta)$ and $(v_x,v_y)=(-\Delta,0)$, respectively. The speeds are calculated such that the centripetal force cancels that of the trapping potential. Similarly, the wave function of a lattice of 2 × 2 solitons is constructed as

$$\psi_{\text{lattice}}=A(\Delta)[e^{-(x-\Delta)^2-(y-\Delta)^2}e^{i[-\Delta(x-\Delta)+\Delta(y-\Delta)]}$$
$$+e^{-x^2-(y-\Delta)^2}e^{-i\Delta(x-\Delta)}e^{i\pi}$$
$$+e^{-(x-\Delta)^2-y^2}e^{i\Delta(y-\Delta)}e^{i\pi}$$
$$+e^{-x^2-y^2}]. \tag{11}$$

This wave function corresponds to four solitons located at the vertices of a square of side Δ with appropriate tangential speeds so that the centripetal force cancels that of the trap. The binding energy for the string and lattice are then calculated in terms of Δ:

$$E_{\text{string}}=\frac{-25\Delta^2+8e^{-5\Delta^2/8}+16e^{-3\Delta^2/8}-32e^{-3\Delta^2/16}+8}{32(1-e^{5\Delta^2/8})} \tag{12}$$

and

$$E_{\text{lattice}}=f_0+f_1\cos\left(\frac{\Delta^2}{2}\right)+f_2\cos(\Delta^2)+f_3\cos(2\Delta^2)$$
$$+f_4\cos\left(\frac{3\Delta^2}{2}\right)+f_5\sin(\Delta^2), \tag{13}$$

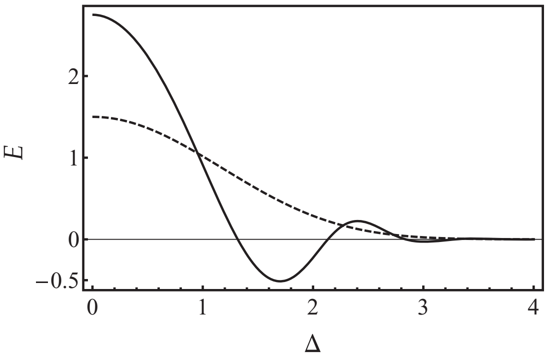

FIG. 11. Binding energy of a two-solitons molecule (dashed curve) and a 2×2 square lattice (solid curve) of solitons versus the soliton separation. The curves are calculated from the analytic expressions of the binding energy, as described in Sec. IV. The parameters are $\lambda = g = 1$.

where

$$f_0 = \frac{\pi}{16A} e^{-5\Delta^2/2}[e^{15\Delta^2/8}(9\Delta^2 - 8) + 16e^{7\Delta^2/16} - 32e^{\Delta^2/2} - 32e^{7\Delta^2/8} - 32e^{3\Delta^2/2} + 32e^{27\Delta^2/16} - e^{5\Delta^2/4}(25\Delta^2 + 4) - 4],$$

$$f_1 = \frac{\pi}{A} e^{-29\Delta^2/16}[8 - 3e^{3\Delta^2/16}],$$

$$f_2 = \frac{\pi}{16A} e^{-9\Delta^2/4}[e^{\Delta^2}(4 - 25\Delta^2) + 32e^{3\Delta^2/16} - 32e^{5\Delta^2/8} + 32e^{23\Delta^2/16} + e^{13\Delta^2/8}(41\Delta^2 - 8) - 32],$$

$$f_3 = -\frac{\pi}{4A} e^{-5\Delta^2/2}[-4e^{7\Delta^2/16} + 2e^{5\Delta^2/4} + 1],$$

$$f_4 = -\frac{\pi}{A} e^{-13\Delta^2/8}, \quad f_5 = \frac{5\pi}{4A} e^{-5\Delta^2/4}\Delta^2,$$

and

$$A = \pi e^{-5\Delta^2/4}[-2e^{5\Delta^2/8} + 2e^{5\Delta^2/4} + (1 - 2e^{5\Delta^2/8})\cos(\Delta^2) + 1].$$

In contrast to the case of the string molecule, the energy of the lattice molecule turns out to contain oscillatory terms, which result in the local minima of the binding energy. This is clear in Fig. 11, where we plot the binding energy for the two cases. It is thus evident that molecules composed of a two-dimensional geometry such as square lattices are expected to be more robust than one-dimensional structures such as a string molecule.

V. CONCLUSION

In Refs. [25,26] the dynamical instabilities of vortices in two- and three-dimensional geometries were studied. By introducing the appropriate symmetry-breaking perturbation, the Bogoliubov–de Gennes equations were used to calculate the energy spectrum and showed that with the perturbation corresponding to the dipole mode, for instance, the vortex splits into two clusters that revolve around the center of the trap and undergo split-merge cycles. This behavior is similar to our finding for the case of a two-soliton molecule evolving into a ring soliton, as shown in Fig. 4, where the hole between the two solitons of this figure can now be regarded as a vortex. As we have seen above, a ring soliton can be generated with any number of bright solitons provided they are out of phase and initially close enough to each other. For instance, nine solitons form the ring soliton of Fig. 7. Similar examples can be considered to generate the quadrupole and octupole modes of Ref. [25,26]. For the case of square lattice out-of-phase solitons, we have seen above that an $n \times n$ lattice undergoes a cycle into an $(n-1) \times (n-1)$ lattice. Within the present context, this is a dynamical instability of the lattice into one of its excited modes. Such a ringlike type (necklace) of molecules was also considered in optical solitons [20,24].

The possibility of obtaining stable two-dimensional structures of solitons, as shown in this paper, lays the groundwork for a realistic study of a two-dimensional soliton gas. One can, for instance, start with solitons distributed randomly and assign a random speed to each soliton in addition to the angular speed required to cancel the force of the trap. The properties of the gas can thus be studied in the rotating frame. Two main features of the soliton interaction are expected to considerably distinguish the resulting dynamics in comparison with a classical gas. These are the phase-dependent interaction and the possibility of solitons crossing each other [30].

Experimentally, the soliton molecules described in this paper can be excited by an appropriate phase mask with a laser beam that has a Gaussian intensity profile. This will mimic the initial wave function that we have employed here.

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APPENDIX: DERIVING THE ASYMPTOTIC FORM OF THE INTERACTION BETWEEN SOLITONS IN TWO DIMENSIONS

The time-independent Gross-Pitaevskii equation takes the form

$$\mu\psi = \frac{1}{2}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\psi, \tag{A1}$$

where $\rho = \sqrt{x^2 + y^2}$ and μ is the chemical potential. The solution of this equation that satisfies the boundary condition $\psi(\rho \to \infty) \to 0$ takes the asymptotic form

$$\psi(\rho,t) = i\sqrt{\frac{8}{\pi\sqrt{2\mu}}} e^{i\mu t} \frac{e^{-\sqrt{2\mu}\rho}}{\sqrt{\rho}}.$$

The interaction energy between two solitons separated by a distance Δ will be proportional to the spacial integral of the overlap between the two tails of the solitons, namely,

$$V(\Delta) \propto \int_0^{\Delta} \rho\, d\rho \int_0^{2\pi} d\phi \frac{e^{-\rho}}{\sqrt{\rho}} \frac{e^{-\sqrt{(\rho-\Delta\cos\phi)^2+(\Delta\sin\phi)^2}}}{\sqrt{(\rho - \Delta\cos\phi)^2 + (\Delta\sin\phi)^2}}, \tag{A2}$$

where ϕ is the angle between Δ and ρ. This gives the result in Eq. (2).

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