Stability of nonlinear normal modes in the Fermi-Pasta-Ulam β chain in the thermodynamic limit

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All possible symmetry-determined nonlinear normal modes (also called simple periodic orbits, one-mode solutions, etc.) in both hard and soft Fermi-Pasta-Ulam β chains are discussed. A general method for studying their stability in the thermodynamic limit as well as its application for each of the above nonlinear normal modes are presented.

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I. INTRODUCTION

This paper is devoted to studying the stability of dynamical objects which are called by very different terms such as one-mode solutions (OMSs) [1,2], simple periodic orbits (SPOs) [3], low-dimensional solutions [4], and one-dimensional bushes [5,6]. Below we refer to them as *nonlinear normal modes* (NNMs). Let us comment on this terminology.

The concept of similar nonlinear normal modes was developed by Rosenberg many years ago [7]. Each NNM represents a periodic vibrational regime in the conservative N-particle mechanical system for which the displacement $x_i(t)$ of every particle is proportional to the displacement of an arbitrary chosen particle, say, the first particle $[x_1(t)]$ at any instant t,

$$x_i(t) = c_i x_1(t), \tag{1}$$

where $\{c_1 = 1, c_2, c_3, \ldots, c_N\}$ are constant coefficients [8].

Note that convenient *linear normal modes* (LNMs) also satisfy Eq. (1) since, for any such mode, one can write,

$$x_i(t) = a_i \sin(\omega t + \phi_0), \quad i = 1, \ldots, N, \tag{2}$$

where a_i are constant amplitudes of individual particles, while ω and ϕ_0 are the frequency and initial phase of the considered mode.

As a rule, NNMs can exist in the mechanical systems with rather specific interparticle interactions, for example, in systems whose potential energy represents a *homogeneous* function with respect to all its arguments. However, in some cases, the existence of NNMs is caused by certain symmetry-related reasons. We refer to such dynamical objects as *symmetry-determined* NNMs. In Ref. [9], we have found all symmetry-determined NNMs in all N-particle mechanical systems with any of 230 space groups. This proved to be possible due to the group-theoretical methods developed in Refs. [10–12] for constructing *bushes* of vibrational modes.

There are many papers that discuss stability of NNMs and other time-periodic regimes in different lattice models. Great difficulties arise in studying the stability of nonlinear vibrations in the thermodynamic limit, i.e., when the number N of dynamical equations tends to infinity ($N \to \infty$). In this case, one can use the well-known Floquet method which leads to a system of N linear differential equations with time-periodic coefficients. This so-called *variational* system can be obtained by linearizing the original nonlinear equations near the considered dynamical regime.

In Refs. [13,14], for physical systems with *discrete symmetry*, we have developed a group-theoretical method for splitting a high-dimensional variational system into a number of independent subsystems of much smaller dimensions and have demonstrated how it works for some dynamical models. It is important to emphasize that this method can be successfully applied to analyze the stability of dynamical regimes of different types—they can be periodic as well as nonperiodic in time.

In this paper, we, first, demonstrate how the above method of splitting the variational system into independent subsystems can be applied in the case where $N \to \infty$ and, second, we develop a specific asymptotic technique to study the stability of zero solution of these subsystems. Note that such an approach can be used to analyze stability in the thermodynamic limit of periodic dynamical regimes in a variety of physical systems with discrete symmetry.

We demonstrate our mathematical techniques with one of the dynamical models that have been proposed in the famous work by Fermi, Pasta, and Ulam [15] in connection with the investigation of energy equipartition between normal modes in weakly nonlinear many-particle physical systems. In fact, it is this work that has been used prominently in computer simulations in modern science. The above models are now called FPU chains of α and β types. These models have played a fundamental role in the development of nonlinear dynamics as an independent scientific discipline. Investigation of the FPU chains dynamics has led to rediscovery of solitons, to detecting some properties of the deterministic (dynamical) chaos, and to revealing a number of new nonlinear dynamical objects (see, for example, Ref. [16]). There are many papers devoted to studying NNMs, discrete breathers, and bushes of vibrational modes in the FPU chains and their various generalizations. Keeping in mind the great role of FPU models in analyzing various nonlinear dynamics problems, we have studied the stability of all symmetry-determined NNMs in the FPU β chain and present here some *analytical* results on their stability properties in the *thermodynamic limit* ($N \to \infty$). We also compare our results with those by different authors when it was possible.

The paper is organized as follows. In Sec. II the concept of *bushes* of NNMs is discussed. Some comments on the stability of NNMs are presented in Sec. III. In Sec. IV all

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possible symmetry-determined NNMs in the FPU β chain are considered, while stability diagrams for these modes can be found in Sec. V. An analytical method for studying the stability of NNMs in the thermodynamical limit is presented in Sec. VI. We list our results on the stability properties for every NNM in Sec. VII. Some technical details are given in the appendices.

II. NONLINEAR NORMAL MODES AND THEIR BUSHES

The concept of bushes of modes was introduced in Ref. [10] (the theory of these dynamical objects can be found in Refs. [12,17]). In a rigorous mathematical sense, they represent symmetry-determined *invariant manifolds* decomposed into the basis vectors of *irreducible representations* of the symmetry group characterizing the considered mechanical system ("parent" group). Because of the specific subject of the present paper, it is sufficient to consider only bushes of vibrational modes in nonlinear monoatomic chains. Such bushes have been discussed in Refs. [5,6]. Let us reproduce here some ideas and results from these papers.

Every bush $B[G]$ describes a certain vibrational regime, and some specific *pattern* of instantaneous displacements of all the particles of the mechanical system corresponds to it. In turn, this pattern is characterized by a symmetry group G (in particular, such group can be trivial) which is a *subgroup* of the symmetry group G_0 of the mechanical system in its equilibrium state. For example, let us consider the *two-dimensional* bush $B[a^4,i]$ in the monoatomic chain with periodic boundary conditions whose displacement pattern $\vec{X}(t)$ can be written as follows:

$$\vec{X}(t) = \{x_1(t),x_2(t),-x_2(t),-x_1(t)| \\ x_1(t),x_2(t),-x_2(t),-x_1(t)|\dots\}. \quad (3)$$

This pattern is determined by two time-dependent functions $x_1(t)$, $x_2(t)$, and the corresponding dynamical regime of the N-particle chain is fully described by displacements inside the *vibrational primitive cell*, which is 4 times larger than that of the equilibrium state. We will refer to the ratio of the primitive cell size of the vibrational state to that of the equilibrium state as *multiplication number* (MN) and, therefore, for the pattern (3), one can write MN = 4.

The symmetry group $G = [a^4,i]$ of the bush $B[a^4,i]$ is determined by two *generators*: the translation (a^4) by four lattice spacings (a) and the inversion (i) with respect to the center of the chain (note that the condition $N \bmod 4 = 0$ must hold for the existence of such a bush).

If we decompose the displacement pattern (3) into the linear normal coordinates [18],

$$\vec{\Psi}_j = \left\{ \frac{1}{\sqrt{N}} \left[\sin\left(\frac{2\pi j}{N} n\right) + \cos\left(\frac{2\pi j}{N} n\right) \right] \middle| n = 1 \dots N \right\} \\ (j = 0 \dots N-1), \quad (4)$$

we get the following form of the bush $B[a^4,i]$ in the *modal space*:

$$\vec{X}(t) = \mu(t)\vec{\Psi}_{N/2} + \nu(t)\vec{\Psi}_{3N/4}, \quad (5)$$

where

$$\vec{\Psi}_{N/2} = \frac{1}{\sqrt{N}}\{-1,1|-1,1|-1,1|-1,1|\dots\}, \quad (6)$$

$$\vec{\Psi}_{3N/4} = \frac{1}{\sqrt{N}}\{-1,-1,1,1|-1,-1,1,1|\dots\}, \quad (7)$$

while $\mu(t)$ and $\nu(t)$ are time-dependent coefficients in front of the normal coordinates $\vec{\Psi}_{N/2}$ and $\vec{\Psi}_{3N/4}$. Thus, only $m = 2$ normal coordinates from the full set (4) contribute to the "configuration vector" $\vec{X}(t)$ corresponding to the given bush and we will refer to m as the *bush dimension*.

In Ref. [6], we developed a simple crystallographic method for obtaining the displacement pattern $\vec{X}(t)$ for any subgroup G of the parent group G_0. Using this method, one can obtain bushes of different dimensions for an arbitrary nonlinear chain. The *one-dimensional bushes* $(m = 1)$ represent symmetry-determined nonlinear normal modes. The displacement pattern $\vec{X}(t)$ corresponding to a given NNM depends on only one (time-periodic) function $\nu(t)$,

$$\vec{X}(t) = \nu(t) \cdot \vec{c}, \quad (8)$$

where $\vec{c}$ is a constant vector, which is formed by the coefficients c_i $(i = 1 \dots N)$ from Eq. (1), while the function $\nu(t)$ satisfies a certain differential equation. This so-called governing equation can be obtained by substitution of the ansatz (8) into the dynamical equations of the considered chain.

In some sense, the concept of bushes of vibrational modes can be considered as a certain *generalization* of the notion of NNMs by Rosenberg. Indeed, if we substitute the ansatz (5) into dynamical equations of the chain, we obviously get two "governing" equations for the functions $\nu(t)$ and $\mu(t)$ that determine the above-discussed two-dimensional bush (note that, in contrast to a NNM, such a dynamical object describes, in general, a *quasiperiodic* motion). Finally, one can conclude that an m-dimensional bush is determined by m time-dependent functions for which m governing differential equations can be obtained from the dynamical equations of the considered mechanical system. Let us emphasize that bushes of modes represent a new type of *exact* excitation in nonlinear systems with discrete symmetries and the excitation energy proves to be trapped in a given bush for the case of Hamiltonian systems.

It is very important to emphasize that there exist only a *finite number* of vibrational bushes of any fixed dimension in every N-particle mechanical system. As a consequence, there is a sufficiently small number of NNMs (one-dimensional bushes) in the FPU chains (three NNMs in the FPU α model and six—in the FPU β model). All possible one-dimensional bushes are explicitly listed in our papers [5,6] (see also Ref. [19]).

III. SOME COMMENTS ON THE STABILITY OF NNMS

The stability of some NNMs in the FPU chains has been studied in Refs. [1–4,20–28] by use of numerical and analytical methods. In Refs. [3,27], T. Bountis and coworkers have investigated the destabilization thresholds (E_{1u} and E_{2u}) of two nonlinear normal modes which they call SPO-1 and SPO-2 by use of numerical methods. The authors of the above papers tried to reveal some relations between the destabilization thresholds E_{1u}, E_{2u} and the origin of the weak

chaos in connection with the famous Fermi-Pasta-Ulam problem of the energy equipartition between different modes. In particular, they conclude that the main role in the weak chaos appearance in the thermodynamic limit ($N \to \infty$) is played by SPO-2, because $E_{2u} \sim \frac{1}{N^2}$, $E_{1u} \sim \frac{1}{N}$, and, therefore, $E_{2u} < E_{1u}$. However, there are some other SPOs in the FPU β chain and there is some interest in their role in the origin of the weak chaos in the thermodynamic limit. Some comments are appropriate at this point.

According to Lyapunov [29], a set of strictly periodic orbits for nonlinear systems can be obtained from the linear normal modes (which are introduced in the harmonic approximation) by continuation with respect to a parameter characterizing the strength of nonlinearity [30]. From this point of view, there exist N different SPOs for longitudinal vibrations of an N particle monoatomic chain. However, only a few of the modes, constructed in such a way, possess an *identical* time dependence of the displacements of all the particles. More exactly, only few of the Lyapunov modes can be written in the form (8), implying a separation of time and space variables that is typical for the Rosenberg nonlinear normal modes. Indeed, in the general case, $x_i(t) = \nu_i(t)c_i$, where $\nu_i(t)$ $(i = 1, \ldots, N)$, there are *different* functions of time with identical periods. Note that in the present paper we consider only *extended* SPOs, but the same problem exists for *localized* periodic modes (discrete breathers) and we have discussed it in Ref. [31].

As far as we aware, all periodic solutions in monoatomic chains that have been studied (see the above cited papers [1–4] and references therein) belong to the class of the Rosenberg nonlinear normal modes determined by Eq. (8). Moreover, the spatial profiles $\vec{c} = \{c_1, c_2, \ldots, c_N\}$ of these modes possess certain symmetry properties. In particular, every such mode can be characterized by a MN determining the enlargement of the primitive cell of the vibrational state in comparison with that of the equilibrium state. As was already noted, we refer to these modes as symmetry-determined NNMs and there exist only a finite number of such modes (even for the case where $N \to \infty$) for each nonlinear chain [5,6,19].

The above considered SPO-1 and SPO-2 and the well-known π mode (zone boundary mode) represent NNMs with multiplication numbers 4, 3, and 2, respectively. However, among *six* symmetry-determined NNMs in the FPU β chain [6] there exist another three NNMs with MN = 3, MN = 4, and MN = 6. The stability of the second NNM with MN = 4 in the thermodynamic limit were studied in Ref. [2] by both numerical and analytical methods.

The stability diagrams for all the nonlinear normal modes in the FPU β chain, as well as for the FPU α chain, can be found in our paper [6]. With the aid of these diagrams, one may reveal many stability properties of NNMs for an arbitrary N, in particular, for the thermodynamic limit ($N \to \infty$). Note that these diagrams were obtained numerically.

IV. NONLINEAR NORMAL MODES IN THE FPU-β CHAIN

As already mentioned, there exist only a *finite* number of symmetry-determined NNMs in any monoatomic chain. Every NNM corresponds to a certain *subgroup* of the symmetry group of the chain dynamical equations. The difference in the number of nonlinear normal modes for the FPU α chain (three NNMs) and for the FPU β chain (six NNMs) is associated with the fact that the symmetry group of the FPU β chain dynamical equations is higher than that of the FPU α chain [5,6,19].

The FPU β model represents a chain of unit masses coupled with each other by the appropriate nonlinear springs. The dynamical equations describing longitudinal vibrations of the FPU β chain can be written in the form,

$$\ddot{x}_i = f(x_{i+1} - x_i) - f(x_i - x_{i-1}), \quad i = 1 \ldots N, \tag{9}$$

where $x_i(t)$ is the displacement of the ith particle from its equilibrium state at the instant t, while the force $f(\Delta x)$ depends on the spring deformation Δx as

$$f(\Delta x) = \Delta x + \beta(\Delta x)^3. \tag{10}$$

The periodic boundary condition is assumed to hold

$$x_{N+1}(t) \equiv x_1(t), \quad x_0(t) \equiv x_N(t). \tag{11}$$

Let us now mention some results in Ref. [6], which are necessary for further discussions.

Every NNM in the FPU β chain can be written as follows [see Eq. (8)]:

$$\vec{X}(t) = \nu(t) \cdot \vec{c},$$

where $\nu(t)$ satisfies the Duffing equation

$$\ddot{\nu} + \omega^2 \nu + \gamma \frac{\beta}{N} \nu^3 = 0 \tag{12}$$

with different values ω and γ for different NNMs. The function $\nu(t)$ describes the time evolution of a given NNM, while the N-dimensional vector $\vec{c}$ determines the pattern of the particle displacements.

Below, we list all possible NNMs in the FPU β chain.

(1) $B[a^2,i]$

$$\vec{c} = \frac{1}{\sqrt{N}}\{1,-1|1,-1|1,-1|\ldots\}, \quad \omega^2 = 4, \quad \gamma = 16 \quad (N \bmod 2 = 0). \tag{13}$$

This is a boundary zone mode or π mode.

(2) $B[a^3,i]$

$$\vec{c} = \frac{3}{\sqrt{6N}}\{1,0,-1|1,0,-1|1,0,-1|\ldots\}, \quad \omega^2 = 3, \quad \gamma = \frac{27}{2} \quad (N \bmod 3 = 0). \tag{14}$$

There exist three "dynamical domains" of this NNM (see below).

(3) $B[a^4,ai]$

$$\vec{c} = \frac{2}{\sqrt{2N}}\{0,1,0,-1|0,1,0,-1|\ldots\}, \quad \omega^2 = 2, \quad \gamma = 4 \quad (N \bmod 4 = 0). \tag{15}$$

There exist two dynamical domains of this NNM.

(4) $B[a^3,iu]$

$$\vec{c} = \frac{1}{\sqrt{2N}}\{1,-2,1|1,-2,1|1,-2,1|\ldots\}, \quad \omega^2 = 3, \quad \gamma = \frac{27}{2} \quad (N \bmod 3 = 0). \tag{16}$$

There exist three dynamical domains of this NNM.

(5) $B[a^4,iu]$

$$\vec{c} = \frac{1}{\sqrt{N}}\{1,-1,-1,1|1,-1,-1,1|\ldots\},$$
$$\omega^2 = 2, \quad \gamma = 8 \quad (N \bmod 4 = 0). \tag{17}$$

There exist two dynamical domains of this NNM.

(6) $B[a^3u,ai]$

$$\vec{c} = \frac{3}{\sqrt{6N}}\{0,1,1,0,-1,-1|0,1,1,0,-1,-1|\ldots\},$$
$$\omega^2 = 1, \quad \gamma = \frac{3}{2} \quad (N \bmod 6 = 0). \tag{18}$$

There exist three dynamical domains of this NNM.

Let us comment on the above-listed NNMs in the FPU β chain. Every NNM, denoted by the $B[G]$, is characterized by the corresponding symmetry group G, which represents a certain subgroup of the symmetry group $G_0 = [a,i,u]$ of the FPU β dynamical equations (9) and (10). We determine every such group by the set of its generators using the following notations: a, the translation of the chain by one lattice spacing; i, the inversion with respect to the center of the chain; and u, the operator, which changes signs of the displacements of all particles without any their transposition.

The symmetry group $G_0 = [a,i,u]$ of the FPU β dynamical equations is described by *three* generators (a, i, and u). The corresponding transformations a, i, and u of N-dimensional vectors $\vec{x} = \{x_1,x_2,\ldots,x_N\}$ *do not* change the dynamical equations (9) and (10) of the FPU β chain.

All the above-listed groups of NNMs are fully described by only *two* generators, but these generators can be written as some *products* of the generators a, i, and u of the group G_0. For example, a^2, a^3, and a^4 are translations of the chain by two, three, and four lattice spacings, respectively. The transformation ai means that we must perform the inversion of the displacement pattern with respect to the chain center and then translate it by one lattice spacing.

Note that transformations a and i do not commute,

$$ia = a^{-1}i \quad \text{or} \quad ia = a^{N-1}i \tag{19}$$

[the relation $a^{-1} = a^{N-1}$ holds because of the periodic boundary condition (11)]. On the other hand, the transformation u does commute with both a and i transformations,

$$ua = au, \quad ui = iu. \tag{20}$$

The transformation a^2u means that we must change signs of all displacements and then translate the displacement pattern $\vec{X}$ by two lattice spacings.

Some simple examples are worth mentioning at this point. For the chain with $N = 6$ particles, we can write the following relations:

$$a\,\{x_1,x_2,x_3,x_4,x_5,x_6\} = \{x_2,x_3,x_4,x_5,x_6,x_1\},$$
$$i\,\{x_1,x_2,x_3,x_4,x_5,x_6\} = \{-x_6,-x_5,-x_4,-x_3,-x_2,-x_1\},$$
$$iu\,\{x_1,x_2,x_3,x_4,x_5,x_6\} = \{x_6,x_5,x_4,x_3,x_2,x_1\},$$
$$a^2i\,\{x_1,x_2,x_3,x_4,x_5,x_6\} = \{-x_4,-x_3,-x_2,-x_1,-x_6,-x_5\},$$
$$a^2u\,\{x_1,x_2,x_3,x_4,x_5,x_6\} = \{-x_3,-x_4,-x_5,-x_6,-x_1,-x_2\},$$

and so on.

The displacement pattern corresponding to a given NNM can be obtained as an *invariant vector* of its symmetry group $G \subset G_0$. For example, let us obtain the displacement pattern for the NNM with $G = [a^4,ai]$ [see Eq. (15)]. For simplicity, we demonstrate the method for obtaining displacement patterns with the case where $N = 8$. Let

$$\vec{X} = \{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\},$$

where x_i $(i = 1\ldots8)$ are arbitrary displacements of eight particles of the chain. The vector $\vec{X}$ must be invariant with respect to the action of our two generators a^4 and ai of the symmetry group of the considered NNM,

$$a^4\vec{X} = \vec{X}, \quad ai\vec{X} = \vec{X}. \tag{21}$$

The former equation is reduced to the following form:

$$a^4\vec{X} = \{x_5,x_6,x_7,x_8|x_1,x_2,x_3,x_4\}$$
$$= \{x_1,x_2,x_3,x_4|x_5,x_6,x_7,x_8\},$$

from which we conclude that

$$x_5 = x_1, \quad x_6 = x_2, \quad x_7 = x_3, \quad x_8 = x_4.$$

This displacement pattern is formed by two vibrational primitive cells whose size is 4 times larger than that of the FPU β chain in its equilibrium state. The sets of the displacements in both cells are identical,

$$\vec{X} = \{x_1,x_2,x_3,x_4|x_1,x_2,x_3,x_4\}. \tag{22}$$

Now let us take into account the second equation (21). Acting on the vector (22) by ai, we obtain

$$ai\vec{X} = a\{-x_4,-x_3,-x_2,-x_1|-x_4,-x_3,-x_2,-x_1\}$$
$$= \{-x_3,-x_2,-x_1,-x_4|-x_3,-x_2,-x_1,-x_4\}.$$

Then, using the equation $ai\vec{X} = \vec{X}$, we get

$$x_1 = -x_3, \quad x_2 = -x_2 = 0, \quad x_4 = -x_4 = 0.$$

Thus, the invariant (under the action of the group $G = [a^4,ai]$) vector $\vec{X}$ depends on *only one* arbitrary parameter, which we denote by x,

$$\vec{X} = \{x,0,-x,0|x,0,-x,0\}. \tag{23}$$

(Note that this vector being invariant with respect to generators of the group $G = [a^4,ai]$ will automatically be invariant relative to all its other elements). Then the NNM corresponding to the invariant vector (23) can be written as follows

$$\vec{X}_{\rm NNM}(t) = \{\nu(t),0,-\nu(t),0|\nu(t),0,-\nu(t),0\}$$
$$= \nu(t)\{1,0,-1,0|1,0,-1,0\}. \tag{24}$$

To find all NNMs, we can try *all subgroups* of the symmetry group $G_0 = [a,i,u]$ to choose those displacement patterns, which depend on *only one* arbitrary parameter.

The patterns depending on m arbitrary parameters with $m > 1$ form the m-dimensional bushes of vibrational modes. In this sense, nonlinear normal modes may be called one-dimensional bushes. In Refs. [10,12], three different group-theoretical methods for constructing the bushes of vibrational modes in *arbitrary* N-particle nonlinear mechanical systems were developed. The most efficient of these methods uses the concept of irreducible representations of the symmetry groups.

TABLE I. Nonlinear normal modes in the FPU β chain.

NNM	Displacement pattern	Modal space representation	Governing equation
$B[a^2,i]$	$\lvert 1,-1\rvert$	$\nu\Psi_{N/2}$	$\ddot{\nu}+4\nu+\frac{16\beta}{N}\nu^3=0$
$B[a^3,i]$	$\lvert 1,-1,0\rvert$	$\nu\frac{1}{\sqrt{2}}(\Psi_{N/3}-\Psi_{2N/3})$	$\ddot{\nu}+3\nu+\frac{27\beta}{2N}\nu^3=0$
$B[a^3,iu]$	$\lvert 1,1,-2\rvert$	$\nu\frac{1}{\sqrt{2}}(\Psi_{N/3}+\Psi_{2N/3})$	$\ddot{\nu}+3\nu+\frac{27\beta}{2N}\nu^3=0$
$B[a^4,ai]$	$\lvert 1,0,-1,0\rvert$	$\nu\frac{1}{\sqrt{2}}(\Psi_{N/4}-\Psi_{3N/4})$	$\ddot{\nu}+2\nu+\frac{4\beta}{N}\nu^3=0$
$B[a^4,iu]$	$\lvert 1,-1,-1,1\rvert$	$\nu\Psi_{N/4}$	$\ddot{\nu}+2\nu+\frac{8\beta}{N}\nu^3=0$
$B[a^3u,ai]$	$\lvert 1,1,0,-1,-1,0\rvert$	$\nu\frac{1}{\sqrt{2}}(\Psi_{N/6}-\Psi_{5N/6})$	$\ddot{\nu}+\nu+\frac{3\beta}{2N}\nu^3=0$

Taking into account the above method that was used for constructing Eq. (24), we conclude that every NNM can be written in the form

$$\vec{X}_{\mathrm{NNM}}(t)=\nu(t)\cdot\vec{c}, \tag{25}$$

where $\vec{c}$ is a certain time-independent vector. Substituting ansatz (25) into the dynamical equations (9) and (10) of the FPU β chain, with explicit forms of the vectors $\vec{c}$ from Eqs. (13)–(18), one can find that FPU β equations are reduced to only one differential equation (governing equation of the corresponding NNM) of the form,

$$\ddot{\nu}+\omega^2\nu+\gamma\frac{\beta}{N}\nu^3=0. \tag{26}$$

This is the Duffing equation with different values ω and γ for different NNMs which are listed in Eqs. (13)–(18).

Above, we have mentioned the existence of so-called dynamical domains of all nonlinear normal modes presented in Eqs. (13)–(18). Let us comment on this notion borrowed from the theory of phase transitions. We have already emphasized that a certain symmetry group G corresponds to every NNM. This group is a subgroup of the symmetry group of the considered mechanical system in its equilibrium state ($G\subset G_0$). If we act on the vector $\vec{X}(t)$ corresponding to a given NNM by operator $g\in G_0$, which *does not* belong to subgroup G, we get the *equivalent* configuration vector $\tilde{\vec{X}}(t)=g\vec{X}(t)$. The equivalent vector $\tilde{\vec{X}}(t)$ corresponds to a new NNM, which is described by the *same* dynamical equations as that of the NNM associated with the vector $\vec{X}(t)$. For example, three dynamical domains are associated with the NNM from Eq. (14),

$$B[a^3,i]:\vec{c}=\frac{3}{\sqrt{6N}}\{1,0,-1|1,0,-1|\ldots\}, \tag{27}$$

$$B[a^3,ai]:\vec{c}=\frac{3}{\sqrt{6N}}\{0,1,-1|0,1,-1|\ldots\}, \tag{28}$$

$$B[a^3,a^2i]:\vec{c}=\frac{3}{\sqrt{6N}}\{1,-1,0|1,-1,0|\ldots\}. \tag{29}$$

All the displacement patterns (27)–(29) differ from each other by a cyclic transposition of the displacements inside each primitive cell of the chain vibrational state. Let us note that the symmetry groups G_j ($j=1,2,3$) of NNMs from Eqs. (27)–(29) prove to be *conjugate* subgroups in the parent group G_0, for example, $G_2=g^{-1}G_1g$ ($g\in G_0$).

Since the above-discussed "domains" possess equivalent dynamical properties, we study below the stability of only one copy of the full set of dynamical domains for every NNM in the FPU β chain. All symmetry-determined NNMs that can exist in the FPU β chain with an appropriate number of particles are listed in Table I.

V. STABILITY DIAGRAMS FOR NNMS IN THE FPU β CHAIN

The detailed numerical analysis of stability of NNMs in the FPU chains can be found in Ref. [6], where results have been presented in the form of certain "stability diagrams." Let us consider the structure of such diagrams and the method of their obtaining.

Every NNM represents a *periodic* dynamical regime and the standard Floquet method can be applied for investigation of its stability. According to this method, we have to linearize the nonlinear FPU β dynamical equations (9) and (10) in the vicinity of a given NNM $\vec{X}_{\mathrm{NNM}}(t)$. To this end, let us introduce an infinitesimal vector

$$\vec{\delta}=\{\delta_1,\delta_2,\delta_3,\ldots,\delta_N\}, \tag{30}$$

which determines a certain *perturbation* of the NNM. Letting

$$\vec{X}(t)=\vec{X}_{\mathrm{NNM}}(t)+\vec{\delta}(t), \tag{31}$$

we substitute $\vec{X}(t)$ into nonlinear equations (9) and (10) and omit all nonlinear terms in $\delta_j(t)$ ($j=1\ldots N$). As a result of this procedure, we get the linearized (variational) system

$$\ddot{\vec{\delta}}=J(t)\,\vec{\delta}, \tag{32}$$

where $J(t)$ is the corresponding Jacobian matrix (see details in Ref. [6] and especially in Ref. [13]). Equation (32) represents an $N\times N$ system of linear differential equations with time-dependent coefficients, which, in turn, are determined by the periodic function $\nu(t)$ describing the time evolution of the considered NNM.

The straightforward way to analyze the stability becomes practically impossible for $N\to\infty$. In Ref. [13], the general group-theoretical method has been developed for splitting (decomposition) of the original system $\ddot{\vec{\delta}}=J(t)\,\vec{\delta}$ of N linear differential equations into certain subsystems of sufficiently small dimensions $N_j\ll N$. For the FPU β chain, these dimensions do not exceed three. We may then apply the Floquet method for such subsystems of small dimensions. Moreover, proceeding in this manner, one can reveal those subsets of the vibrational modes which are responsible for the first loss of stability of the considered NNM. As a consequence of this approach, it proves to be possible to construct very transparent diagrams which demonstrate explicitly stability properties of each FPU nonlinear normal mode [6].

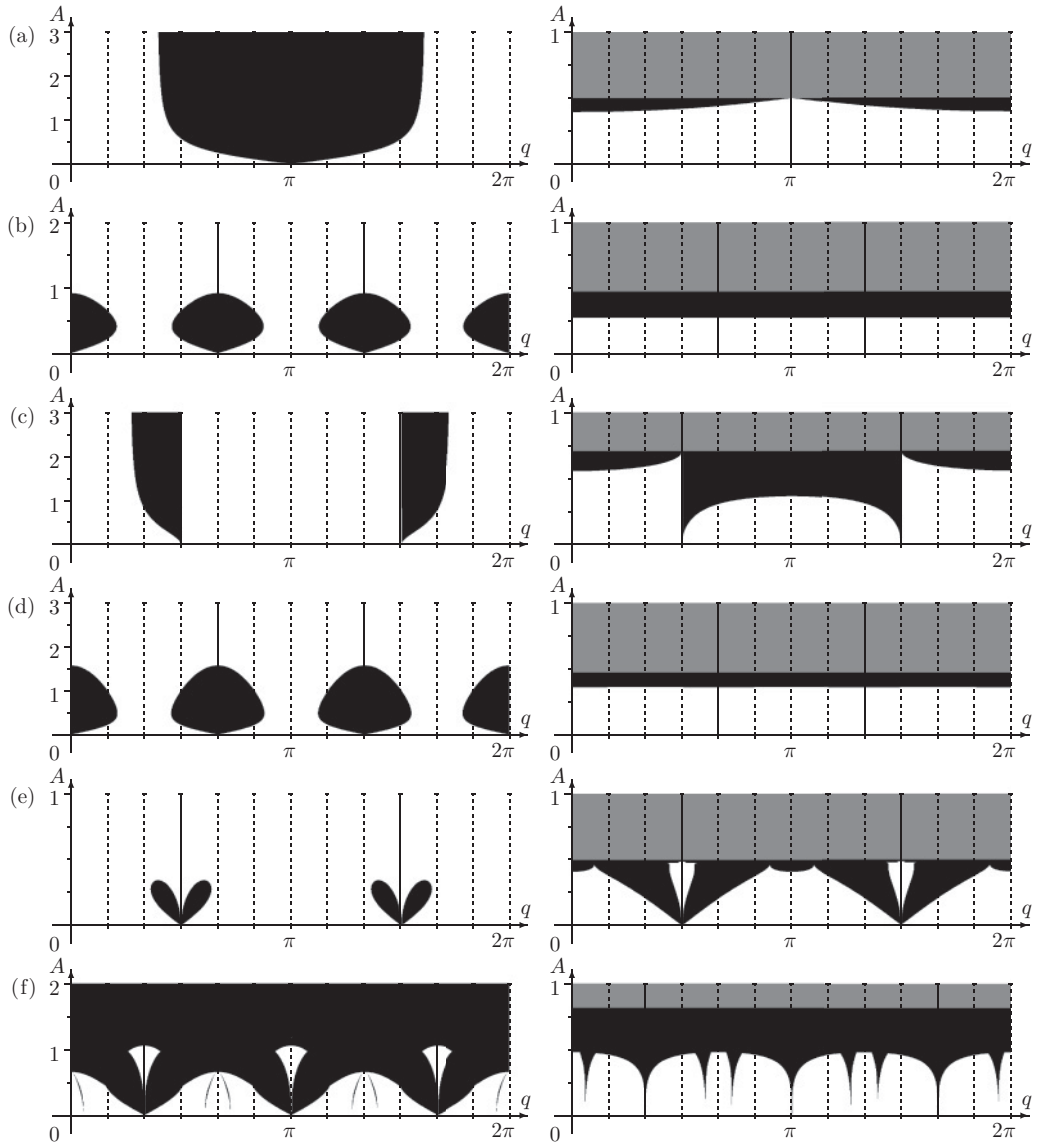

FIG. 1. Stability diagrams for NNMs in the FPU β chain (left $\beta > 0$, right $\beta < 0$): (a) $B[a^2,i]$; (b) $B[a^3,i]$; (c) $B[a^4,ai]$; (d) $B[a^3,iu]$; (e) $B[a^4,iu]$; and (f) $B[a^3u,ai]$.

In the explicit form, the above splitting can be found for all NNMs in the FPU β chain in Table 8 of Ref. [6]. Let us comment on that table using as an example the nonlinear normal mode $B[a^4,iu]$ for $\beta > 0$, whose stability diagram resembling rabbit ears is depicted in Fig. 1.

In this case, the variational system can be split into $(\frac{N}{2} - 2)$ two-dimensional systems of the following form:

$$\begin{aligned}
&\ddot{\delta}_j + 4\sin^2\left(\frac{\pi j}{N}\right)\left[1 + \frac{6\beta}{N}\nu^2(t)\right]\delta_j \\
&\quad = -\frac{12\beta}{N}\nu^2(t)\sin\left(\frac{2\pi j}{N}\right)\delta_{j'}, \\
&\ddot{\delta}_{j'} + 4\sin^2\left(\frac{\pi j'}{N}\right)\left[1 + \frac{6\beta}{N}\nu^2(t)\right]\delta_{j'} \\
&\quad = -\frac{12\beta}{N}\nu^2(t)\sin\left(\frac{2\pi j}{N}\right)\delta_j. \qquad (33)
\end{aligned}$$

Here $j' = \frac{N}{2} - j$ and $j = 1, \ldots, (\frac{N}{2} - 1)$.

Thus, the jth component of the infinitesimal vector $\vec{\delta}$ (after the appropriate orthogonal transformation) turns out to be coupled only to its j' component, where $j' = \frac{N}{2} - j$. Therefore, the investigation of the stability of the nonlinear normal mode $B[a^4,iu]$ reduces the study of the stability of the *zero solution* of Eqs. (33) with different values of the indices j and $j' = \frac{N}{2} - j$. It can be seen from Eqs. (33) that the pairs of modes whose indices are situated symmetrically with respect to the index $j_0 = \frac{N}{4}$ of the considered NNM are coupled. Note that Eqs. (33) represent the system of two coupled differential equations with time-periodic coefficients determined by the solution $\nu(t)$ of the Duffing equation (26). It is well known that equations of such a type possess domains of stable and unstable motion [32].

For the system of Eqs. (33), we can construct the stability diagram in the plane (j-A), where A is the amplitude of the NNM [i.e., the amplitude of the function $\nu(t)$ from Eq. (26)], while j is its index. Such diagrams provide us with the most

valuable information for the case of the "thermodynamic limit" when the number of lattice cells tends to infinity ($N \to \infty$).

When $N \to \infty$, the number of modes also tends to infinity and it is convenient to introduce the wave number k ($0 \leqslant k < 2\pi$) instead of the mode index j:

$$k = \left(\frac{2\pi}{N}\right) j, \quad j = 1 \ldots N. \tag{34}$$

In Eq. (4) there is a normalizing coefficient $1/\sqrt{N}$ originating from the conventional definition of the mode norm $|\vec{\Psi}_j|^2 = 1$. As a consequence, displacements of all particles corresponding to a given normal mode with fixed amplitude tend to zero with $N \to \infty$, while the coefficient $1/N$ appears in Eqs. (26) and (33), etc., in combination with the coefficient β.

Hereafter, we use a more convenient (for our purposes) normalization of normal modes (normalization to the "volume" of the system), namely we assume $|\vec{\Psi}_j|^2 = N$. This permits us to exclude the coefficient $1/N$ coupled to β. Moreover, using a trivial coordinate rescaling, one can set $|\beta| = 1$. As a result, Eqs. (33) are transformed to the following forms:

$$\begin{aligned} &\ddot{\delta}_k + 4\sin^2\left(\frac{k}{2}\right)[1 \pm 6v^2(t)]\delta_k = \mp 12v^2(t)\sin(k)\,\delta_{k'}, \\ &\ddot{\delta}_{k'} + 4\cos^2\left(\frac{k}{2}\right)[1 \pm 6v^2(t)]\delta_{k'} = \mp 12v^2(t)\sin(k)\,\delta_k. \end{aligned} \tag{35}$$

Here the upper and lower signs correspond to $\beta > 0$ and $\beta < 0$, respectively.

Following this idea, we construct our stability diagrams in the (k-A) plane, where A is the amplitude of a normal mode subjected to the above-mentioned normalizing condition $|\vec{\Psi}_j|^2 = N$. For any fixed N, the permissible values of the wave number k represent the equidistant set inside the interval $[0,2\pi]$. When $N \to \infty$, these values of k form a *dense* set on the above interval, depicted on the horizontal axis of diagrams in Fig. 1. Rich information about stability properties of NNMs can be obtained from diagrams of such a type. Let us consider this idea in more detail using as an example the stability diagram of the NNM B$[a^4,iu]$ depicted in Fig. 1(e) for $\beta > 0$.

In this diagram, we denote the permissible values of the wave number k for $N = 12$ by the dotted vertical lines ($k = \gamma, 2\gamma, 3\gamma, \ldots, 12\gamma$, where $\gamma = 2\pi/12$). The black color corresponds to the regions of unstable motion in the plane (k-A). From Fig. 1(e) one can see that for the FPU β chain with $N = 12$ particles the NNM $B[a^4,iu]$ with $\vec{c} = \{1,-1,-1,1|1,-1,-1,1|1,-1,-1,1\}$ turns out to be *stable* for *every* amplitude A up to 2 (actually, at least, up to $A = 20$). Indeed, the dotted vertical lines do not cross the black (unstable) regions in the form of the rabbit ears. Obviously, the mode $B[a^4,iu]$ is also stable for the chains with $N = 4$ and $N = 8$ particles since the vertical dotted lines, similar to those depicted in Fig. 1(e), are *more distant* from each other than for $N = 12$, and the regions of instability situate fully between the neighboring lines.

In the case where $N = 16$, there exist such dotted vertical lines that are *tangents* to the rabbit-ears unstable regions, while for $N > 16$ (note that the relation $N \bmod 4 = 0$ must hold), these lines begin to *cross* the unstable regions. Therefore, we can conclude that the considered nonlinear normal mode in the FPU β chains with $N > 16$ particles becomes unstable for the vibrational amplitudes, which fall into the black regions in Fig. 1(e). The case where $N = 16$ represents the "boundary" case between stable and unstable behavior of the $B[a^4,iu]$ NNM for appropriate amplitudes.

From Fig. 1(e), it can be seen that the *critical* amplitude A_c of the nonlinear normal mode $B[a^4,iu]$, for which this mode loses its stability, decreases with increasing N and we can evaluate numerically the corresponding scaling law of the function $A_c(N)$ in the thermodynamic limit $N \to \infty$.

Let us consider this question in more detail. The instability regions touch the k axis at the points $\frac{\pi}{2}$ ($j = \frac{N}{4}$) and $\frac{3\pi}{2}$ ($j = \frac{3N}{4}$), which correspond to the NNM $B[a^4,iu]$ and its dynamical domain $B[a^4,a^2iu]$, respectively. Specifically, near these points the considered NNMs loses their stability in the thermodynamic limit $N \to \infty$.

Let us focus on the point ($k = \frac{\pi}{2}$), which corresponds to the index $j_0 = \frac{N}{4}$ of the mode $B[a^4,iu]$. All the neighboring k points correspond to the "sleeping" modes, i.e., modes with zero amplitudes for the stable behavior of the mode $B[a^4,iu]$. If we increase the amplitude A of the NNM $B[a^4,iu]$ from zero, the sleeping modes, which become excited first, are those with indices $j' = \frac{N}{4} \pm 1$, i.e., the modes closest in number to the index $j_0 = \frac{N}{4}$ of the NNM $B[a^4,iu]$.

We denote by $A_c(N)$ the critical value of the amplitude A of the considered nonlinear normal mode for which the loss of its stability takes place, i.e., $A_c(N)$ is the threshold of the $B[a^4,iu]$ NNM stability for a given N. Note that we can speak about the threshold value $A_c^{(j)}$ for the *excitation* of the sleeping j mode as a result of its interaction with the $B[a^4,iu]$ NNM whose index is equal to $j_0 = \frac{N}{4}$. From Fig. 1(e) we then obtain

$$A_c(N) = \min_j A_c^{(j)} = A_c^{(\frac{N}{4}\pm 1)}. \tag{36}$$

For the index j situated near $j_0 = \frac{N}{4}$ ($k = \frac{\pi}{2}$), one can see from Fig. 1(e), that

$$A_c^{(j)} = \Delta k \tan\alpha, \tag{37}$$

where $\Delta k = \frac{2\pi}{N}$ is the minimal distance between the neighboring permitted points on the k axis and $\tan\alpha$ is determined by the tangent to the black unstable region near the point $k_0 = \frac{\pi}{2}$.

The scaling law of the function $A_c(N)$ for $N \to \infty$ can be deduced from Eqs. (36) and (37):

$$A_c(N) = \frac{2\pi}{N}\tan\alpha \quad (N \to \infty). \tag{38}$$

Here $\tan\alpha \sim 1$, as shown in Fig. 1(e).

The above-discussed threshold function $A_c(N)$ for $N \to \infty$ can be calculated analytically. For the nonlinear normal mode $B[a^4,iu]$, such a calculation has been recently performed in Ref. [2]. The similar analytical results for the π mode ($j_0 = \frac{N}{2}$) were earlier obtained in Refs. [1,20,22–26]. In the present paper, we present the *analytical* estimations of stability thresholds for $N \to \infty$ for *all those possible* in the FPU β ($\beta > 0$ and $\beta < 0$) nonlinear normal modes (see Sec. VI).

Let us return to the stability diagram in Fig. 1(e) for the NNM $B[a^4,iu]$ for $\beta > 0$. It is very interesting that for the *sufficiently large* amplitudes ($A > A_c \approx 0.346$) the nonlinear normal mode $B[a^4,iu]$ again becomes stable *even in the thermodynamic limit* $N \to \infty$. Such results cannot be obtained

by use of the analytical methods applied in Ref. [2] and in the present paper. However, the stability of NNMs for nonlinear chains in the case of large amplitudes can be analyzed by use of the numerical method used in Ref. [6] which provides us with the stability diagrams depicted in Fig. 1.

In a recent paper [28], the existence of the "second stability threshold" E_c for the NNM $B[a^4,iu]$ was also revealed. Beyond E_c, the stability of the mode $B[a^4,iu]$ (in Ref. [28], it is called $\pi/2$ mode) is restored. This result was obtained by numeric methods. In term of the partial energy per one particle of the FPU β chain, this threshold turns out to be equal to $E_c = 0.14715$. On the other hand, we can find an approximate value for E_c directly from diagram depicted in Fig. 1(e): $E_c \approx 0.15$. This value is in good agreement with that from Ref. [28].

Moreover, one can reveal such a "surprising behavior" (as has been written in Ref. [28]) directly from our stability diagrams presented in Ref. [6] for NNMs $B[a^3,i]$ and $B[a^3,iu]$. We reproduce these diagrams in Figs. 1(b) and 1(d) of the present paper, and they show the approximate values of the "second stability thresholds" for the above-mentioned nonlinear normal modes: $E_c(B[a^3,i]) \approx 3.70$ and $E_c(B[a^3,iu]) \approx 24.8$.

In the right column of Fig. 1, we present new results on the stability of all possible NNMs in the FPU β chain with $\beta < 0$. In contrast to the case where $\beta > 0$, the chain with $\beta < 0$ is unstable itself for the mode amplitudes which exceed a certain critical value (this case corresponds to the region depicted by gray color in Fig. 1). Note that beyond $E = 0.092$ for the NNM $B[a^3,i]$ and $E = 0.14$ for the NNM $B[a^3,iu]$, all sleeping modes turn out to be excited *simultaneously*. We have previously revealed such a remarkable behavior for the stability properties of the FPU α chain (see Ref. [5]).

VI. ANALYTICAL METHOD FOR STABILITY ANALYSIS OF NONLINEAR NORMAL MODES IN THE THERMODYNAMIC LIMIT

For studying the stability of NNMs in the thermodynamic limit ($N \to \infty$), we use a method which is similar, in some sense, to that developed in Ref. [2]. However, our method is more general and, as a consequence, it can be applied to all nonlinear normal modes in the FPU β chain for the case where $N \to \infty$. This method is described in detail below.

The standard linear stability analysis of a given NNM leads us to Eq. (32) representing an $N \times N$ system of differential equations with periodic coefficients. Obviously, such a straightforward method proves to be especially complicated when $N \to \infty$.

On the other hand, we can apply the general group-theoretical method [13] for splitting the system (32) into some independent subsystems L_j of small dimensionalities n_j due to the symmetry properties of the considered NNM. In Ref. [6], it has been shown that for all NNMs in the FPU β chain these dimensionalities (n_j) do not exceed 3. This fact is extremely useful for stability studying and we have already used it in Ref. [6] for numerical construction of the stability diagrams for NNMs in the FPU β model with $\beta > 0$.

At this point it is appropriate to dwell on the physical cause of the stability loss of nonlinear normal modes. This loss of stability can be treated in terms of "parametric interactions" between a given excited NNM and other modes with zero amplitudes ("sleeping" modes) (see Ref. [12]).

In the simplest case, such an interaction can be described by the Mathieu equation. Indeed, studying the stability of the π mode in the FPU β chain, we deal with splitting of the N-dimensional variational system into individual *scalar* equations,

$$\ddot{v}_j + 4\left[1 + \frac{12\beta}{N}v^2(t)\right]v_j \sin^2 k = 0, \tag{39}$$

where $v_j = v_j(t)$ is a sleeping mode, while the solution $v(t)$ of the Duffing equation (26) describes the time evolution of the π mode.

In some approximation, we can replace the exact function $v(t)$ by its first Fourier harmonic. After substitution of this approximate function into Eq. (39), the latter can be transformed to the standard form of the Mathieu equation, which possesses an infinite set of stable and unstable regions. Depending on the value of the considered NNM amplitude [the function $v(t)$] one can get into a region of unstable motion and then infinitesimal solution of the Mathieu equation begins exponentially increase in amplitude over time evolution. This means that the original NNM loses its stability and transforms into a two-dimensional bush of vibrational modes.

The above phenomenon represents a parametric resonance: the parameter in square brackets in Eq. (39) changes periodically, which leads, under certain conditions, to excitation of the sleeping degree of freedom described by the function $v_j(t)$. Let us note that different sleeping modes $v_j(t)$ turn out to be unequal in relation to excitation by the given NNM $v(t)$. Indeed, in Fig. 1(a) it can be seen that $v_j(t)$ with wave numbers k closer to $k_{\text{res}} = \pi/2$ prove to be excited earlier, i.e., for smaller amplitudes A of the π mode. The values k_{res} for different NNMs in the limit $N \to \infty$ can be revealed in Fig. 1: They correspond to the points that contact with the k axis. We need to study vicinities of the points ($k = k_{\text{res}}$, $A = 0$) for obtaining scaling of the stability thresholds in the thermodynamic limit ($N \to \infty$).

In the general case, parametric interaction with the original (active) NNM leads to exciting not only one sleeping mode but also a certain set of such modes (for the FPU β this set can consist of one, two, or three sleeping modes). For example, the variational system for NNM $B[a^4,iu]$ can be decomposed into two-dimensional subsystems $L_j^{(2)}$ [see Eqs. (33) and the discussion below this equation]. Each of these subsystems contains two sleeping modes $v_j(t)$ and $v_{j'}(t)$, where $j' = \frac{N}{2} - j$ [in terms of wave number (34), these modes are denoted by $v_k(t)$ and $v_{\frac{\pi}{2}-k}(t)$]. This means that when NNM $v(t)$ loses its stability, both sleeping modes $v_k(t)$ and $v_{\frac{\pi}{2}-k}(t)$ are excited *simultaneously*.

Similarly, the variational system for the NNMs $B[a^3,i]$ and $B[a^3,iu]$ can be decomposed into three-dimensional independent subsystems $L_j^{(3)}$. This means that these nonlinear normal modes lose their stability simultaneously relative to the three sleeping modes with wave numbers k, $k + \frac{2\pi}{3}$, $k + \frac{4\pi}{3}$. Since there are no analytical results on parametric resonance in such cases, we have to use the general Floquet stability analysis

to obtain thresholds of the stability loss of the corresponding NNMs.

In terms of the Floquet method, the stability loss of a given NNM can be interpreted in the following manner. When we vary the amplitude A of NNM in the stability region, Floquet multipliers move on the unit circle. Some of them can leave this circle after colliding with each other, and this phenomenon means that the considered NNM loses stability. On the other hand, Floquet multipliers represent eigenvalues of the monodromy matrix $\hat{X}(\pi)$, and they can be found as the roots λ_j of its *characteristic polynomial*.

A bifurcation from stability to instability of the considered NNM takes place when some roots of this polynomial coincide with each other. This fact can be revealed by vanishing of the *discriminant* D of the monodromy matrix characteristic polynomial. Indeed, in the general case,

$$D = \prod_{i<j} (\lambda_i - \lambda_j)^2 \tag{40}$$

and coincidence of every pair of eigenvalues (λ_i, λ_j) leads to the relation $D = 0$.

Now let us consider some technical details of our method for studying stability of NNMs in the thermodynamic limit. The explicit form of the independent subsystems L_j for all FPU β NNMs can be found in Ref. [6]. In the present paper, we write L_j for each NNM in the following matrix-vector form:

$$\ddot{\vec{\mu}} + [a\hat{\omega}^2 + bv^2(t)\hat{M}]\vec{\mu} = 0. \tag{41}$$

Here $\hat{\omega}^2$ and $\hat{M}$ are $n_j \times n_j$ constant matrices depending on the harmonic frequencies of the variables coupled by Eq. (41). In Table II, we present the matrices $\hat{\omega}^2$ and $\hat{M}$ for each of six NNMs permissible in the FPU-β chain in terms of the frequencies $\tilde{\omega}_1 = \sin(\frac{k}{2})$, $\tilde{\omega}_2 = \sin(\frac{k}{2} + \frac{\pi}{3})$, $\tilde{\omega}_3 = \sin(\frac{k}{2} + \frac{2\pi}{3})$, and $\tilde{\omega}_4 = \sin(\frac{k}{2} + \frac{\pi}{2})$.

The time-depending function $v(t)$ entering Eq. (41) is a periodic solution to the governing Eq. (12). For every NNM in the FPU β chain, this governing equation represents the Duffing equation

$$\ddot{v} + av + bv^3 = 0, \tag{42}$$

TABLE II. Input data for stability analysis of nonlinear normal modes in the FPU β chain.

NNM	Energy vs. NNM's amplitude	$\hat{\omega}$	$\hat{M}$
$B[a^2,i]$	$2A^2 + \frac{4\beta}{N}A^4$	$\tilde{\omega}_1$	$3\,\tilde{\omega}_1^2$
$B[a^3,i]$	$\frac{3}{2}A^2 + \frac{27\beta}{8N}A^4$	$\frac{2}{\sqrt{3}}\begin{pmatrix}\tilde{\omega}_1 & & \\ & \tilde{\omega}_2 & \\ & & \tilde{\omega}_3\end{pmatrix}$	$\frac{4}{3}\begin{pmatrix}2\tilde{\omega}_1^2 & -\tilde{\omega}_1\tilde{\omega}_2 & \tilde{\omega}_1\tilde{\omega}_3 \\ -\tilde{\omega}_1\tilde{\omega}_2 & 2\tilde{\omega}_2^2 & -\tilde{\omega}_2\tilde{\omega}_3 \\ \tilde{\omega}_1\tilde{\omega}_3 & -\tilde{\omega}_2\tilde{\omega}_3 & 2\tilde{\omega}_3^2\end{pmatrix}$
$B[a^3,iu]$	$\frac{3}{2}A^2 + \frac{27\beta}{8N}A^4$	$\frac{2}{\sqrt{3}}\begin{pmatrix}\tilde{\omega}_1 & & \\ & \tilde{\omega}_2 & \\ & & \tilde{\omega}_3\end{pmatrix}$	$\frac{4}{3}\begin{pmatrix}2\tilde{\omega}_1^2 & \tilde{\omega}_1\tilde{\omega}_2 & -\tilde{\omega}_1\tilde{\omega}_3 \\ \tilde{\omega}_1\tilde{\omega}_2 & 2\tilde{\omega}_2^2 & \tilde{\omega}_2\tilde{\omega}_3 \\ -\tilde{\omega}_1\tilde{\omega}_3 & \tilde{\omega}_2\tilde{\omega}_3 & 2\tilde{\omega}_3\end{pmatrix}$
$B[a^4,ai]$	$A^2 + \frac{\beta}{N}A^4$	$\sqrt{2}\tilde{\omega}_1$	$6\,\tilde{\omega}_1^2$
$B[a^4,iu]$	$A^2 + \frac{2\beta}{N}A^4$	$\sqrt{2}\begin{pmatrix}\tilde{\omega}_1 & \\ & \tilde{\omega}_4\end{pmatrix}$	$3\begin{pmatrix}\tilde{\omega}_1^2 & \tilde{\omega}_1\tilde{\omega}_4 \\ \tilde{\omega}_1\tilde{\omega}_4 & \tilde{\omega}_4^2\end{pmatrix}$
$B[a^3u,ai]$	$\frac{1}{2}A^2 + \frac{3\beta}{8N}A^4$	$2\begin{pmatrix}\tilde{\omega}_1 & & \\ & \tilde{\omega}_2 & \\ & & \tilde{\omega}_3\end{pmatrix}$	$4\begin{pmatrix}2\tilde{\omega}_1^2 & -\tilde{\omega}_1\tilde{\omega}_2 & \tilde{\omega}_1\tilde{\omega}_3 \\ -\tilde{\omega}_1\tilde{\omega}_2 & 2\tilde{\omega}_2^2 & -\tilde{\omega}_2\tilde{\omega}_3 \\ \tilde{\omega}_1\tilde{\omega}_3 & -\tilde{\omega}_2\tilde{\omega}_3 & 2\tilde{\omega}_3^2\end{pmatrix}$

where a is the squared frequency of the harmonic approximation, while $b = \frac{\beta}{N}\gamma$ is a nonlinearity coefficient. Equation (42) is called the *hard* (*soft*) Duffing equation if $b > 0$ ($b < 0$). For both cases, analytical solutions of this equation are known (see Appendix A).

Our further stability analysis of NNMs reduces to investigating the stability of the zero solution of Eq. (41). The analysis consists of the following steps.

A. Step 1: Simplification of Eq. (41) in the thermodynamic limit

In this case $b \sim 1/N$ and we can decompose the coefficients of Eq. (41) into power series with respect to the small dimensionless parameter,

$$\varepsilon = \frac{bA^2}{a}. \tag{43}$$

This very cumbersome decomposition has been performed with the aid of the MAPLE mathematical package. The corresponding result can be written as follows:

$$\begin{aligned}\frac{d^2}{d\tau^2}\vec{\mu} + \Bigg\{\hat{\omega}^2 + \Bigg[-\frac{3}{4}\hat{\omega}^2 + \left(\frac{1}{2} + \frac{1}{2}\cos 2\tau\right)\hat{M}\Bigg]\varepsilon \\ + \Bigg[\frac{75}{128}\hat{\omega}^2 + \left(-\frac{13}{32} - \frac{3}{8}\cos 2\tau + \frac{1}{32}\cos 4\tau\right)\hat{M}\Bigg]\varepsilon^2 \\ + \Bigg[-\frac{243}{512}\hat{\omega}^2 + \left(\frac{87}{256} + \frac{597}{2048}\cos 2\tau - \frac{3}{64}\cos 4\tau \right. \\ \left. + \frac{3}{2048}\cos 6\tau\right)\hat{M}\Bigg]\varepsilon^3\Bigg\}\vec{\mu} + O(\varepsilon^4) = 0.\end{aligned} \tag{44}$$

B. Step 2: Solving Eq. (44)

This equation represents a system of differential equations with time-periodic coefficients and to construct the corresponding monodromy matrix we must obtain its solution for $t = \pi$. On the other hand, for small time intervals, the solution of Eq. (44) can be found by a simple perturbation theory. To that end we decompose $\vec{\mu}(t)$ into a formal series,

$$\vec{\mu}(t) = \sum_{n=0}^{\infty} \varepsilon^n \vec{\mu}_n(t), \tag{45}$$

substitute it into Eq. (44) and equate to zero the terms with every fixed power of the small parameter ε. As a result, we get the following set of differential equations

$$\ddot{\vec{\mu}}_0 + \hat{\omega}^2\vec{\mu}_0 = 0, \tag{46a}$$

$$\ddot{\vec{\mu}}_1 + \hat{\omega}^2\vec{\mu}_1 = -\left[-\tfrac{3}{4}\hat{\omega}^2 + \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\tau\right)\hat{M}\right]\vec{\mu}_0, \tag{46b}$$

$$\begin{aligned}\ddot{\vec{\mu}}_2 + \hat{\omega}^2\vec{\mu}_2 = &-\left[-\tfrac{3}{4}\hat{\omega}^2 + \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\tau\right)\hat{M}\right]\vec{\mu}_1 \\ &- \Bigg[\frac{75}{128}\hat{\omega}^2 + \Big(-\frac{13}{32} - \frac{3}{8}\cos 2\tau \\ &+ \frac{1}{32}\cos 4\tau\Big)\hat{M}\Bigg]\vec{\mu}_0, \ldots .\end{aligned} \tag{46c}$$

Because of the diagonal form of the matrix $\hat{\omega}^2$ (see Table II), these equations determine certain sets of harmonic oscillators with different time-periodic external forces. Each of these oscillators is described by equation

$$\ddot{x} + \omega^2 x = f(\tau).$$

The general solution to this equation, obtained by the method of variation of arbitrary constants, can be written in the form

$$x(\tau) = c_1 \sin\omega\tau + c_2 \cos\omega\tau + \frac{1}{\omega}\int_0^\tau f(t)\sin[\omega(\tau - t)]dt, \tag{47}$$

where $f(t)$, in our case, represents a *sum* of time-periodic functions whose frequencies are *incommensurable*. Indeed, for the most NNMs from Table II, $\hat{\omega}^2$ are matrices with different diagonal elements and, therefore, the components of the vector $\vec{\mu}_0(\tau)$ from (46a) vibrate with different frequencies. Substituting $\vec{\mu}_0(\tau)$ into (46b) leads to mixing its time-dependent components because of multiplying by the matrix $\hat{M}$, and such mixing produces more and more complicated terms in the right-hand side of Eqs. (46) when we take into account higher orders in the decomposition (44).

C. Step 3: Constructing monodromy matrix

The usual way to study the stability of a given periodic dynamical regime is by use of the Floquet method. In this method, we linearize nonlinear equations of motion in the vicinity of the periodic solution and calculate the *monodromy matrix* $\hat{X}(T)$ by integrating $2n$ times the linearized equations with time-periodic coefficients over one period T using specific initial conditions [n is the number of equations in Eq. (44)]. These conditions are determined by the successive columns of a $2n \times 2n$ identity matrix.

Solving Eqs. (46) in a step-by-step manner, we can construct the approximate analytical solution to Eq. (44) up to a fixed order of the small parameter ε. With the aid of this solution, we are able to obtain the monodromy matrix $\hat{X}(\pi)$ for Eq. (44), where π is the period of its coefficients.

The stability of the considered periodic solution is determined by Floquet multipliers representing eigenvalues of the monodromy matrix. If all these multipliers lie on the unit circle, the solution is *linear stable*. In other case, the solution *linear unstable*.

D. Step 4: Obtaining discriminants of characteristic polynomials

We can obtain eigenvalues of the monodromic matrix $\hat{X}(\pi)$ as the roots of its characteristic polynomial, which, in turn, can be expressed via different traces of $\hat{X}(\pi)$.

For the discriminants D_1, D_2, and D_3 of the characteristic polynomials of $n = 1, 2, 3$ degrees we obtained the following formulas:

$$D_1 = (a_1 + 2)(a_1 - 2), \tag{48}$$

$$D_2 = (a_2 + 2a_1 + 2)(a_2 - 2a_1 + 2)\left(8 + a_1^2 - 4a_2\right)^2, \tag{49}$$

$$\begin{aligned} D_3 = {} & (a_3 + 2a_2 + 2a_1 + 2)(a_3 - 2a_2 + 2a_1 - 2) \\ & \cdot \left(9a_1^2 + 54a_1a_3 - 27a_3^2 - 42a_1^2a_2 + 18a_1a_2a_3 \right. \\ & \left. -4(a_2 - 3)^3 + 8a_1^4 + a_1^2a_2^2 - 4a_1^3a_3\right)^2, \end{aligned} \tag{50}$$

where coefficients a_1, a_2, a_3 are determined via traces of the monodromy matrix $\hat{X}(\pi)$:

$$\begin{aligned} a_1 &= \mathrm{Tr}\hat{X}(\pi), \\ a_2 &= \tfrac{1}{2}[\mathrm{Tr}^2\hat{X}(\pi) - \mathrm{Tr}\hat{X}^2(\pi)], \\ a_3 &= \tfrac{1}{6}[\mathrm{Tr}^3\hat{X}(\pi) - 3\mathrm{Tr}\hat{X}(\pi)\mathrm{Tr}\hat{X}^2(\pi) + 2\mathrm{Tr}\hat{X}^3(\pi)]. \end{aligned} \tag{51}$$

E. Some examples

We now present some illustrations of the above-discussed technique.

Example 1. Nonlinear normal mode $B[a^2,i]$ (π mode): $|x,-x|$.

First, let us discuss the case where $\beta > 0$. One can see that there exist modes, corresponding to the left and right sides of the black region in Fig. 1(a), which are not excited by parametric interactions with the π mode. This fact was revealed analytically in Ref. [22] with the aid of the rotating wave approximation (RWA). In Ref. [24], in the framework of the same approximation, the following relation between the amplitude threshold value A_c and the wave number k [i.e., the boundary curve of the black region in Fig. 1(a)] was obtained as follows:

$$A_c = \frac{\left|\sin\frac{k-\pi}{2}\right|}{\sqrt{9\cos^2\frac{k-\pi}{2} - 3}},$$

This analytical formula is in good agreement with the numerical results.

Let us now consider the stability threshold of the π mode in the thermodynamic limit $N \to \infty$ using the above-discussed method. We have to consider the vicinity of the point $(\frac{\pi}{2},0)$ on the $(k\text{-}A)$ plane ($k_{\mathrm{res}} = \pi/2$). The one-dimensional constant matrices of the decoupled variational system can be found in Table II,

$$\hat{\omega}_k^2 = \sin^2\frac{k}{2}, \quad \hat{M}_k = 3\sin^2\frac{k}{2}.$$

The monodromy matrix also depends on the wave number k and we can decompose its trace $\mathrm{Tr}\hat{X}(\pi)$ into the Taylor series in two small parameters, $\Delta k = k - k_{\mathrm{res}}$ and $\varepsilon = bA^2/a$ (in our case, $a = 4$, $b = 16\beta/N$ and, therefore, $\varepsilon = 4\beta A^2/N$). This decomposition reads,

$$\begin{aligned} \mathrm{Tr}\hat{X}(\pi) = {} & \left(-2 + \tfrac{1}{64}\pi^2\Delta k^4 - \tfrac{1}{1536}\pi^2\Delta k^6 + \cdots\right) \\ & + \left(-\tfrac{3}{32}\pi^2\Delta k^2 + \tfrac{7}{512}\pi^2\Delta k^4 + \cdots\right)\varepsilon \\ & + \tfrac{27}{256}\pi^2\Delta k^2\varepsilon^2 + \cdots. \end{aligned}$$

In the considered case, $n = 1$ and the corresponding discriminant is

$$D_1 = (a_1 + 2)(a_1 - 2),$$

where $a_1 = \mathrm{Tr}\hat{X}(\pi)$.

The condition $D_1 = 0$ leads to the equations

$$a_1 + 2 = 0 \quad \text{or} \quad a_1 - 2 = 0. \tag{52}$$

From the first of these equations, we find

$$\begin{aligned} a_1 + 2 = {} & a\left(\tfrac{1}{64}\pi^2\Delta k^4 - \tfrac{1}{1536}\pi^2\Delta k^6\right) \\ & + \left(-\tfrac{3}{32}\pi^2\Delta k^2 + \tfrac{7}{512}\pi^2\Delta k^4\right)\varepsilon + \tfrac{27}{256}\pi^2\Delta k^2\varepsilon^2 = 0, \end{aligned}$$

and we then obtain

$$\varepsilon = \tfrac{1}{6}\Delta k^2 + \tfrac{7}{144}\Delta k^4 + \mathrm{O}(\Delta k^6).$$

Substitution of $\varepsilon = \frac{4\beta}{N}A_c^2$ and $\Delta k = \frac{2\pi}{N}$ leads us to the following result:

$$A_c = \frac{\pi}{\sqrt{6\beta N}} + \frac{7\pi^3}{12N^2\sqrt{6\beta N}} + \mathrm{O}(N^{-9/2}).$$

The corresponding energy per one particle is

$$\frac{E_c}{N} = \frac{1}{N}\left(\frac{aA_c^2}{2} + \frac{bA_c^4}{4}\right) = \frac{\pi^2}{3\beta N^2} + \frac{11\pi^4}{36\beta N^4} + \mathrm{O}(N^{-6}).$$

The second equation (52) leads to a contradiction with the condition of smallness of the parameter ε and, therefore, this equation does not produce instability of the π mode in the limit $N \to \infty$. Note that the analytical dependence $E_c/N = \pi^2/(3\beta N)$ was revealed in Ref. [21] and later was recovered in Refs. [1,23].

In the case where $\beta < 0$, the stability properties of the π mode differ utterly. Indeed, one can see in Fig. 1(a) (right column) that this mode turns out to be stable up to the finite value of its amplitude A_c. Using a numerical method, we found that in the thermodynamic limit $N \to \infty$ $A_c \approx 0.393$ ($E_c \approx 0.213$).

Example 2. Nonlinear normal mode $B[a^4,iu]$: $|x,x,-x,-x|$.

The variational system for this NNM is decoupled into 2×2 subsystems with time-periodic coefficients. As a result, for studying the stability loss of the mode B$[a^4,iu]$ we have to vanish the discriminant D_2 from (49).

The first factor of D_2 near the resonance wave number $k_{\mathrm{res}} = \pi/2$ is equal to

$$a_2 + 2a_1 + 2 = \tfrac{9}{128}\varepsilon^2\pi^4\Delta k^2 + \tfrac{1}{16}\pi^4\Delta k^4 + \cdots. \quad (53)$$

Being positive, this factor cannot lead to the condition $D_2 = 0$.

The second factor of D_2 from Eq. (49) reads,

$$a_2 - 2a_1 + 2 = 16 + \mathrm{O}(\varepsilon^2,\Delta k^2), \quad (54)$$

which also does not vanish in the thermodynamic limit $N \to \infty$.

Only the last factor of the discriminant D_2,

$$\tfrac{9}{64}\varepsilon^2\pi^4\Delta k^2 - \tfrac{1}{16}\pi^4\Delta k^6 + \cdots, \quad (55)$$

can lead to fulfillment of the condition $D_2 = 0$. This yields

$$\varepsilon = \pm\tfrac{2}{3}\Delta k^2,$$

and, therefore,

$$A_c = \frac{\sqrt{2}\pi}{\sqrt{3|\beta|N}}, \quad E_c = \frac{2\pi^2}{3|\beta|N^2}. \quad (56)$$

Note that this NNM was investigated thoroughly in Ref. [2]. Above, we simply reproduced the main result of this paper by use of our method.

Example 3. Nonlinear normal modes $B[a^3,i]$: $|x,0,-x|$ and $B[a^3,iu]$: $|x,-2x,x|$.

The variational systems for these NNMs can be decoupled into 3×3 independent subsystems whose matrices are presented in Table II. One now has to vanish the discriminant D_3 from Eq. (50). With the aid of our method, we get the following results.

For both modes $B[a^3,i]$ and $B[a^3,iu]$, we have obtained identical scaling for the case $N \to \infty$,

$$A_c = \frac{2\pi}{3\sqrt{\beta N}}, \quad E_c = \frac{2\pi^2}{3\beta N^2}. \quad (57)$$

Some numerical results on stability of the NNM $B[a^3,i]$ have been found in Ref. [3], but we do not know any results on stability properties of the NNM $B[a^3,iu]$. In conclusion, let us consider another scenario of the stability loss of NNMs. Indeed, up to this point, we have discussed only the loss of stability associated with parametric interactions of a given NNM with other (linear) normal modes of the FPU β chain.

Some NNMs in the FPU β chain, when $\varepsilon = bA^2/a \to 0$, transform not into *one* linear normal mode (LNM) but into a certain *superposition* of such modes. For example, NNM $B[a^4,ai]$

$$\vec{X}(t) = \nu(t)\frac{2}{\sqrt{2N}}\{0,1,0,-1\,|\,0,1,0,-1\,|\,\ldots\} \quad (58)$$

transforms, in the case $\varepsilon \to 0$, into the linear combination

$$\xi_1\vec{\Psi}_{N/4}(t) + \xi_2\vec{\Psi}_{3N/4}(t), \quad \xi_1 = -\xi_2 = \frac{1}{\sqrt{2}} \quad (59)$$

of two linear normal modes

$$\vec{\Psi}_{N/4}(t) = \vec{c}_{N/4}\cos(\omega t), \quad \vec{\Psi}_{3N/4}(t) = \vec{c}_{3N/4}\cos(\omega t), \quad (60)$$

where

$$\begin{aligned}\vec{c}_{N/4} &= \frac{1}{\sqrt{N}}\{1,-1,-1,1\,|\,1,-1,-1,1\,|\,\ldots\},\\ \vec{c}_{3N/4} &= \frac{1}{\sqrt{N}}\{-1,-1,1,1\,|\,-1,-1,1,1\,|\,\ldots\},\\ \omega^2 &= 2.\end{aligned} \quad (61)$$

One can also say that NNM $B[a^4,ai]$ is the result of the continuation of the superposition (59) with respect to the nonlinearity parameter ε of the considered FPU β chain. We have to emphasize that the continuation of an *arbitrary* linear combination of the two above-discussed LNMs does not represent an exact solution to nonlinear dynamical equations of the FPU β chain, while the superposition with $\xi_1 = -\xi_2 = \frac{1}{\sqrt{2}}$ produces an exact solution.

The following question then arises: If we increase the parameter ε from zero, is it possible that the discussed NNM (58) loses stability because of transformation into a *two-dimensional bush*,

$$\nu(t)\vec{c}_{N/4} + \mu(t)\vec{c}_{3N/4}, \quad (62)$$

with two *different* functions of time $\nu(t)$, $\mu(t)$? In general, such a bush describes not periodic but a *quasiperiodic* dynamical regime in the FPU β chain. In contrast to the previous case, this stability loss scenario does not imply any extension of the vibrational modes set, but it means breaking the correlation (59) between two LNMs, $\nu_{N/4}(t)\vec{\Psi}_{N/4}$ and $\nu_{3N/4}(t)\vec{\Psi}_{3N/4}$. We do not discuss this scenario of the stability loss in the present paper. However, our analysis allows us to assert that such a scenario is inessential for all six symmetry-determined NNMs in the FPU β chain.

TABLE III. Asymptotic behavior of the critical parameters of NNMs in the thermodynamic limit for the FPU β chain with $\beta > 0$.

NNM	A_c	E_c/N	Refs.
$B[a^2,i]$	$\frac{\pi}{\sqrt{6\beta N}}$	$\frac{\pi^2}{3\beta N^2}$	[1,20–27]
$B[a^3,i]$	$\frac{2\pi}{3\sqrt{\beta N}}$	$\frac{2\pi^2}{3\beta N^2}$	[3]
$B[a^4,ai]$	$\sqrt{\frac{2\pi}{3\beta}}$	$\frac{2\pi}{3\beta N}$	[3,27]
$B[a^3,iu]$	$\frac{2\pi}{3\sqrt{\beta N}}$	$\frac{2\pi^2}{3\beta N^2}$	
$B[a^4,iu]$	$\frac{\sqrt{2}\pi}{\sqrt{3\beta N}}$	$\frac{2\pi^2}{3\beta N^2}$	[2]
$B[a^6,ai,a^3u]$	$\frac{2\pi}{\sqrt{3\beta N}}$	$\frac{2\pi^2}{3\beta N^2}$	

VII. RESULTS AND DISCUSSION

With the aid of the above-discussed method, we investigated scaling of stability thresholds in the thermodynamic limit $N \to \infty$ for all possible nonlinear normal modes in the FPU β chain for both cases $\beta > 0$ and $\beta < 0$. The summary results are presented in Tables III and IV. Let us comment on these results.

In the first column of Tables III and IV, the symbols of NNMs are given. In two next columns, we present the scaling laws for $N \to \infty$ of stability thresholds for the amplitude of each NNM, $A_c(N)$, and for its specific energy, $\varepsilon_c(N)$ (the energy per one particle of the chain). In the last column of the above tables, we give references to the papers in which stability thresholds of the corresponding NNMs are also discussed.

A. The case $\beta > 0$

First, let us pay attention to the following interesting fact: four of six NNMs in the FPU β chain, namely,

$$\begin{aligned} B[a^3,i] &= \nu(t)\frac{3}{\sqrt{6N}}\{1,0,-1\,|\,1,0,-1\,|\cdots\},\\ B[a^3,iu] &= \nu(t)\frac{1}{\sqrt{2N}}\{1,-2,1\,|\,1,-2,1\,|\cdots\},\\ B[a^4,iu] &= \nu(t)\frac{1}{\sqrt{N}}\{1,-1,-1,1\,|1,-1,-1,1\,|\cdots\},\\ B[a^3u,ai] &= \nu(t)\frac{3}{\sqrt{6N}}\{0,1,1,0,-1,-1|0,1,1,0,-1,-1|\cdots\}, \end{aligned} \tag{63}$$

possess *identical* scaling of the stability threshold in the limit $N \to \infty$:

$$\varepsilon_c(N) = \frac{2\pi^2}{3\beta N^2}. \tag{64}$$

TABLE IV. Asymptotic behavior of the critical parameters of NNMs in the thermodynamic limit for the FPU β chain with $\beta < 0$.

NNM	A_c	E_c/N	Refs.
$B[a^4,ai]$	$\sqrt{\frac{2\pi}{3\lvert\beta\rvert}}$	$\frac{2\pi}{3\lvert\beta\rvert N}$	
$B[a^4,iu]$	$\frac{\sqrt{2}\pi}{\sqrt{3\lvert\beta\rvert N}}$	$\frac{2\pi^2}{3\lvert\beta\rvert N^2}$	[2]
$B[a^6,ai,a^3u]$	0	0	

Note that time dependence of these modes, i.e., the function $\nu(t)$, obeys different Duffing equations (12)–(18).

The scaling of $\varepsilon_c(N)$ for the π mode

$$B[a^2,i] = \nu(t)\frac{1}{\sqrt{N}}\{1,-1\,|\,1,-1\,|\ldots\} \tag{65}$$

is *exactly* twice less than that determined by Eq. (64).

Only for the mode

$$B[a^4,ai] = \nu(t)\frac{2}{\sqrt{2N}}\{0,1,0,-1\,|\,0,1,0,-1\,|\ldots\} \tag{66}$$

does the scaling law of the stability threshold turns out to differ cardinally,

$$\varepsilon_c(N) = \frac{2\pi}{3\beta N}. \tag{67}$$

A qualitative difference in scaling between this mode and all other NNMs for the FPU β chain with $\beta > 0$ can be seen in the left column of Fig. 1.

B. The case $\beta < 0$

The stability properties of the same NNMs in the FPU β chain with $\beta < 0$ prove to differ essentially, as one can see in the right column of Fig. 1.

First, for three NNMs, namely $B[a^2,i]$, $B[a^3,i]$, and $B[a^3,iu]$, the stability thresholds $\varepsilon_c(N)$ do not tend to zero when $N \to \infty$. These modes lose their stability for a certain *finite* value A_c of the NNM's amplitude. For the above listed nonlinear normal modes, these values are equal respectively to 0.213, 0.092, and 0.138. The fundamental difference between scaling of $\varepsilon_c(N)$ for the modes $B[a^4,ai]$ and $B[a^4,iu]$ takes place in the case where $\beta < 0$, as well as for the above-discussed case $\beta > 0$: $\varepsilon_c(N) \sim 1/N$ for $B[a^4,ai]$ and $\varepsilon_c \sim 1/N^2$ for $B[a^4,iu]$ (see Table IV).

Studying the stability threshold for the NNM $B[a^3u,ai]$ proves to be more difficult. In this case, the loss of stability is determined by the second scenario discussed at the end of Sec. VI. Normalizing the variational equations and describing the dynamics of the vibrational modes with basis vectors $\vec{\Psi}_{N/2}$, $\frac{1}{\sqrt{2}}(\vec{\Psi}_{N/6} \pm \vec{\Psi}_{5N/6})$ simultaneously with the Duffing equation, corresponding to NNM $B[a^3u,ai]$, we infer that for $\beta < 0$ there exists an exponential *detuning* between the above modes for an arbitrary small amplitude of the investigated NNM. This means that $\varepsilon_c(N)$ turns out to be equal to zero, as indicated in Table IV.

VIII. SUMMARY

In the present paper, a certain asymptotic technique for studying the stability loss of nonlinear normal modes in the FPU β chain in the thermodynamic limit $N \to \infty$ is developed. Using this technique, we were able to obtain the scaling laws of the stability threshold $\varepsilon_c(N)$ for all six symmetry-determined NNMs that are possible in the FPU β chain for both cases $\beta > 0$ and $\beta < 0$.

The general method [13] for splitting the variational system for a given dynamical regime in a physical system with discrete symmetry into independent subsystems of small dimensions

was applied for investigating the stability of NNMs in the FPU β chain. The above dimensions for the considered case turn out to be equal to 1, 2, and 3. This splitting allows us to construct numerically the stability diagrams (Fig. 1) that can help to reveal many interesting properties of nonlinear normal modes, such as the qualitative behavior of the stability thresholds $\varepsilon_c(N)$ in the thermodynamic limit $N \to \infty$, the existence of the "second stability threshold" for some NNMs, the existence of finite limits of $\varepsilon_c(N)$ for certain modes, and so on.

APPENDIX A: SOLUTION OF THE DUFFING EQUATION

For initial conditions $v(0) = A$, $\dot{v}(0) = 0$, the solution of the hard Duffing equation can be written in the form

$$v(t) = A\mathrm{cn}(\Omega t, \kappa^2). \tag{A1}$$

Here

$$\Omega^2 = a/(1 - 2\kappa^2), \tag{A2}$$

while modulus κ of the Jacobi elliptic cosine is determined by the relation

$$2\kappa^2 = bA^2/(a + bA^2). \tag{A3}$$

The solution (A1) represents a periodic function with the period

$$T = 4K(\kappa)/\Omega, \tag{A4}$$

where $K(\kappa)$ is the complete elliptic integral of the first kind.

For the same initial conditions, the solution of the soft Duffing equation ($b < 0$) can be written in the form

$$v(t) = A\mathrm{sn}(\Omega t + K(\kappa), \kappa^2), \tag{A5}$$

with

$$\Omega^2 = a/(1 + \kappa^2), \tag{A6}$$

$$\kappa^2 = |b|A^2/(2a - |b|A^2), \tag{A7}$$

$$T = 4K(\kappa)/\Omega. \tag{A8}$$

It is convenient to introduce the time scaling

$$\tau = \frac{2\pi}{T}t = \frac{\pi\Omega}{2K(\kappa)}t, \tag{A9}$$

which transforms Eq. (41) into the equation with π-periodic coefficients. As a result of this scaling the form of Eq. (41) does not change, but the constant a and b, entering this equation, must be multiplied by $4K^2(\kappa)/(\pi^2\Omega^2)$. We do not change notations in Eq. (41); however, we imply that the above transformations are already fulfilled.

APPENDIX B: ASYMPTOTIC EXPANSION OF SOME ELLIPTIC FUNCTIONS

The function $v(t)$ in the form (A1) for the case $\beta > 0$ and in the form (A5) for $\beta < 0$ must be substituted into Eq. (41) taking into account that modulus κ goes to zero when $N \to \infty$. To simplify $v(t)$, we use the following formulas from the theory of elliptic functions [33]:

$$\mathrm{cn}(u,\kappa^2) = \frac{2\pi}{\kappa K(\kappa)} \sum_{n=1}^{\infty} \frac{q^{\frac{2n-1}{2}}}{1 + q^{2n-1}} \cos\frac{(2n-1)\pi u}{2K(\kappa)}, \tag{B1}$$

$$\mathrm{sn}(u,\kappa^2) = \frac{2\pi}{\kappa K(\kappa)} \sum_{n=1}^{\infty} \frac{q^{\frac{2n-1}{2}}}{1 - q^{2n-1}} \sin\frac{(2n-1)\pi u}{2K(\kappa)}, \tag{B2}$$

where

$$K(\kappa) = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2\kappa^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\kappa^4 + \cdots + \left[\frac{(2n-1)!!}{2^n n!}\right]^2\kappa^{2n} + \cdots\right\} \tag{B3}$$

is the complete elliptic integral of the first kind, while

$$q = q(\kappa) = \exp\left[-\pi\frac{K(\kappa')}{K(\kappa)}\right]$$

($\kappa' = \sqrt{1-\kappa^2}$ is the complimentary modulus of the elliptic functions). Note that the modulus κ depends on bA^2 in a different manner for the cases $\beta > 0$ and $\beta < 0$ [see Eqs. (A3) and (A7), respectively].

We then have to decompose the left-hand-side of Eq. (41) into the power series with respect to the small parameter $\varepsilon = bA^2/a$.

APPENDIX C: NEWTON FORMULAS

According to the Newton formulas, the coefficients of the characteristic polynomial

$$\det(\hat{A} - \lambda\hat{E}) \equiv (-1)^N(\lambda^N + p_1\lambda^{N-1} + \cdots + p_{N-1}\lambda + p_N)$$

of any $N \times N$ matrix $\hat{A}$ can be expressed via the sums

$$s_k = \sum_{i=1}^{N} \lambda_i^k, \quad k = 1, \ldots, N \tag{C1}$$

with the aid of the recurrence relation

$$p_k = -(s_k + p_1 s_{k-1} + p_2 s_{k-2} + \cdots + p_{k-1}s_1)/k.$$

Thus, we have

$$p_1 = -s_1, \quad p_2 = -(s_2 + p_1 s_1)/2,$$
$$p_3 = -(s_3 + p_1 s_2 + p_2 s_1), \ldots.$$

On the other hand, all sums s_k in Eq. (C1) can be found directly by means of traces of the matrix $\hat{A}$:

$$s_k = \mathrm{Tr}(\hat{A}^k), \quad k = 1, \ldots, N.$$

It is well known that, in the case of any Hamiltonian system with n degree of freedom, Floquet multipliers λ_i form pairs $\lambda_i, \lambda_i^{-1}$ $(i = 1, \ldots, N)$ and, as a consequence, the characteristic polynomial $f(\lambda)$ of the monodromic matrix $\hat{X}(\pi)$ proves to be *palindromic*:

$$f(\lambda) = \lambda^{2n} - a_1\lambda^{2n-1} + a_2\lambda^{2n-2} - \cdots + a_2\lambda^2 - a_1\lambda + 1 = 0 \tag{C2}$$

with the following coefficients a_i:

$$a_1 = \mathrm{Tr}\hat{X}(\pi), \quad a_2 = \tfrac{1}{2}[\mathrm{Tr}^2\hat{X}(\pi) - \mathrm{Tr}\hat{X}^2(\pi)], \ldots.$$

One can express discriminant (40) explicitly via these coefficients of the characteristic polynomial. With the aid of Newton formulas applied to polynomial (C2) and using the mathematical package MAPLE, we, finally, find Eqs. (48)–(50) from the main text of the paper.

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