Dynamical critical exponents for the mean-field Potts glass

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In this paper we study the critical behavior of the fully connected p-color Potts spin glass at the dynamical transition. In the framework of mode coupling theory (MCT), the time autocorrelation function displays a two-step relaxation, with two exponents governing the approach to the plateau and the exit from it. Exploiting a relation between static and equilibrium dynamics which has been recently introduced, we are able to compute the critical slowing down exponents at the dynamical transition with arbitrary precision and for any value of the number of colors p. When available, we compare our exact results with numerical simulations. In addition, we present a detailed study of the dynamical transition in the large p limit, showing that the system is not equivalent to a random energy model.

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I. INTRODUCTION

Mean-field spin-glass models can be divided into two main classes, the ones which undergo a continuous transition and the ones which, instead, display a jump in the order parameter. In systems belonging to the former class, at a certain temperature T_s a second order phase transition takes place, with a continuous growth of the Edwards-Anderson (EA) order parameter $q_{EA} = \frac{1}{N}\sum_i \overline{\langle S_i\rangle^2}$ and zero magnetization (in absence of magnetic field): the spins are essentially frozen in a random direction so that the global mean magnetization vanishes while the mean square magnetization is finite. In the low temperature phase the replica symmetry is broken with a continuous pattern [full replica symmetry breaking (RSB)] or with a steplike pattern (one-step RSB) according to the Parisi scheme [1], and the order parameter is, in fact, a nontrivial function $q(x)$. One can also study the Langevin dynamics of these systems, showing that exactly at the thermodynamic transition temperature T_s there is ergodicity breaking; therefore we can say that, in correspondence to the static transition, a dynamical transition takes place too.

There exists another class of mean-field spin-glass model (like the p-spin or the p-color Potts model [2,3]) which display two different transitions: at a temperature T_s there is a thermodynamic phase transition which is second order in terms of potentials but can be discontinuous in the EA order parameter. The low temperature phase is (at least in the vicinity of the critical temperature) one-step replica symmetry broken. At a temperature $T_d > T_s$ a dynamical phase transition occurs, where the system's relaxation time becomes infinite and the ergodicity is broken [2,4]. This is due to the fact that at the dynamical transition the equilibrium state splits into a large (exponential in the system size) number of excited states, represented by free energy local minima. Since in the mean field the barriers between these states become infinitely high in the thermodynamic limit, the equilibrium dynamics remains stuck forever in one of them, and the overlap cannot relax to zero.

This second class of mean-field systems has been shown to share some relevant properties of structural glasses [5–8]; more specifically, the dynamical equations are exactly equivalent to those predicted by the mode coupling theory (MCT) above the mode coupling temperature T_{mc} where ergodicity breaking occurs. The analogy between structural glass models (with self-induced frustration) and proper mean-field spin glasses (with quenched disorder) has been widely studied and has provided rather accurate predictions [9–12]. In systems with continuous transition, above T_s the spin-spin time correlation function $C(t) = \langle\sigma_i(0)\sigma_i(t)\rangle$ decays exponentially at large times, which means that the system is ergodic. Lowering the temperature the relaxation time grows until it diverges exactly at T_s (the static transition temperature), so that the ergodicity is broken, and the relaxation (at large times) follows a power law $C(t) \sim t^{-\nu}$ with some exponent ν.

The systems belonging to the discontinuous class introduced above behave quite differently: above T_d the time correlation function displays at first a fast decay to a plateau and then a slow decay to zero (in the absence of a magnetic field) [4]; the length of the plateau grows, lowering the temperature until it diverges at T_d. According to MCT the approach to the plateau and the decay from it are both characterized by a power law behavior, respectively,

$$C(t) \simeq q_d + ct^{-a}, \tag{1}$$

$$C(t) \simeq q_d - c't^{b}, \tag{2}$$

where q_d is the height of the plateau, and the two exponents satisfy the exact MCT relation

$$\frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \lambda, \tag{3}$$

where λ is usually treated as a tunable parameter (see, for example, [13]). The exponents a and b have been computed exactly only for the spherical p-spin model [4] because the dynamical equations are particularly simple and correspond to the so-called schematic MCT models. In most cases it is instead very difficult or impossible to compute the exponents in a purely dynamical framework, either analytically or through Monte Carlo simulations.

Numerical simulations are often difficult to interpret and give a quite poor indication of the value of the exponents due to strong finite size effects; if the system is not infinite, barriers between metastable states cannot be infinitely high, the dynamics does not remain stuck in a single state, and all the observables eventually relax to their equilibrium value since, through activated processes, the configuration is able to explore the whole phase space. The extent of this effect depends on the specific model we consider and, in particular, on how fast the barriers between metastable states grow with the size of the system.

Recently, a connection between the mode coupling exponents and some purely thermodynamic quantities has been introduced [14]; this connection suggests a quite simple recipe to compute the dynamical exponents exactly starting from the static mean-field theory which is much easier to work out for all the reasonable models one can think of.

The aim of this paper is to apply this technique to the mean-field Potts spin glass and compute the MCT exponents for any value of the number of colors p. The Potts glass is particularly interesting because, as will be pointed out in the following sections, the parameter p allows us to switch from a continuous transition ($p \leqslant 4$) to a discontinuous transition ($p > 4$); moreover in the latter case it works as a tuning parameter for the magnitude and separation of the static and dynamical transitions. In the following we will not make use of the symplectic representation which is widely exploited in literature [15–17]. The outline of this paper is as follows: in Sec. II we give a sketch of the technique used to compute the MCT exponents in a generic model; in Sec. III we summarize some of the necessary known results about the Potts model. In Sec. IV we compute the dynamical exponents for the Potts model for arbitrary value of the parameter p. In Sec. V we compare our theoretical exact results with numerical simulations, and in Sec. VI we give our conclusions and final remarks.

II. HOW TO COMPUTE THE EXPONENT

Given a fully connected model, it is possible to compute the Gibbs free energy as a function of the order parameter, which, in the case of a spin-glass transition, is the well known overlap matrix Q. The thermodynamic value of the order parameter can be determined by minimizing the Gibbs free energy functional. It can then be expanded around the replica symmetric saddle point solution, giving rise to eight different kinds of third order terms. For our purposes, only two of them will be relevant, namely,

$$w_1 \mathrm{Tr}(\delta Q^3) = w_1 \sum_{a,b,c} \delta Q_{ab} \delta Q_{bc} \delta Q_{ca}, \tag{4}$$

and

$$w_2 \sum_{a,b} \delta Q_{ab}^3. \tag{5}$$

In the case of continuous transitions it has been found [14] that there exists a quite simple relation between the exponent ν and the two coefficients w_1 and w_2:

$$\frac{\Gamma^2(1-\nu)}{\Gamma(1-2\nu)} = \frac{w_2(T_s)}{w_1(T_s)}. \tag{6}$$

In the case of discontinuous transitions it can be shown [14] that a relation analogous to (6) holds at the dynamical transition, which, again, gives the connection between the dynamical exponents a and b and the static coefficients, namely,

$$\frac{\Gamma^2(1-a)}{\Gamma(1-2a)} = \frac{\Gamma^2(1+b)}{\Gamma(1+2b)} = \frac{w_2(T_d)}{w_1(T_d)}, \tag{7}$$

where, different from the former case, the expansion of the Gibbs free energy has to be performed around the dynamical overlap (the height of the infinite plateau at the dynamical transition).

In order to compute the two coefficients w_1 and w_2 one must determine the expression of the Gibbs free energy as a function of the overlap and then expand it to third order around the replica symmetric (RS) thermodynamic value q. The reason why the expansion has to be performed around a replica symmetric solution will be clarified in Sec. IV. In fully connected models, introducing a replicated external field ε, the free energy reads

$$f(\varepsilon) = -\frac{1}{\beta n N} \ln \int dQ \exp N(\mathcal{S}[Q] + \mathrm{Tr}\, \varepsilon Q), \tag{8}$$

which, for $N \to \infty$, can be evaluated at the saddle point,

$$f(\varepsilon) = -\frac{1}{\beta n} \mathrm{extr}_Q(\mathcal{S}[Q] + \mathrm{Tr}\, \varepsilon Q). \tag{9}$$

We can immediately notice that the equation above exactly defines $f(\varepsilon)$ as the *anti-Legendre transform* $\overline{\mathscr{L}}$ of the effective action,

$$f(\varepsilon) = \overline{\mathscr{L}}(\mathcal{S}[Q]), \tag{10}$$

and, again, by definition the Gibbs free energy $\Gamma(Q)$ is the *Legendre transform* $\mathscr{L}$ of $f(\varepsilon)$, yielding

$$\Gamma(Q) \equiv \mathscr{L}(f(\varepsilon)) = \mathscr{L}(\overline{\mathscr{L}}(\mathcal{S}[Q])) = \mathcal{S}[Q]. \tag{11}$$

This implies that the functional form of the Gibbs free energy is exactly the same as the effective action. In fully connected models, we can then directly expand the latter.

The general form of the third order term in the effective action reads

$$\mathcal{S}^{(3)} = \sum_{(ab)(cd)(ef)} W_{ab,cd,ef}\, \delta Q_{ab} \delta Q_{cd} \delta Q_{ef}. \tag{12}$$

Since $a \neq b$, $c \neq d$, and $e \neq f$ and the coefficients W are computed in a RS ansatz, we can have eight different vertices:

$$\begin{aligned} W_{\alpha\beta,\beta\gamma,\gamma\alpha} &= W_1, & W_{\alpha\beta,\alpha\beta,\alpha\beta} &= W_2, \\ W_{\alpha\beta,\alpha\beta,\alpha\gamma} &= W_3, & W_{\alpha\beta,\alpha\beta,\gamma\delta} &= W_4, \\ W_{\alpha\beta,\beta\gamma,\gamma\delta} &= W_5, & W_{\alpha\beta,\alpha\gamma,\alpha\delta} &= W_6, \\ W_{\alpha\gamma,\beta\gamma,\delta\mu} &= W_7, & W_{\alpha\beta,\gamma\delta,\mu\nu} &= W_8. \end{aligned} \tag{13}$$

Following Ref. [18], Eq. (12) can be rephrased in the following way:

$$\mathcal{S}^{(3)} = w_1 \sum_{\alpha\beta\gamma} \delta Q_{\alpha\beta}\delta Q_{\beta\gamma}\delta Q_{\gamma\alpha} + w_2 \sum_{\alpha\beta} \delta Q_{\alpha\beta}\delta Q_{\alpha\beta}\delta Q_{\alpha\beta} + w_3 \sum_{\alpha\beta\gamma} \delta Q_{\alpha\beta}\delta Q_{\alpha\beta}\delta Q_{\alpha\gamma} + w_4 \sum_{\alpha\beta\gamma\delta} \delta Q_{\alpha\beta}\delta Q_{\alpha\beta}\delta Q_{\gamma\delta}$$
$$+ w_5 \sum_{\alpha\beta\gamma\delta} \delta Q_{\alpha\beta}\delta Q_{\alpha\gamma}\delta Q_{\beta\delta} + w_6 \sum_{\alpha\beta\gamma\delta} \delta Q_{\alpha\beta}\delta Q_{\alpha\gamma}\delta Q_{\alpha\delta} + w_7 \sum_{\alpha\beta\gamma\delta\mu} \delta Q_{\alpha\gamma}\delta Q_{\beta\gamma}\delta Q_{\delta\mu} + w_8 \sum_{\alpha\beta\gamma\delta\mu\nu} \delta Q_{\alpha\beta}\delta Q_{\gamma\delta}\delta Q_{\mu\nu}, \quad (14)$$

with

$$w_1 = W_1 - 3W_5 + 3W_7 - W_8, \quad w_2 = \frac{1}{2}W_2 - 3W_3 + \frac{3}{2}W_4 + 3W_5 + 2W_6 - 6W_7 + 2W_8,$$
$$w_3 = 3W_3 - 3W_4 - 6W_5 - 3W_6 + 15W_7 - 6W_8, \quad w_4 = \frac{3}{4}W_4 - \frac{3}{2}W_7 + \frac{3}{4}W_8, \quad w_5 = 3W_5 - 6W_7 + 3W_8, \quad (15)$$
$$w_6 = W_6 - 3W_7 + 2W_8, \quad w_7 = \frac{3}{2}W_7 - \frac{3}{2}W_8, \quad w_8 = \frac{1}{8}W_8.$$

It is therefore sufficient to compute the eight W coefficients and use Eq. (15) to get w_1 and w_2.

III. THE POTTS MODEL: SUMMARY OF KNOWN RESULTS

We consider the p-color disordered Potts Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij}\, \eta(\sigma_i,\sigma_j), \quad (16)$$

with

$$\eta(a,b) = p\,\delta_{a,b} - 1, \quad (17)$$

where p is the number of colors and $\sigma = 0,1,\ldots,p-1$. The sum is extended over all the possible couples taken from N spins, and the couplings J_{ij} are independent Gaussian random variables with mean J_0/N and variance J^2/N, where the normalization is needed in order to obtain a finite thermodynamic limit. As usual, we are interested in computing the mean-field free energy, exploiting the well known replica trick in order to average over the disorder

$$\overline{\ln Z} = \lim_{n\to 0} \frac{1}{n} \ln \overline{Z^n}. \quad (18)$$

Carrying on the computation, we obtain the replicated partition function in a functional integral form:

$$\overline{Z^n} = \int D\underline{Q}\, D\underline{m}\, \exp(-N\mathcal{S}[m,Q]), \quad (19)$$

where the "effective action" $\mathcal{S}[m,Q]$ is a function of two order parameters: the magnetization m_r^α and the overlap $Q_{rs}^{\alpha\beta}$, with Greek replica indices $\alpha,\beta = 1,\ldots,n$ and Latin color indices $r,s = 1,\ldots,p$.

$$\mathcal{S}[m,Q] = \frac{\beta^2 J^2}{4}(1-p) + \frac{\beta^2 J^2}{2p^2} \sum_{\alpha<\beta} \sum_{r,s} \left(Q_{rs}^{\alpha\beta}\right)^2$$
$$\times \frac{\beta}{2p}\left[J_0 + \beta J^2 \frac{p-2}{2}\right]$$
$$\times \sum_\alpha \sum_r \left(m_r^\alpha\right)^2 - \ln \mathrm{Tr}_{\{\sigma\}} e^{\mathcal{H}[m,Q,\{\sigma\}]}, \quad (20)$$

$$\mathcal{H}[m,Q,\{\sigma\}] = \frac{\beta^2 J^2}{p^2} \sum_{\alpha<\beta} \sum_{r,s} Q_{rs}^{\alpha\beta}\, \eta(\sigma^\alpha,r)\eta(\sigma^\beta,s)$$
$$+ \frac{\beta}{p}\left[J_0 + \beta J^2 \frac{p-2}{2}\right] \sum_\alpha \sum_r m_r^\alpha\, \eta(\sigma^\alpha,r). \quad (21)$$

In order to determine the order parameters we can use the two saddle point equations, which read

$$Q_{rs}^{ab} = \langle\langle \eta(\sigma^\alpha,r)\eta(\sigma^\beta,s)\rangle\rangle, \quad (22)$$

$$m_r^\alpha = \langle\langle \eta(\sigma^\alpha,r)\rangle\rangle, \quad (23)$$

where $\langle\langle\cdots\rangle\rangle$ is the average taken with respect to the measure

$$\mu(\{\sigma\}) = \frac{e^{\mathcal{H}[m,Q,\{\sigma\}]}}{\mathrm{Tr}_{\{\tau\}} e^{\mathcal{H}[m,Q,\{\tau\}]}}. \quad (24)$$

The order parameters are clearly redundant; in fact, they satisfy the following constraints:

$$\sum_r Q_{rs}^{\alpha\beta} = 0 \quad \forall s, \quad \sum_r m_r^\alpha = 0. \quad (25)$$

In the particular case $p = 2$ one recovers the Sherrington-Kirkpatrick (SK) model solution [1].

For $p > 2$ ferromagnetic ordering is always preferred below some temperature T_F [3]. An upper bound T_E for the temperature T_F below which ferromagnetic ordering appears is [19] (from now on we consider $J = 1$)

$$T_E = \frac{p-2}{2(1-J_0)}. \quad (26)$$

For $p > 4$, in order to prevent ferromagnetic ordering from occurring at a higher temperature than the spin-glass one, the couplings should be antiferromagnetic on average, with J_0 less than some (negative) threshold value. A lower bound for the critical mean value is $J_F = (4-p)/2$ [19]. Under this condition the magnetization is zero, and it is straightforward to show that, as a consequence, the overlap has the symmetry $Q_{rs}^{\alpha\beta} = Q^{\alpha\beta}\eta(r,s)$. Then it is possible to write the Gibbs free energy as a function of a unique overlap matrix in the following

way [19]:

$$\Gamma[Q] = \frac{1}{2}(p-1)\beta^2 \sum_{\alpha<\beta} Q_{\alpha\beta}^2 - \ln \mathrm{Tr} \exp\left[\beta^2 \sum_{\alpha<\beta} Q_{\alpha\beta}\eta(\sigma^\alpha,\sigma^\beta)\right]. \quad (27)$$

Differentiating with respect to $Q_{\alpha\beta}$, one obtains the saddle point equation,

$$Q_{\alpha\beta} = \frac{1}{p-1}\frac{\mathrm{Tr}\,\eta(\sigma^\alpha,\sigma^\beta)\exp(\mathcal{U}[Q,\sigma])}{\mathrm{Tr}\exp(\mathcal{U}[Q,\sigma])} = \frac{1}{p-1}\langle\langle\eta(\sigma^\alpha,\sigma^\beta)\rangle\rangle, \quad (28)$$

with

$$\mathcal{U}[Q,\sigma] = \beta^2 \sum_{\alpha<\beta} Q_{\alpha\beta}\eta(\sigma^\alpha,\sigma^\beta). \quad (29)$$

It has been shown [3] that for $2.8 < p < 4$ the system undergoes a continuous transition at a temperature $T_s = 1$ with one-step RSB. The breaking point is $\overline{m} = (p-2)/2$.

For $p > 4$ the transition occurs at a temperature $T_s > 1$; it is discontinuous, and the RSB scheme is one step with breaking parameter $\overline{m} = 1$ at criticality. In this case there exists a (dynamical) glass transition, associated with the static one, occurring at some temperature T_d greater than T_s. The static and dynamical transition temperatures and overlap can be determined numerically with great accuracy using the marginality condition and the techniques described in Ref. [19]. We briefly summarize the results here.

We can compute the free energy (27) in the one-step RSB ansatz with $q_1 = q$ and $q_0 = 0$ and expand it at first order around $\overline{m} = 1$ as $\Gamma_0 + (\overline{m}-1)\Gamma_1(q)$,

$$\Gamma(q) = \tfrac{1}{4}\beta^2(1-p) - \ln(p) + (\overline{m}-1) \times \left(\tfrac{1}{4}\beta^2(p-1)q^2 + \tfrac{1}{2}\beta^2 q(p+1) + \ln(p) - I_2\right), \quad (30)$$

where the integral I_2 is given by

$$I_2 = \exp\left(-\frac{\beta^2 pq}{2}\right)\int_{-\infty}^{\infty}\prod_{r=1}^{p}\left(\frac{dy_r}{\sqrt{2\pi}}e^{-\frac{y_r^2}{2}}\right) \times e^{\beta\sqrt{qp}y_1}\ln\left[\left(\sum_{r=1}^{p}\exp\left(\beta(qp)^{\frac{1}{2}}y_r\right)\right)\right]. \quad (31)$$

For $m = 1$ expression (30) gives the high-temperature free energy, which is independent of q. This general expansion allows us to determine the static and the dynamic transitions.

The static temperature is determined by imposing that a solution q_s exists that satisfies the following conditions:

$$\left(\frac{\partial\Gamma}{\partial q}\right)_{q=q_s} = \left(\frac{\partial\Gamma_1}{\partial q}\right)_{q=q_s} = 0, \quad (32)$$

$$(\Gamma_1)_{q=q_s} = 0. \quad (33)$$

On the other hand, for the dynamical transition temperature, we must search for a marginal stability, and the conditions become

$$\left(\frac{\partial\Gamma}{\partial q}\right)_{q=q_d} = \left(\frac{\partial\Gamma_1}{\partial q}\right)_{q=q_d} = 0, \quad (34)$$

$$\left(\frac{\partial^2\Gamma}{\partial q^2}\right)_{q=q_d} = \left(\frac{\partial^2\Gamma_1}{\partial q^2}\right)_{q=q_d} = 0. \quad (35)$$

In the language of the Franz-Parisi potential [20] the two conditions above correspond, respectively, to the appearance of a local minimum (horizontal flex) for the dynamical transition and to the fact that this minimum reaches the same height as the paramagnetic one for the static transition.

As we will see in the following, the p-dimensional integral I_2 in Eq. (30) is extremely hard to evaluate numerically as soon as $p > 2$. Therefore in Ref. [19] the authors use the identity

$$\ln(1+A) = \int_0^\infty \frac{dx}{x}e^{-x}(1-e^{-Ax}), \quad (36)$$

and taking

$$A = \sum_{r=1}^{p}\exp\left(\beta(qp)^{\frac{1}{2}}y_r\right) - 1, \quad (37)$$

they obtain the result

$$I_2 = \int_0^\infty \frac{dx}{x}e^{-x}\{1 - e^x w\left(xe^{\beta^2 qp}\right)w^{p-1}(x)\}, \quad (38)$$

with

$$w(x) = \int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}y^2 - x\exp\left(\beta(pq)^{\frac{1}{2}}y\right)\right), \quad (39)$$

which is much easier to evaluate numerically [21].

A. Infinite number of colors

It has been pointed out [3] that the Potts model becomes a random energy model (REM) in the limit $p \to \infty$, with a critical temperature that diverges like

$$T_s = \frac{1}{2}\sqrt{\frac{p}{\ln(p)}}. \quad (40)$$

In the following, we show that the limit model is not exactly a REM. The first of Eqs. (32), which is satisfied at both the dynamical and statical transitions, can be written in the following way:

$$q = \frac{1}{p-1}(p\,L^{(p)}(\beta,q) - 1), \quad (41)$$

with

$$L^{(p)}(\beta,q) = \left(\int \mathcal{G}_p(\underline{z})\sum_{r=1}^{p}\exp\left(\beta(pq)^{1/2}z_r\right)\right)^{-1} \times \int \mathcal{G}_p(\underline{z})\left(\frac{\sum_{r=1}^{p}\exp\left(2\beta(pq)^{1/2}z_r\right)}{\sum_{r=1}^{p}\exp\left(\beta(pq)^{1/2}z_r\right)}\right)$$

and

$$\mathcal{G}_p(\underline{z}) = \left(\prod_{r=1}^{p}\frac{dz_r}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}z_r^2\right)\right). \quad (42)$$

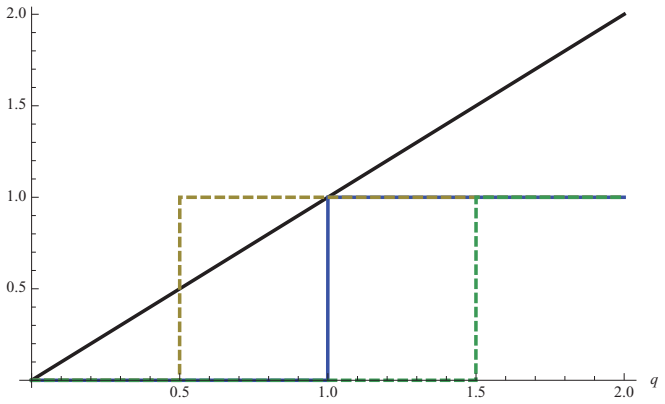

FIG. 1. (Color online) Black solid line: $y = q$. Yellow (left) and green (right) dashed lines: $y = L^{(\infty)}(\xi,q)$ for $\xi = 4$ and $\xi = 4/3$. Blue (gray) solid line: $y = L^{(\infty)}(\xi,q)$ for $\xi = 2$. The dynamical transition is located at $\xi = 2$, for which the line $y = q$ is tangent to the curve $y = L^{(\infty)}(\xi,q)$.

We have been able to show that, as a function of the rescaled temperature

$$\xi = \beta^2 \frac{p}{\ln(p)}, \tag{43}$$

the right hand side of Eq. (41) tends to a Heaviside function in the infinite p limit (see the Appendix),

$$L^{(\infty)}(\xi,q) = \theta\left(q - \frac{2}{\xi}\right) \tag{44}$$

with breaking point $q = 2/\xi$.

As can be easily seen from Fig. 1 both q_d and q_s go to 1 in the limit $p \to \infty$, and the dynamical transition is located at the rescaled temperature such that the breaking point of $L^{(\infty)}$ is 1:

$$\xi_d = 2 \implies T_d = \sqrt{\frac{p}{2\ln(p)}}. \tag{45}$$

While in a REM the ratio between T_d and T_s is formally infinite [22], in the large p Potts model this ratio tends to a finite value, namely,

$$\frac{T_d}{T_s} \xrightarrow[p\to\infty]{} \sqrt{\frac{\xi_s}{\xi_d}} = \sqrt{2} \approx 1.414\ldots \tag{46}$$

Therefore, the limit model is still a "glassy" model with dynamic and static transitions. This is at variance with the Ising p-spin model in the $p \to \infty$ limit that goes to a REM [22].

IV. THE POTTS MODEL: MCT EXPONENTS

The determination of the mode coupling exponents follows essentially the steps described in Sec. II. Expanding the effective action (27) to third order around the replica symmetric saddle point, we obtain the following eight coefficients:

$$\begin{aligned}
W_1 &= R_1 - 3(p-1)qM_2 + 2(p-1)^3q^3,\\
W_2 &= R_2 - 3(p-1)qM_1 + 2(p-1)^3q^3,\\
W_3 &= R_3 - (p-1)qM_1 - 2(p-1)qM_2 + 2(p-1)^3q^3,\\
W_4 &= R_4 - (p-1)qM_1 - 2(p-1)qM_3 + 2(p-1)^3q^3,\\
W_5 &= R_5 - 2(p-1)qM_2 - (p-1)qM_3 + 2(p-1)^3q^3,\\
W_6 &= R_6 - 3(p-1)qM_2 + 2(p-1)^3q^3,\\
W_7 &= R_7 - (p-1)qM_2 - 2(p-1)qM_3 + 2(p-1)^3q^3,\\
W_8 &= R_8 - 3(p-1)qM_3 + 2(p-1)^3q^3,
\end{aligned} \tag{47}$$

where the replica symmetric overlap is determined through the saddle point equation

$$q = \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\rangle\rangle, \tag{48}$$

the "mass matrix" can assume three different values

$$\begin{aligned}
M_1 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\alpha,\sigma^\beta)\rangle\rangle,\\
M_2 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\alpha,\sigma^\gamma)\rangle\rangle,\\
M_3 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\gamma,\sigma^\delta)\rangle\rangle,
\end{aligned} \tag{49}$$

and the six-replica cumulants are given by

$$\begin{aligned}
R_1 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\beta,\sigma^\gamma)\eta(\sigma^\gamma,\sigma^\alpha)\rangle\rangle,\\
R_2 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\alpha,\sigma^\beta)\rangle\rangle,\\
R_3 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\alpha,\sigma^\gamma)\rangle\rangle,\\
R_4 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\gamma,\sigma^\delta)\rangle\rangle,\\
R_5 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\beta,\sigma^\gamma)\eta(\sigma^\gamma,\sigma^\delta)\rangle\rangle,\\
R_6 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\alpha,\sigma^\gamma)\eta(\sigma^\alpha,\sigma^\delta)\rangle\rangle,\\
R_7 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\alpha,\sigma^\gamma)\eta(\sigma^\delta,\sigma^\mu)\rangle\rangle,\\
R_8 &= \langle\langle \eta(\sigma^\alpha,\sigma^\beta)\eta(\sigma^\gamma,\sigma^\delta)\eta(\sigma^\mu,\sigma^\nu)\rangle\rangle.
\end{aligned} \tag{50}$$

Given the relationship [Eq. (15)] between w_1, w_2, and the W coefficients, one obtains

$$\begin{aligned}
w_1 &= R_1 - 3R_5 + 3R_7 - R_8,\\
w_2 &= \tfrac{1}{2}[R_2 - 6R_3 + 3R_4 + 6R_5 + 4(R_6 - 3R_7 + R_8)],
\end{aligned} \tag{51}$$

where only the disconnected cumulants are left.

If the thermodynamic phase transition is continuous, then it coincides with the dynamical one (as in the SK model). In this case dynamical quantities at infinite time relax to their static value [23], and the averages above can be computed in a replica symmetric ansatz taking finally the limit $n \to 0$. If, instead, the transition is discontinuous, then the coefficients have to be computed at the dynamical transition, where quantities at infinite time do *not* relax to their equilibrium (thermodynamic) value but remain stuck at their value inside the most excited metastable states. The averages should then be computed *inside* a single state; this corresponds to taking a one-step RSB ansatz with breaking parameter $m \to 1$ or, if the mutual overlap between different states is 0 as in our case, a RS ansatz with the number of replicas $n \to 1$ [24]. Finally, we can assume replica symmetry and leave n unspecified, obtaining the following expressions for the two coefficients:

$$w_1 = p^3(L_3 - 3L_4 + 3L_{23} - L_{222}) \tag{52}$$

and

$$w_2 = \frac{p^2}{2}(1-q) + \frac{1}{2}p^3(q - 6L_3 + 10L_4 + 3L_{22} - 12L_{23} + 4L_{222}),$$

where, exploiting the fact that $(\eta(\sigma^a,\sigma^b))^2 = (p-1) + (p-2)\eta(\sigma^a,\sigma^b)$, the saddle point equation becomes

$$q = \frac{1}{p-1}\left[p \frac{\int \mathcal{G}_p(\underline{z}) \left(\sum_{r=1}^{p} \exp\left(\beta (pq)^{1/2} z_r\right)\right)^{n-2} \left(\sum_{r=1}^{p} \exp\left(2\beta (pq)^{1/2} z_r\right)\right)}{\int \mathcal{G}_p(\underline{z}) \left(\sum_{r=1}^{p} \exp\left(\beta (pq)^{1/2} z_r\right)\right)^{n}} - 1 \right], \tag{53}$$

and we have defined the class of integrals

$$L_{klh} = \frac{1}{\int \mathcal{G}_p(\underline{z}) \left(\sum_{r=1}^{p} \exp\left(\beta (pq)^{1/2} z_r\right)\right)^{n}} \int \mathcal{G}_p(\underline{z}) \left(\sum_{r=1}^{p} \exp\left(\beta (pq)^{1/2} z_r\right)\right)^{n-k-l-h} \left(\sum_{r=1}^{p} \exp\left(k\beta (pq)^{1/2} z_r\right)\right)$$
$$\times \left(\sum_{r=1}^{p} \exp\left(l\beta (pq)^{1/2} z_r\right)\right) \left(\sum_{r=1}^{p} \exp\left(h\beta (pq)^{1/2} z_r\right)\right), \quad L_{kl} = \frac{1}{p} L_{kl0}, \quad L_k = \frac{1}{p^2} L_{k00}, \tag{54}$$

with $\mathcal{G}_p(\underline{z})$ given in Eq. (42). As already pointed out, the above result holds for both continuous and discontinuous transitions, with the only difference that in the former case q and the L integrals are computed at $n = 0$, while in the latter we consider $n = 1$.

A. The continuous transition

If $p < 4$ the phase transition is second order [3], with $q(x)$ being continuous for $p = 2$ and steplike for $p = 3$. In the case of continuous transitions we have to consider $n = 0$ and $q = 0$, and the result (already found in Ref. [3]) is very simple, namely,

$$\frac{w_2}{w_1} = \frac{p-2}{2}, \tag{55}$$

which yields $\nu_2 = 0.5$ and $\nu_3 \simeq 0.395$. As in the case of the fully connected model, it can be proven [25] that, on the Bethe lattice, for $p \leqslant 4$ the phase transition is second order. The difference on the Bethe lattice is that, for $p = 3$ and low enough connectivity, the order parameter $q(x)$ is a continuous (Parisi type) function, while for high connectivity it becomes a steplike function (as in the fully connected case). This does not affect the result, which is again (for $p \leqslant 4$) given by Eq. (55).

B. The discontinuous transition

We are interested here in the case $p > 4$, when the system undergoes a dynamical transition; therefore we must take the limit $n \to 1$ in order to compute the exponents correctly. The computation of the overlap q and the third order coefficients w_1 and w_2 involves p-dimensional integrals (54), which become very difficult to evaluate numerically as soon as p is greater than 2. In order to overcome this issue, using the identity

$$\frac{1}{A^k} = \frac{1}{(k-1)!} \int_0^\infty x^{k-1} e^{-Ax}, \tag{56}$$

the integrals (54) can be rewritten in following form, which is more suitable for numerical evaluation:

$$L_{klh} = \frac{e^{-\frac{1}{2}\beta^2 pq}}{p\,(k+l+h-2)!} \Bigg[p\, e^{\frac{1}{2}(k+l+h)^2\beta^2 pq} \int_0^\infty dx\, x^{k+l+h-2}\, w^{p-1}(x)\, w\left(x\, e^{(k+l+h)\beta^2 pq}\right)$$
$$+ p(p-1)\, e^{\frac{1}{2}(k+l)^2\beta^2 pq} e^{\frac{1}{2}h^2\beta^2 pq} \int_0^\infty dx\, x^{k+l+h-2}\, w^{p-2}(x)\, w\left(x\, e^{(k+l)\beta^2 pq}\right) w\left(x\, e^{h\beta^2 pq}\right)$$
$$+ p(p-1)\, e^{\frac{1}{2}(k+h)^2\beta^2 pq} e^{\frac{1}{2}l^2\beta^2 pq} \int_0^\infty dx\, x^{k+l+h-2}\, w^{p-2}(x)\, w\left(x\, e^{(k+h)\beta^2 pq}\right) w\left(x\, e^{l\beta^2 pq}\right)$$
$$+ p(p-1)\, e^{\frac{1}{2}(l+h)^2\beta^2 pq} e^{\frac{1}{2}k^2\beta^2 pq} \int_0^\infty dx\, x^{k+l+h-2}\, w^{p-2}(x)\, w\left(x\, e^{(l+h)\beta^2 pq}\right) w\left(x\, e^{k\beta^2 pq}\right)$$
$$+ p(p-1)(p-2)\, e^{\frac{1}{2}(k^2+l^2+h^2)\beta^2 pq} \int_0^\infty dx\, x^{k+l+h-2}\, w^{p-3}(x)\, w\left(x\, e^{k\beta^2 pq}\right) w\left(x\, e^{l\beta^2 pq}\right) w\left(x\, e^{h\beta^2 pq}\right) \Bigg], \tag{57}$$

with $w(x)$ given in Eq. (39).

Through identity (56) we have been able to reduce p-dimensional to "sort of" two-dimensional integrals. They are not technically two-dimensional integrals because in formula (57) for each value of the integration variable x we have to perform two, three, or four integrations to obtain the function w in different points (instead of just one integration, which would be needed in a regular two-dimensional integral).

Some care is needed in the computation of $w(x)$, especially for small x's, since the integrand has an extremely steep growth near zero and the integration step must be taken very small. In order to go to very large values of p it should be better to recast the integrals (57) in their asymptotic form using Eqs. (A5) and

TABLE I. The dynamical transition temperature, the dynamical overlap, the exponent parameter, and the a exponent for different values of p ranging from 5 to 100.

p	T_d	q_d	λ	a
5	1.0101	0.09507	0.8764	0.2290
7	1.0577	0.2206	0.8236	0.2651
10	1.1420	0.3238	0.8052	0.2759
12	1.1970	0.3665	0.8002	0.2787
15	1.2748	0.4114	0.7962	0.2810
20	1.3926	0.4598	0.7930	0.2827
30	1.5941	0.5142	0.7904	0.2841
40	1.7648	0.5455	0.7895	0.2846
100	2.4964	0.6187	0.7892	0.2848

(A8) given in the Appendix. The results for different values of the number of colors p are reported in Table I and in Fig. 2.

V. COMPARISON WITH NUMERICAL SIMULATIONS

For $p = 10$ Monte Carlo simulations were performed in Refs. [26,27] to investigate the finite size effects on the glass transition. References [26,27] find that the thermodynamic static quantities such as the energy, the entropy, the susceptibility, and the overlap distribution display very strong finite size effects. It is found also that the system remains always ergodic and the *plateau* in the equilibrium spin-spin correlation function $C(t)$ is almost invisible even at temperatures close to the dynamical transition T_G and for big system sizes. Since for $N \to \infty$ the physics of the system should be described by the exact mode coupling equations, Refs. [26,27] expect a divergence of the relaxation time $\tau(t)$ with a power law behavior at the dynamical transition:

$$\tau \propto \left(\frac{T}{T_d} - 1\right)^{-\gamma}, \tag{58}$$

with an exponent γ which, in mode coupling theory, is related to the exponents a and b through the exact relation

$$\gamma = \frac{1}{2a} + \frac{1}{2b}. \tag{59}$$

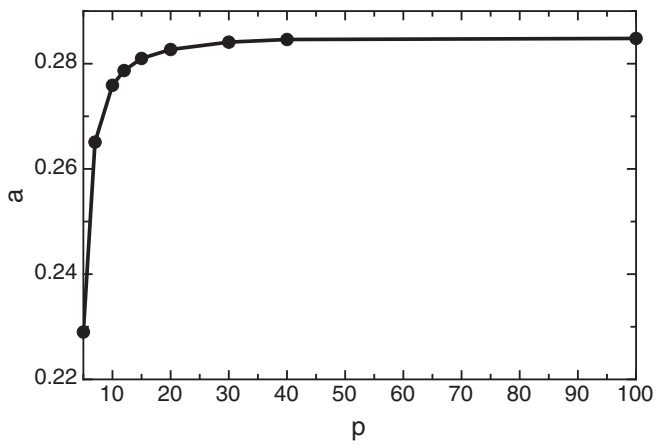

FIG. 2. Joined circles: the exponent a computed for different values of the number of colors ranging from $p = 5$ to $p = 100$ in the discontinuous regime.

Refs. [26,27] plot $\tau^{-\frac{1}{\gamma}}$ for a set of reasonable trial values of γ and find that the data are linearized in the region $1.1 \leqslant T \leqslant 1.4$ for $\gamma = 2.0 \pm 0.5$. This value of γ gives, through relation (59), an indirect estimate for $a(\gamma) \approx 0.36$. Two different kinds of finite size scaling are considered in order to perform extrapolations of $C(t,N)$ at $N \to \infty$. References [26,27] find that only one of the two gives a $C(t)$ which is suitably compatible with a power law behavior of the type in Eq. (1). In this way they can make a rough direct estimate of the exponent, obtaining $a = 0.33 \pm 0.04$, which, despite the difficulties (identification of the plateau, extrapolation, etc.), is close (within 2σ) to our exact computation.

VI. CONCLUSIONS AND REMARKS

In the first part of this paper we presented a review of some known results about the disordered Potts model. In the second part, exploiting a technique that has been recently developed [14], we computed the dynamical exponents of the autocorrelation decay in the cases of continuous and discontinuous transitions in a completely static framework.

In Fig. 2 we show the plot of the exponent a as a function of the parameter p from $p = 5$ up to $p = 100$.

Since our computation is nonperturbative, the exponents can be determined with arbitrary precision, and they can be taken as a reference in numerical simulations. Knowing a *priori* the exponents in the thermodynamic limit, one has an additional tool for studying, for example, the finite size effects and the deviations from MCT in a numerical simulation of a finite (fully connected) system.

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APPENDIX: INFINITE p LIMIT

In this Appendix we compute the dynamical critical temperature T_d and overlap q_d in the limit $p \to \infty$ and the first correction. First of all, note that the saddle point equation (41) in the infinite p limit becomes

$$q = L^{(\infty)}(\beta, q). \tag{A1}$$

Using identity (56), the p-dimensional integral L of Eq. (41) can be rewritten in the following way:

$$L^{(p)}(\beta,q) = \int_0^\infty dx \left[w\left(x e^{-\frac{3}{2}\beta^2 pq}\right)\right]^{p-1} w\left(x e^{\frac{1}{2}\beta^2 pq}\right). \tag{A2}$$

Setting $\alpha \equiv \beta p^{1/2} q^{1/2}$ and making the change of variables

$$x = e^{\frac{1}{2}\alpha^2 + \alpha\overline{x}}, \tag{A3}$$

the integral becomes

$$\int_{-\infty}^\infty d\overline{x}\, \alpha\, e^{\frac{1}{2}\alpha^2 + \alpha\overline{x}} w\left(e^{\alpha^2+\alpha\overline{x}}\right)\left[w\left(e^{-\alpha^2+\alpha\overline{x}}\right)\right]^{p-1}. \tag{A4}$$

Through standard manipulations it can be shown that

$$\alpha\, e^{\frac{1}{2}\alpha^2+\alpha\overline{x}} w\left(e^{\alpha^2+\alpha\overline{x}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\overline{x}^2} F(\overline{x}), \tag{A5}$$

where $F(\overline{x})$ tends to unity in the $\alpha \to \infty$ limit:

$$F(\overline{x}) = \int_{-\infty}^{\infty} dy \left[e^{y-e^y}\right] e^{-\frac{y^2}{2\alpha^2}+\frac{y\overline{x}}{\alpha}}. \tag{A6}$$

Moreover we have clearly

$$w\left(e^{-\alpha^2+\alpha\overline{x}}\right) \to 1. \tag{A7}$$

The behavior of w^{p-1} and, consequently, of the integral L is now determined by the leading order of the first correction $\Delta \equiv w - 1$; in fact

$$\left[w\left(e^{-\alpha^2+\alpha\overline{x}}\right)\right]^{p-1} \simeq \exp[p\ \ln(1+\Delta)]. \tag{A8}$$

We have to compute the leading behavior of

$$\Delta \equiv \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\left[\exp\left(-e^{-\alpha^2+\alpha(y+\overline{x})}\right) - 1\right]; \tag{A9}$$

making the change of variables

$$y = z + \alpha - \overline{x}, \tag{A10}$$

we obtain the form

$$\Delta = e^{-\frac{1}{2}\alpha^2+\alpha\overline{x}} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\overline{x})^2} e^{-\alpha z}[\exp(-e^{\alpha z}) - 1], \tag{A11}$$

which in the limit of large α goes to the following form:

$$\begin{aligned}\Delta &\simeq -e^{-\frac{1}{2}\alpha^2+\alpha\overline{x}} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\overline{x})^2}\theta(-z) \\ &= -\frac{1}{2} e^{-\frac{1}{2}\alpha^2+\alpha\overline{x}} \operatorname{erfc}\left(\frac{\overline{x}}{\sqrt{2}}\right).\end{aligned} \tag{A12}$$

Given this correction, Eq. (A8) becomes

$$w^{p-1} \simeq \exp\left[-\frac{p}{2} e^{-\frac{1}{2}\alpha^2+\alpha\overline{x}} \operatorname{erfc}\left(\frac{\overline{x}}{\sqrt{2}}\right)\right]. \tag{A13}$$

Using a rescaled inverse temperature, $\beta^2 = \xi\ \ln(p)/p$, we have

$$\alpha^2 = \xi q \ln(p), \tag{A14}$$

and substituting into Eq. (A13), we obtain the following expression:

$$w^{p-1} \simeq \exp\left[-\frac{1}{2} p^{1-\frac{\xi q}{2}} e^{\overline{x}\sqrt{\xi q \ln(p)}} \operatorname{erfc}\left(\frac{\overline{x}}{\sqrt{2}}\right).\right] \tag{A15}$$

Independently of the value of $\overline{x}$, the quantity (A15) clearly goes to 1 if $\xi q > 2$, while it goes to 0 if $\xi q < 2$. Therefore, the integrand in Eq. (A2) converges *uniformly* to a normalized Gaussian if $q > 2/\xi$ or to 0 if $q < 2/\xi$, and since the convergence is uniform, the limit can be taken *before* the integration, yielding (see Fig. 3)

$$L^{(\infty)}(\xi,q) \equiv \lim_{p\to\infty} L^{(p)}\left(\sqrt{\xi\frac{\ln(p)}{p}},q\right) = \theta\left(q - \frac{2}{\xi}\right). \tag{A16}$$

The dynamical transition will be located at the temperature for which the two curves $y = q$ and $y = L^{(\infty)}(\xi,q)$ are tangent, that is, when the breaking point of the step function is 1. For this reason we have

$$q_d^{(\infty)} = 1, \quad \xi_d^{(\infty)} = 2. \tag{A17}$$

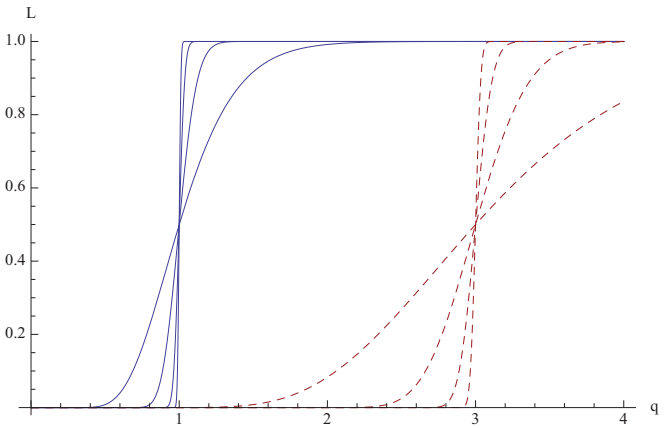

FIG. 3. (Color online) The right-hand side of Eq. (A21) for $p = 10^{10}, 10^{10^2}, 10^{10^3}, 10^{10^4}$. Solid blue lines: $\xi = 2$. Dashed orange lines: $\xi = 2/3$.

We have obtained the desired result in the infinite p limit, and now we want to compute the leading correction both to the critical temperature and to the critical overlap, starting again from Eq. (A13).

In order to obtain a finite result for finite $\overline{x}$ we must have

$$\ln(p) - \frac{\alpha^2}{2} + t\alpha = 0 \tag{A18}$$

for some value of t which now becomes our variable. Under this condition w^{p-1} behaves like $\theta(t - \overline{x})$, the equation for the overlap now reads

$$q = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{t}{\sqrt{2}}\right)\right), \tag{A19}$$

and Eq. (A18) with the substitution (A14) becomes

$$\xi q - \frac{2t}{\sqrt{\ln(p)}}(\xi q)^{\frac{1}{2}} - 2 = 0. \tag{A20}$$

Substituting Eq. (A20) into Eq. (A19), we get

$$q = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{1}{2\sqrt{2}}(\xi q - 2)\sqrt{\frac{\ln(p)}{\xi q}}\right)\right). \tag{A21}$$

Equation (A21) can be used to obtain approximate solutions in the large p limit.

Considering that $1 \ll t \ll \ln(p)$, from Eqs. (A19) and (A20), we have at leading order

$$q \simeq 1 - \frac{e^{-t^2/2}}{t\sqrt{2\pi}}, \quad \xi q \simeq 2 + 2\sqrt{2}\frac{t}{\sqrt{\ln(p)}} + \cdots, \tag{A22}$$

and we can then define

$$\varepsilon = 1 - q, \quad \mu = \xi - 2, \tag{A23}$$

satisfying the two coupled equations

$$\varepsilon = \frac{e^{-t^2/2}}{t\sqrt{2\pi}}, \quad \mu - 2\varepsilon = 2\sqrt{2}\frac{t}{\sqrt{\ln(p)}}. \tag{A24}$$

We can obtain t^2 from the second expression in Eqs. (A24) and plug it into the logarithm of the first one, getting at leading order

$$\ln(\varepsilon) = -\tfrac{1}{16}\ln(p)(\mu - 2\varepsilon)^2. \tag{A25}$$

At criticality the derivative of Eq. (A25) must hold as well, giving the second constraint necessary to determine both ε and μ:

$$\frac{1}{\varepsilon} = \frac{1}{4}(\mu - 2\varepsilon)\ln(p). \tag{A26}$$

Substituting this last equation into (A25), we obtain

$$\varepsilon^2 \ln(\varepsilon) = -\frac{1}{\ln(p)}, \tag{A27}$$

which at leading order gives

$$\varepsilon = \left[\frac{2}{\ln(p)\ln(\ln(p))}\right]^{\frac{1}{2}}. \tag{A28}$$

Substituting into (A26) and taking the leading order, we get the other correction,

$$\mu = 2\sqrt{2}\left[\frac{\ln(\ln(p))}{\ln(p)}\right]^{\frac{1}{2}}. \tag{A29}$$

Given ε and μ, we can write the dynamical overlap and critical temperature with the leading correction for large p:

$$\begin{aligned} q_d &= 1 - \left[\frac{2}{\ln(p)\ln(\ln(p))}\right]^{\frac{1}{2}}, \\ T_d^2 &= \frac{p}{2\ln(p)}\left[1 - \sqrt{2}\left[\frac{\ln(\ln(p))}{\ln(p)}\right]^{\frac{1}{2}}\right]. \end{aligned} \tag{A30}$$

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