Two-dimensional *XXZ*-Ising model with quartic interactions

J. S. Valverde
Instituto de Matemática, Estatística e Física, Universidade Federal do Rio Grande, Av. Itália km 8, Bairro Carreiros, CEP: 96.203-900, Rio Grande, RS, Brazil

(Received 10 February 2012; published 25 May 2012)

In this work we study a two-dimensional *XXZ*-Ising spin-1/2 model with quartic interactions. The model is composed of a two-dimensional lattice of edge-sharing unitary cells, where each cell consists of two triangular prisms, converging in a basal plane with four Ising spin-1/2 (open circles); the apical positions are also occupied by four Heisenberg spin-1/2 (solid circles). Interaction of the base plane containing the multispin Ising interaction has the parameter J_4, and the other pairwise interactions have parameter J. For the proposed model we construct the phase diagram at zero temperature and give all possible spin configurations. In addition, we investigate two regions where the model can be solved exactly, the *free fermion condition* (FFC) and the *symmetrical eight-vertex condition* (SEVC). For this purpose we perform a straightforward mapping for a zero-field eight-vertex model. The necessary conditions for the equivalence are analyzed for all ranges of the interaction parameters. Unfortunately, the present model does not satisfy the FFC unless the trivial case; however, it was possible to give a region where the model can be solved approximately. We study the SEVC and verify that this condition is always satisfied. We also explore and discuss the critical conditions giving the region where these critical points are relevant.

DOI: 10.1103/PhysRevE.85.051135 PACS number(s): 05.50.+q, 75.10.Jm

I. INTRODUCTION

It is well known that exact solutions for 2*d* lattices can help us to understand more about phase transitions and critical behavior of classical spin systems. Quantum spin models have a richer internal structure and are implemented as toy models for study of low-dimensional lattices; in this sense they remain as the most interesting models where several mathematical techniques can be used in order to explore the possibility of obtaining exact solutions. The first works in this direction were developed by Fisher [1] and Syozi [2] where the *star-triangle* and *decoration* transformation was performed. Actually in Fisher's work, the central decorated Ising site is generalized to an arbitrary mechanical system by mapping it into another equivalent Ising model. This procedure is accomplished by introducing a new interaction parameter set in the partition function. In such a way the resulting system can be used to explore the physical properties of the decorated model. In recent years several results for 2*d* Ising-Heisenberg models connected with their exact solutions were realized and investigation of physical properties of such systems was obtained [3–6]. In these works a multispin interaction describing the whole lattice is considered in the Hamiltonian and by mapping it into the square Ising model, all these new interaction parameters are calculated. It is worthwhile to point out that not all 2*d* models are exactly solved, however, in some of these cases it is possible to find out a region where an approximate solution takes place [7–9].

On the other hand, multispin interactions are important for several reasons; for example, higher-order interactions may exhibit rich phase diagrams and at the same time may describe phase transitions and physical behavior not observed in usual spin systems. Spin models with multispin interactions are also interesting because they display the nonuniversal critical phenomena [10]. Other Ising spin models with multispin interaction have been studied with different theoretical methods, as mean-field theory [11,12], effective field theory [13], and renormalization group methods [14]. It has been shown that for some compounds multispin interactions play a forthright role even more important than two-spin interactions. For instance, models with pair and quartic interactions were used to explain the existence of first-order phase transition in the squaric acid crystal ($H_2C_2O_4$) [15] and have been applied to describe thermodynamical properties of hydrogen-bonded ferroelectrics $PbHPO_4$ and $PbDPO_4$ [16]. Moreover, quartic interactions play a central role in explaining the thermodynamic properties of two-dimensional antiferromagnet La_2CuO_4 [17,18], a relevant compound when the superconductivity at high temperature is studied. In this way, self-spin interactions can be included by means of quartic interaction terms, affecting the magnetic properties of several copper compounds. All efforts for obtaining exact results of these models can be useful in the investigation of some important questions related to these magnetic properties [19]. For this reason the study of suchlike models have great interest from the theoretical and experimental points of view.

With this motivation we propose a two-dimensional Ising-Heisenberg model with quartic Ising interaction assigned to the outer spin-1/2 sites and explore the conditions under which it is possible to obtain an exact solution. The model is composed of two-dimensional lattice of edge-sharing unitary cells, where each cell consists of two triangular prisms converging in a basal plane with four Ising spin-1/2 (open circles); the apical positions are also occupied by four Heisenberg spin-1/2 (solid circles). Interaction of the base plane containing the quartic Ising interaction has the J_4 parameter, and the other two-site interactions have the J parameter. We construct the phase diagram for the model at zero temperature. In order to solve the model we perform the summation over the inner sites in each unitary cell of the whole lattice. The best way to achieve it is by fixing the set of spin values of the outer sites. In such a way we obtain a complete set of 16 eigenvalues for the unitary cell, where some of them are degenerated. It should be observed that using the rotation and spin inversion symmetry only three spin configurations are relevant. In addition, the model under consideration will be straightforwardly mapped to the exactly solved eight-vertex model and the conditions, under which an exact solution is possible, will be investigated. This procedure was already discussed in several works where the decoration-iteration transformation, as well as their generalization was performed. Indeed, the main idea of similar transformation is to establish an equivalent form to write down the original partition function by means of a new interaction parameter set.

The work is organized as follows. In Sec. II we explicitly present the two-dimensional Hamiltonian of the *XXZ*-Ising model with quartic interaction. Section III is devoted to the phase diagrams study where different ground states are found as a function of the anisotropy Δ parameter. In Sec. IV we perform a straightforward mapping of our model to the zero-field eight-vertex model, followed by a detailed analysis of the exact solution. Finally, in Sec. V some concluding remarks are given.

II. THE *XXZ*-ISING SPIN-1/2 MODEL WITH QUARTIC INTERACTION

We study a two-dimensional lattice composed of edge-sharing unitary square cells with spin-1/2; inside this square cell we consider four sites with *XXZ*-Ising interactions. Let us begin writing the Hamiltonian for the *XXZ*-Ising spin-1/2 model introducing the unitary cell $\mathcal{H}_u$. This unitary cell is displayed in Fig. 1(a) where the dashed lines represent the interactions of the Ising type and for the solid lines we indicate the Heisenberg interactions. In this unitary cell the four apical Heisenberg spin-1/2 sites $\{\sigma_i\}$ interact together by the pairwise *XXZ* interaction, while at the same time they are engaged two by two in quartic interaction with the corresponding two Ising spin $\{s_i\}$. The complete Hamiltonian of the whole lattice can be written as the sum of all unitary cell $\mathcal{H} = \sum_u \mathcal{H}_u$, with $\mathcal{H}_u$ given by

$$\mathcal{H}_u = \sum_{\langle i,j\rangle} \left(\Delta J\left(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y\right) + J\sigma_i^z\sigma_j^z\right) + J_4 \sum_{(i,j)} s_i s_j \sigma_i \sigma_j. \quad (1)$$

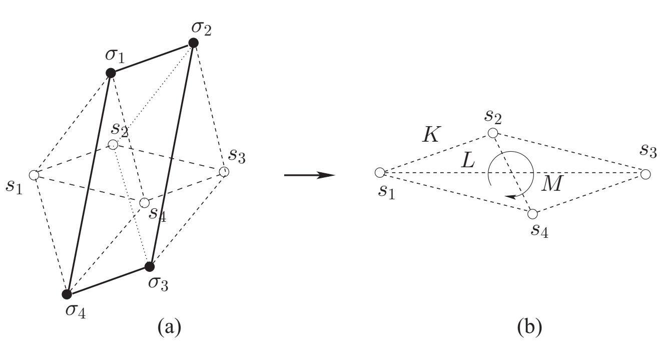

FIG. 1. (a) Illustration of the unitary cell of the *XXZ*-Ising spin-1/2 model with quartic interaction. The dashed lines indicate interactions of the Ising type while the solid lines are connected with the *XXZ* interaction. The open circles denote Ising positions and are connected with other unitary cells of the whole lattice. The solid circles denote the Heisenberg sites inside each unitary cell. (b) Display of the mapping of the unitary cell to the effective two-dimensional eight-vertex model.

The first sum $\langle i,j\rangle$ runs over the nearest neighbor site and the second one (i,j) runs over the next-nearest neighbor site. In the above Hamiltonian (1) we assume that $J_x = J_y = \Delta J$ and $J_z = J$, with Δ measuring a relative strength of the exchange anisotropy in the *XXZ* interaction.

III. PHASE DIAGRAMS

In this section we construct the phase diagram for the two-dimensional *XXZ*-Ising model with quartic interaction at zero temperature. Thereby we perform the summation of the Heisenberg σ_i sites on each unitary cell of the whole lattice. The best way to achieve it, is by fixing the set of Ising spin values $\{s_1,s_2,s_3,s_4\}$; in this sense it is not difficult to obtain a complete set of 16 eigenvalues for the unitary cell depicted in Fig. 1(b). First of all it should be observed that using the rotation and spin inversion symmetry only three spin configurations are relevant, thus it is enough to analyze the following spin configurations (i) $\{+,+,+,+\}$, (ii) $\{+,+,-,-\}$ and (iii) $\{+,+,+,-\}$.

A. Configuration {+, + , + ,+}

We begin discussing the energy levels of the first spin configuration $\{+,+,+,+\}$. For this purpose we perform the summation on the Heisenberg sites $\{\sigma_i\}$; afterwards we diagonalize the total Hamiltonian (1). In this way it is not difficult to obtain all 16 energy eigenvalues for this configuration. These eigenvalues are displayed in the first column of Table I, while the second column indicates the degeneracy order of the corresponding eigenvalues. From those eigenvalues we have some possible energies that become ground states, for example, for the region with $J > 0$ and any value of the parameters J_4 and Δ, the energy $\varepsilon_{FI_1} = -2J - \sqrt{8\Delta^2J^2 + (J-J_4)^2}$, is the lowest eigenvalue with the corresponding eigenvector given by

$$|FI_1\rangle = (1+\mathsf{R})\left|\begin{matrix} {}^{+} & - & +^{+} \\ {}_{+} & + & -_{+}\end{matrix}\right\rangle + a^{(-)}\sum_{r=0}^{3}\mathsf{R}^r\left|\begin{matrix} {}^{+} & - & +^{+} \\ {}_{+} & - & +_{+}\end{matrix}\right\rangle, \quad (2)$$

and $a^{(-)}$ given by

$$a^{(\mp)} = \frac{1}{4\Delta J}\left(J - J_4 - \sqrt{8\Delta^2J^2 + (J \mp J_4)^2}\right). \quad (3)$$

In this notation with the largest (+) and (−) signal (inner signs) we represent sites with spin σ. The magnetization of the unitary cell is neither null nor saturated and corresponds to the ferrimagnetic state with magnetization 1/4; we represent this state as FI_1. By R we represent the rotation operator

TABLE I. The energy levels for *XXZ*-Ising model with quartic interaction for the configuration $\{+,+,+,+\}$.

Energy $\{+,+,+,+\}$	Degeneracy
$-2J \pm \sqrt{8\Delta^2J^2 + (J-J_4)^2}$	1
$\pm 4\Delta J$	2
$4J + 2J_4$	2
$-4J + 2J_4$	1
$-2J_4$	3
0	4

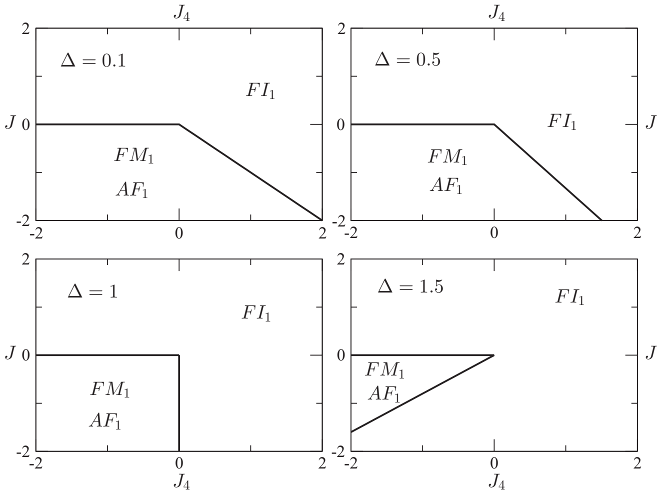

FIG. 2. In this figure we illustrate some ground states for different values of the anisotropy Δ parameter. It can be shown that for large values of $\Delta \gg 1$ only the ferrimagnetic state of type I with magnetization $1/4$ remains.

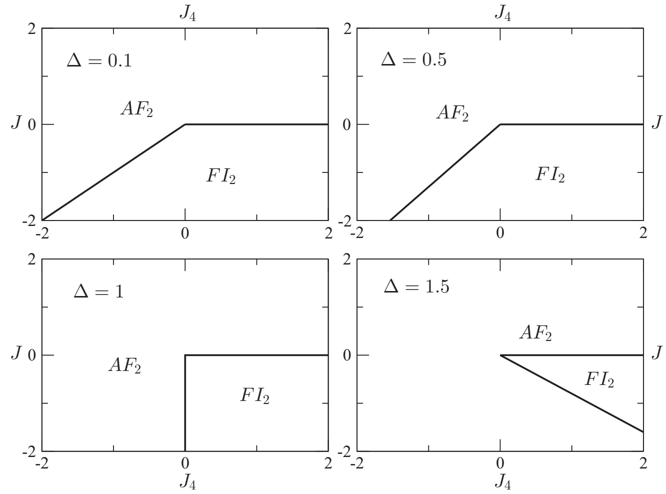

FIG. 3. In this figure the ground states are depicted as a function of the parameter Δ. We note in this phase diagram that for large values of the parameter $\Delta \gg 1$ only the ground state AF_2 is maintained. On the hand, for small values of the parameter Δ the two-degenerated state FI_2 with magnetization equal to $1/4$ appears.

acting only on Heisenberg interaction sites with spin σ; each rotation is performed in $\frac{\pi}{2}$, around the axis perpendicular to the plane of lattice. On the other hand, in the region with $J < 0$, we have the ground state as a function of the parameter Δ. In this case for large values of $\Delta \gg 1$ only the FI_1 state is present, while for small values of $\Delta < 1$ we have additionally two other states, ferromagnetic (FM_1) and antiferromagnetic (AF_1) states which we called type I. These states are degenerated and have the same eigenvalue, $\varepsilon_{FM_1} = \varepsilon_{AF_1} = 4J + 2J_4$. The corresponding eigenvectors are given by

$$|FM_1\rangle = \left| \begin{smallmatrix} + & + & + & + \\ + & + & + & + \end{smallmatrix} \right\rangle, \tag{4}$$

$$|AF_1\rangle = \left| \begin{smallmatrix} + & - & - & + \\ + & - & - & + \end{smallmatrix} \right\rangle. \tag{5}$$

These two states have magnetization equal to $1/2$ for the FM_1 and 0 for the AF_1 state. In Fig. 2 we depict the different ground states as a function of the Δ parameter.

B. Configuration {+, + , − ,−}

The next configuration that becomes eventually a ground-state energy is $\{+, + , - ,-\}$. Proceeding as in the previous case, we diagonalized the corresponding Hamiltonian, afterwards we found all 16 eigenvalues listed in Table II. Two

TABLE II. The energy levels for *XXZ*-Ising model with quartic interaction, for the configuration $\{+, + , - ,-\}$.

Energy $\{+, + , - ,-\}$	Degeneracy
$-2J \pm \sqrt{8\Delta^2 J^2 + (J + J_4)^2}$	1
$\pm 4\Delta J$	2
$4J - 2J_4$	2
$-4J - 2J_4$	1
$2J_4$	3
0	4

regions are also analyzed in relation to the sign of the J parameter. First, we have that for possitive values of $J > 0$ and any value of the parameters J_4 and Δ, the antiferromagnetic state, which we called type II (AF_2), becomes the ground state. It has the energy, $\varepsilon_{AF_2} = -2J - \sqrt{8\Delta^2 J^2 + (J + J_4)^2}$, and the eigenvector,

$$|AF_2\rangle = (1 + \mathsf{R}) \left| \begin{smallmatrix} - & - & + & - \\ + & + & - & + \end{smallmatrix} \right\rangle + a^{(+)} \sum_{r=0}^{3} \mathsf{R}^r \left| \begin{smallmatrix} - & - & - & - \\ + & + & + & + \end{smallmatrix} \right\rangle, \tag{6}$$

where $a^{(+)}$ is given by the relation (3). In the other region with negative values of $J < 0$ we have different situations for different values of the parameter Δ, for example, for large values of the parameter $\Delta \gg 1$ only the AF_2 given by Eq. (6) is present, while for small values of the parameter Δ, we have additionally a new ferrimagnetic state of type II (FI_2). This state has the energy value, $\varepsilon_{FI_2} = 4J - 2J_4$, and the corresponding eigenvector given by

$$|FI_2\rangle = \left| \begin{smallmatrix} - & + & + & - \\ + & + & + & + \end{smallmatrix} \right\rangle. \tag{7}$$

This is a two-degenerated state with magnetization equal to $1/4$. The other eigenvector state with the same energy is equivalent to (7) and it is obtained by applying the spin inversion operator to the whole unitary cell; magnetization of this state is equal to $-1/4$. In Fig. 3 all ground states are represented as a function of the parameter Δ.

C. Configuration {+, + , + ,−}

Finally, we study the last configuration that could have ground-state energies. After some manipulations we found all 16 eigenvalues of the configuration $\{+, + , + ,-\}$. All these values, as well as the degeneracy, are listed in Table III. In this case we have different situations depending on values of the anisotropy parameter Δ. For example, for $\Delta = 1$, we see that the ground-state energies are given by an antiferromagnetic

TABLE III. The energy levels for *XXZ*-Ising model with quartic interaction, for the configuration $\{+,+,+,-\}$.

Energy $\{+,+,+,-\}$	Degeneracy
$\pm 2\sqrt{J_4^2+4\Delta^2 J^2}$	2
$-2J(1\pm\sqrt{1+8\Delta^2})$	1
$\pm 2J_4$	2
$-4J$	1
$4J$	2
0	3

state ($AF^{(+)}$), with energy $\varepsilon_{AF^{(+)}} = -2J(1+\sqrt{1+8\Delta^2})$, and two other degenerated states [i.e., a ferrimagnetic state of type III (FI_3) and an antiferromagnetic state of type III (AF_3)]; these states have energy, $\varepsilon_{FI_3} = \varepsilon_{AF_3} = -2\sqrt{J_4^2+4\Delta^2 J^2}$. This situation is illustrated in Fig. 4(b). The corresponding eigenvectors in this case are given by

$$|AF^{(\pm)}\rangle = (1+\mathsf{R})\left|\begin{smallmatrix} - & - & + & + \\ + & + & - & + \end{smallmatrix}\right\rangle + c^{(\pm)}\sum_{r=0}^{3}\mathsf{R}^r\left|\begin{smallmatrix} - & - & - & + \\ + & + & + & + \end{smallmatrix}\right\rangle, \quad (8)$$

$$|FI_3\rangle = (1+c\mathsf{R})(1+\mathsf{R}^2)\left|\begin{smallmatrix} - & - & + & + \\ + & + & + & + \end{smallmatrix}\right\rangle, \quad (9)$$

$$|AF_3\rangle = (1+c\mathsf{R})(1+\mathsf{R}^2)\left|\begin{smallmatrix} - & + & - & + \\ + & - & - & + \end{smallmatrix}\right\rangle, \quad (10)$$

where $c^{(\pm)}$ and c are equal to

$$c^{(\pm)} = -\frac{1}{4\Delta}(1\pm\sqrt{1+8\Delta^2}), \quad (11)$$

$$c = -\frac{1}{2\Delta J}\left(J_4+\sqrt{J_4^2+4\Delta^2 J^2}\right). \quad (12)$$

For small values of the anisotropy parameter, as, for example, $\Delta = 0.5$, a new ferrimagnetic state ($FI^{\pm}$) appears as depicted in Fig. 4(a). The energy of this state is two degenerated and equal to $\varepsilon_{FI^{(\pm)}} = 4J$, and the corresponding eigenvector,

$$|FI^{(\pm)}\rangle = \left|\begin{smallmatrix} - & \pm & \pm & + \\ + & \pm & \pm & + \end{smallmatrix}\right\rangle. \quad (13)$$

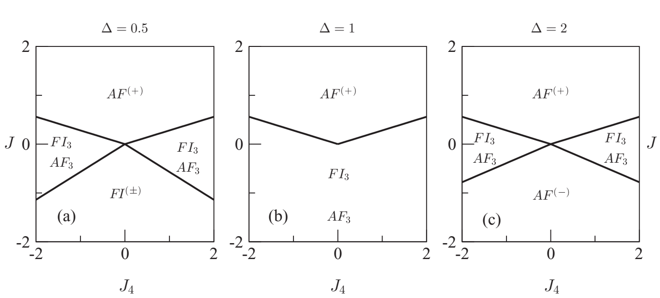

FIG. 4. We illustrate the ground-state energies for different values of the anisotropy parameter Δ. In (b) for $\Delta = 1$ we have three ground states: $AF^{(+)}$, FI_3, and AF_3. For values of the parameter $\Delta < 1$ or $\Delta > 1$, three new ground states appear, $FI^{(\pm)}$ and $AF^{(-)}$, respectively. This is illustrated in (a) and (c) for two fixed values of Δ.

For values of $\Delta > 1$, as, for example, $\Delta = 2$, another antiferromagnetic state $AF^{(-)}$ appears; this ground state has the eigenvalue, $\varepsilon_{AF^{(-)}} = -2J(1+\sqrt{1-8\Delta^2})$, with the eigenvector given by the relation (8). These ground states are depicted in Fig. 4(c).

IV. EQUIVALENCE TO THE ZERO-FIELD EIGHT-VERTEX MODEL

In this section we turn our attention to the analysis of the finite-temperature behavior of the *XXZ*-Ising model. In this sense we proceed to study which conditions are necessary to obtain an exact solution for the proposed model given by the Hamiltonian (1). The best way to achieve it is by performing a straightforward mapping to the exactly solved zero-field eight-vertex model. This procedure was already discussed in several works [1,4,5] where a decorated transformation was applied to Ising-Heisenberg models. Actually, the main idea of similar transformation is to establish an equivalent form to write down the original partition function by means of a new interaction parameter set. Considering that any two unitary cells of the whole lattice commute with each other (i.e., $[\mathcal{H}_u,\mathcal{H}_{u'}]=0$), it is possible to establish a simplified relationship for the partition function. We begin writing the partition function as the following:

$$\mathcal{Z} = \sum_{\{s\}}\prod_{u=1}^{N} w(\{s\}), \quad (14)$$

where N is the number of unitary square cells in the whole lattice and $w(\{s\})$ are defined as the Boltzmann weights assigned to the uth unitary cell. They are given by

$$w(\{s\}) \equiv \mathrm{Tr}_{\{\sigma\}}(e^{-\beta\mathcal{H}_u}). \quad (15)$$

Here $\mathcal{H}_u$ is the Hamiltonian of the unitary cell and is given by the relation (1). By $\beta = 1/kT$ we denote the inverse of temperature, k is the Boltzmann constant, and the $\mathrm{Tr}_{\{\sigma\}}$ indicates the trace on the spin-1/2 sites inside the unitary square cell. As is shown in the works [1,4] it can be established a complete equivalence between the partition function of the original Ising-Heisenberg model and the partition function of the zero-field eight-vertex Ising model on a square lattice. This transformation is illustrated in Fig. 1(b). For this purpose we introduce the effective Boltzmann weight $\widetilde{w}$, defined as

$$\widetilde{w}(\{s\}) = f e^{-\beta\widetilde{\mathcal{H}}_u}, \quad (16)$$

here f is a new constant and $\widetilde{\mathcal{H}}_u$ is the new effective Hamiltonian of the effective square unitary cell. The eight different spin arrangements of these eight Boltzmann weights are schematically depicted in the Fig. 5. The next step is to

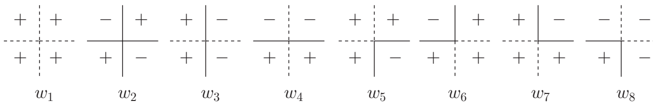

FIG. 5. We Illustrate the eight different possible spin arrangements corresponding to different Boltzmann weights. Inversion of all spins corresponds to the same vertex. The sign ($\pm$) denotes the spin state $\sigma_z = \pm 1/2$.

write down the total effective Hamiltonian as

$$\widetilde{\mathcal{H}}_u = K \sum_{(k,k')} s_k s_{k'} + L \sum_{\langle k,k'\rangle} s_k s_{k'} + M s_1 s_2 s_3 s_4, \qquad \widetilde{\mathcal{H}} = \sum_{\text{all square}} \widetilde{\mathcal{H}}_u. \tag{17}$$

Here K, L, and M represent a new interaction parameter set related to the effective Ising square model and $\widetilde{\mathcal{H}}$ is the total effective Hamiltonian. The first sum in Eq. (17), with $k,k' = 1..4$, runs over the nearest-neighbor spin-1/2 Ising site of the effective unitary cell, while the second one runs over the next-nearest-neighbor spin-1/2 Ising site. Is a matter that the Boltzmann weights contained in the expression (15) and the effective Boltzmann weight given by Eq. (16) are equivalent. In such a way it is not tricky to conclude that an equivalence between both partition functions also happens. In this sense we can write the effective partition function as

$$\widetilde{\mathcal{Z}} = f^N \mathcal{Z}_0. \tag{18}$$

In the above relation $\mathcal{Z}_0$ is the partition function for spin-1/2 of the eight-vertex model. After some algebraic manipulations, we find the following values for the interaction parameters of the effective two dimensional square Ising model:

$$f = \left(w_1 w_3 w_5^2\right)^{1/4}, \tag{19}$$

$$\beta L = \ln\left(\frac{w_3}{w_1}\right)^{1/4}, \tag{20}$$

$$\beta M = \ln\left(\frac{w_5^2}{w_1 w_3}\right)^{1/4}, \tag{21}$$

$$K = 0, \tag{22}$$

where the Boltzman weights defined by (15) take the form,

$$w_1 = 2e^{2\beta J}\mathrm{ch}\left(2\beta\sqrt{8\Delta^2 J^2 + (J - J_4)^2}\right) + e^{-2\beta J_4}\left(e^{4\beta J} + 2e^{-4\beta J}\right) + 3e^{2\beta J_4} + 4\mathrm{ch}(4\beta\Delta J) + 4, \tag{23}$$

$$w_3 = 2e^{2\beta J}\mathrm{ch}\left(2\beta\sqrt{8\Delta^2 J^2 + (J + J_4)^2}\right) + e^{2\beta J_4}\left(e^{4\beta J} + 2e^{-4\beta J}\right) + 3e^{-2\beta J_4} + 4\mathrm{ch}(4\beta\Delta J) + 4, \tag{24}$$

$$w_5 = 4\mathrm{ch}\left(2\beta\sqrt{J_4^2 + 4\Delta^2 J^2}\right) + 2e^{2\beta J}\mathrm{ch}\left(2\beta J\sqrt{1 + 8\Delta^2}\right) + 4\mathrm{ch}(2\beta J_4) + e^{4\beta J} + 2e^{-4\beta J} + 3. \tag{25}$$

The other Boltzmann weights can be obtained from the above relations taking into account the following identities:

$$w_1 = w_2, \qquad w_3 = w_4, \qquad w_5 = w_6 = w_7 = w_8. \tag{26}$$

At this stage we would like to make some remarks about the Boltzmann weights (23)–(25). First of all, it directly results from these relations that the greatest Boltzmann weight is given by w_1 or w_3, so the w_5 cannot be the maximum value of these Bolztmann weights. This fact turns relevant when the critical conditions of the model under consideration are studied. Secondly, from these relations it is possible to observe the symmetry of the Boltzmann weights w_1 and w_3 in relation to the parameter J_4, namely $w_1(\pm J_4) = w_3(\mp J_4)$; furthermore, because w_5 is an even function in J_4 we can assume in the next lines a positive value of $J_4 > 0$ without loss of generality. Finally, we remark that in the case where $w_1 = w_3$, called the disorder solution, immediately it is followed by the zero value of the L parameter given by the relation (20), leaving us only the quartic interaction in the effective model.

A. The exactly solved model

An extensive study of the exactly solvable model can be found in Ref. [3]. On the other hand, recently two-dimensional Ising-Heisenberg models with quartic interaction were solved by mapping into the zero-field eight-vertex model [19]. In general this mapping is possible only for some values of the interaction parameters. In our case, the model defined by the Hamiltonian (1) is mapped into the eight-vertex model in order to explore the range of the interaction parameter values where this mapping is successful. In the following lines we discuss the conditions for obtaining an exactly solvable model in detail.

1. Free fermion condition (FFC)

The study of the partition function with arbitrary Boltzmann weights was considered in detail by Fun and Wu [7] where the partition function was written in a different way as the vacuum expectation value of a linear combination of products of fermion operators satisfying the fermi algebra. It is appropriate to mention that such a model can be regarded as a free-fermion system on a lattice. Using the anticommuting properties and after some manipulations, they found that the model has an exact solution only in the case $\Theta = 0$, with Θ given by

$$\Theta = w_1 w_2 + w_3 w_4 - w_5 w_6 - w_7 w_8. \tag{27}$$

This condition, called the free fermion condition, will apply now to our model. Unfortunately, when imposing the FFC we cannot find an exact solution. However, as was pointed out by Fun and Tang, it is possible to obtain a region where the model can be solved approximately. This happens when the condition $\Theta/w_{\text{max}}^2 \ll 1$ takes place; this procedure is detailed in the works [7,8]. On the other hand, we can verify that in this situation Θ has a positive amount of Θ/w_{max}^2. In Fig. 6(a) we give the region where the FFC is satisfied approximately for the fixed value $\beta J_4 = 0.1$. This condition is represented by the white area where the approximation is taken for $\Theta/w_{\text{max}}^2 \ll 0.001$. Actually, the maximum value of the Boltzmann weight is also a function of the anisotropy parameter Δ; to remark this fact we consider positive values of $J_4 > 0$ and fixed the parameter $\beta J_4 = 0.1$. This is illustrated by the dashed line in

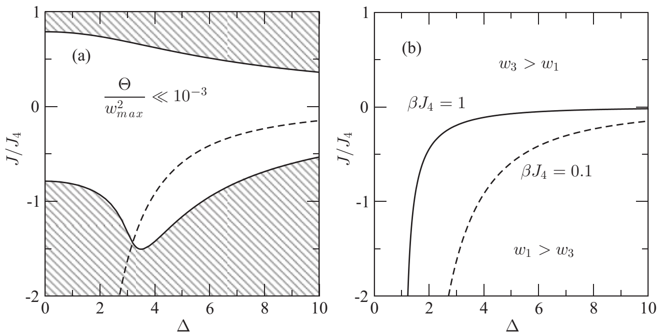

FIG. 6. (a) Illustration of the region where the FFC is satisfied approximately for a fixed value $\beta J_4 = 0.1$. The white region satisfies the condition $\Theta/w_{\max}^2 \ll 0.001$ whereas the shadow region is for the case $\Theta/w_{\max}^2 \gg 0.001$. The dashed line separates two sectors; upward of this line we have the maximum Boltzmann weight w_3, and downward of this line the maximum Boltzmann weight w_1. (b) The maximum Boltzmann weight as a function of Δ and J/J_4. We consider positive values of $J_4 > 0$, and for the sake of comparison we fixed two values $\beta J_4 = 0.1$ and $\beta J_4 = 1$. For negative values of $J_4 < 0$, we have the axis J/J_4 going to $-J/J_4$ and the Boltzmann weights changing as $w_1 \to w_3$.

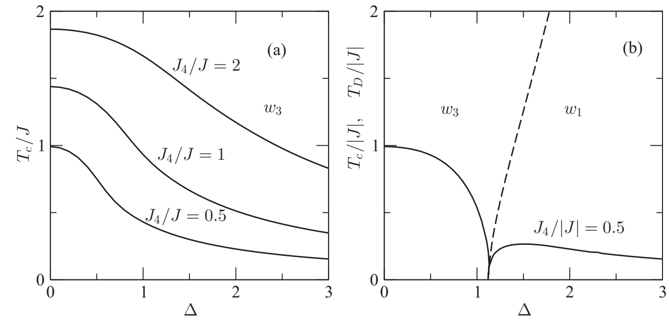

FIG. 7. In this figure we consider only positive values of $J_4 > 0$. In (a) we give the dependence of the critical temperature T_c/J as a function of the anisotropy parameter Δ at three different values of relative strengths of the quartic Ising interaction, J_4/J. This case considers only positive values of J. In the (b) the solid lines represent the values where the critical condition is satisfied for $J < 0$. The left wing takes into account that the maximum Boltzmann weight is given by w_3 while the right wing considers the maximum Boltzmann weight as w_1. The dashed line represents the values for the disorder temperature $T_D/|J|$ as a function of Δ, where the disorder condition $w_1 = w_3$ is satisfied.

the Fig. 6(a). Upward of this line the maximum Boltzmann weight is given by w_3, whereas downward of this line the maximum value results to be w_1. To compare the behavior of the maximum Boltzmann value as a function of Δ we fixed two values of βJ_4; this is represented in Fig. 6(b). It is worthwhile to remark that the symmetry observed for the Boltzmann weights [(23) and (24)] in relation to the shift of J_4 is also observed in Fig. 6. In this sense, if we take negative values of $J_4 < 0$ and invert the axis J/J_4 the Boltzmann weights are also inverted as $w_1 \to w_3$.

2. The symmetric eight-vertex model condition (SEVC)

We discuss the second branch where the model has an exact solution; this condition is called the *symmetric eight-vertex condition* (SEVC) given by

$$w_1 = w_2, \quad w_3 = w_4, \quad w_5 = w_6 \quad w_7 = w_8. \tag{28}$$

It is not tricky to see from the relations (26) that this condition is fully satisfied for any values of the interaction parameters J, J_4 and all values of the anisotropy parameter Δ.

B. The critical line

It is also possible to discuss the critical behavior even when the exactly solvable condition is not satisfied. For the FFC the critical points are given by the following relation:

$$w_1 + w_2 + w_3 + w_4 = 2\max(w_1, w_2, w_3, w_4). \tag{29}$$

Unfortunately, this condition is not satisfied for any value of the interaction parameter even approximately. The second branch for obtaining the critical points is the region where the SEVC condition is satisfied. In this case the Boltzmann weights are related by

$$w_1 + w_3 + w_5 + w_7 = 2\max(w_1, w_3, w_5, w_7). \tag{30}$$

The above relation gives us different regions where the interaction parameters J, J_4 are connected to each other in order to satisfy this condition. In Fig. 7 we display the critical lines giving T_c/J as a function of Δ, with T_c being the critical temperature. In this case we consider only positive values of the parameter $J_4 > 0$. In Fig. 7(a), we consider positive values of the parameter $J > 0$ and depict the critical lines for three different values of J_4/J. In this region the maximum Boltzmann value results to be w_3. In Fig. 7(b) the negative values of the parameter $J < 0$ are considered. To illustrate this situation we fixed the value $J_4/|J| = 0.5$ and displayed the critical points represented by the solid lines in Fig. 7(b). The left wing of critical lines takes into account that the maximum Boltzmann value is given by w_3 while the right wing of the critical lines has the Boltzmann weight w_1 as the maximum value. For the fixed value $J_4/|J| = 0.5$ adopted in this figure we have that at zero critical temperature $T_c/|J| = 0$ the anisotropy parameter is $\Delta = 1.11939$, however, for large negative values of $J \ll 0$ or $J_4/|J| \to 0$ the anisotropy parameter $\Delta \to 1$. It is worth mentioning that for negative values of $J_4 < 0$ it would be necessary to consider negative values of the relative strength of quartic interaction J_4/J. This results in replacing the Bolztmann weight w_1 by w_3, obtaining the same critical lines for this case.

1. Disorder solution

It is also interesting to analyze the case where $w_1 = w_3$. This condition, called the disorder solution, implies an effective reduction of the number of parameters and ensures the disordered nature of the *XXZ*-Ising model. In this case we observe that only for negative values of $J < 0$ the disorder condition is satisfied. Moreover, for very small values of temperature $T \to 0$ and large negative values of $J \ll 0$ the anisotropy parameter approximates to 1 (i.e., $\Delta \to 1$). In Fig. 7(b) we display the disorder temperature $T_D/|J|$ as a function of the parameter Δ and represented by the dashed line. It is quite noticeable that the disorder (dashed) line shown in this figure has almost a constant tangent that becomes more evident for $T \to 0$. In other words it can be possible to obtain by an approximation a linear dependence of the disorder temperature T_D in relation to the anisotropy parameter Δ. In this regard we point out that this condition gives us a zero value for the parameter L defined by (20).

V. CONCLUSIONS

In the present work we proposed a two-dimensional *XXZ*-Ising model with quartic interaction. We have discussed the ground-state energy of the model and plotted the phase diagram at zero temperature as a function of the anisotropy parameter Δ. To study the different ground states we separate all possible configurations fixing the set of Ising spins (s_1,s_2,s_3,s_4). Analysis of these configurations leads us to conclude that only three of them are relevant in relation to the minimal energy. The first configuration $(+,+,+,+)$ gives us three different states: a ferrimagnetic state of type I (FI_1), a ferromagnetic state of type I (FM_1), and an antiferromagnetic state of type I (AF_1). The second configuration $(+,+,-,-)$ results in two other states: an antiferromgnetic state of type II (AF_2) and a ferrimagnetic state of type II (FI_2). The last configuration $(+,+,+,-)$ results in six new states: two antiferromgnetic states that we called $AF^{\pm}$, two ferrimagnetic states called $FI^{\pm}$, one ferrimagnmetic state of type III (FI_3), and one antiferromgnetic state of type III (AF_3). From the phase diagram we figured out that the ground states are deeply connected with the anisotropy parameter Δ.

Then, we explored the conditions under which the model can be exactly solved. With this aim the Boltzmann weights were calculated and the two different constraints for the *free fermion condition* and the *symmetric eight-vertex conditions* were discussed. The first attempt to obtain an exact solution was performed in the region defined by the FFC condition. Unfortunately, the model is not exactly solved for this case. However, it was possible to plot up the region where the model can be solved approximately; we have done this figure with an approximation of $\Theta/w_{\max}^2 \ll 0.001$. For the other region where an exact solution is possible, namely SEVC, we verify that these conditions are satisfied in an unrestricted manner (i.e., for any value of the interaction parameters J, J_4 and the anisotropy parameter Δ). Furthermore, the critical conditions were explored; for the case of SEVC condition we depicted the critical lines where the critical temperature T_c/J appears as a function of the anisotropy parameter Δ.

ACKNOWLEDGMENTS

J. S. Valverde thanks Fundação de Amparo à Pesquisa do Estado do Rio Grande do Sul (FAPERGS) for full financial support.

[1] M. E. Fisher, Phys. Rev. **113**, 969 (1958).
[2] I. Syozi, in *Phase Transitions and Critical Phenomena*, Vol. 1, edited by C. Domb and M. S. Green (Academic, New York, 1972), pp. 269–329.
[3] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, New York, 1982).
[4] J. S. Valverde, Onofre Rojas, and S. M. de Souza, Physica A **388**, 1419 (2009).
[5] J. S. Valverde, Onofre Rojas, and S. M. de Souza, Phys. Rev. E **79**, 041101 (2009).
[6] Jozef Strecka, Phys. Lett. A **374**, 3718 (2010).
[7] C. Fan and F. Yu. Wu, Phys. Rev. **179**, 560 (1969); Phys. Rev. B **2**, 723 (1970).
[8] Kun-Fa Tang, J. Phys. A: Math. Gen. **21**, L1097 (1988).
[9] F. Yu. Wu, Phys. Rev. **168**, 539 (1968); **183**, 604 (1969); Phys. Rev. B **4**, 2312 (1971).
[10] L. P. Kadanoff and F. J. Wegner, Phys. Rev. B **4**, 3989 (1971).
[11] C. L. Wang, Z. K. Quin, and D. L. Lin, J. Magn. Magn. Mater. **88**, 87 (1990).
[12] K. G. Chakraborty, J. Magn. Magn. Mater. **114**, 155 (1992).
[13] M. Saber, Phys. Stat. Sol. **178**, K99 (1993).
[14] F. Lee, H. H. Chen, and F. Y. Wu, Phys. Rev. B **40**, 4871 (1989).
[15] C. L. Wang, Z. K. Qin, and D. L. Lin, Phys. Rev. B **40**, 680 (1989); J. Magn. Magn. Mater. **88**, 87 (1990).
[16] W. Chunlei, Q. Zikai, and Z. Jingbo, Ferroelectrics **77**, 21 (1988).
[17] Y. Honda, Y. Kuramoto, and T. Watanabe, Phys. Rev. B **47**, 11329 (1993).
[18] R. Coldea, S. M. Hayden, G. Aeppli, T. G. Perring, C. D. Frost, T. E. Mason, S. W. Cheong, and Z. Fisk, Phys. Rev. Lett. **86**, 5377 (2001).
[19] Jozef Strecka, Lucia Canova, and Kazuhiko Minami, Phys. Rev. E **79**, 051103 (2009); AIP Conf. Proc. **1198**, 156 (2009).