Fractional Brownian motors and stochastic resonance

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We study fluctuating tilt Brownian ratchets based on fractional subdiffusion in sticky viscoelastic media characterized by a power law memory kernel. Unlike the normal diffusion case, the rectification effect vanishes in the adiabatically slow modulation limit and optimizes in a driving frequency range. It is shown also that the anomalous rectification effect is maximal (stochastic resonance effect) at optimal temperature and can be of surprisingly good quality. Moreover, subdiffusive current can flow in the counterintuitive direction upon a change of temperature or driving frequency. The dependence of anomalous transport on load exhibits a remarkably simple universality.

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I. INTRODUCTION

Diverse research fields such as anomalous diffusion and transport [1–4], Brownian ratchets [5–9], and stochastic resonance (SR) [10] have attracted much attention over the past two decades with a huge amount of research produced and a number of insightful reviews written which address both fundamental aspects of nonequilibrium statistical physics and various interdisciplinary applications in physics, chemistry, biology, and technology. The existing ratchet literature is restricted mostly to normal diffusion ratchets. Here, a rectification current can emerge for the particles diffusing in some periodic and unbiased on average potential due to breaking the symmetry of thermal detailed balance by an external time-dependent driving. This in turn requires breaking some spatiotemporal symmetry [8,9], for example, the spatial inversion symmetry as in Fig. 1 in the case of a fluctuating tilt ratchet [6,7] driven by harmonic force, which we consider in this work. The emergence of net directed motion in unbiased on average systems is a strongly nonequilibrium and nonlinear effect that is absent, for example, within the linear response approximation or linear Onsager regime of nonequilibrium thermodynamics. A characteristic feature of any true ratchet is its ability to sustain a load, that is, a force directed against the transport direction. The presence of a nonzero stopping force distinguishes the genuine ratchets or Brownian motors capable of doing useful work from the futile ones or pseudoratchets [8,9]. In any isothermal Brownian motion which never ceases, the dissipative loss of energy is compensated at thermal equilibrium by the energy (heat) gain due the thermal noise of environment so that on average the classical Brownian particle has a kinetic energy $k_B T/2$ per degree of freedom. This ensures the absence of a net directed motion and of the total heat exchange between the particle and its environment. The directed Brownian motion requires an external source of energy—a part of it will be put into the directed motion and a part dissipated as an *excess* heat to the environment. The thermal noise plays a constructive role here, as a sort of lubricant to smooth the friction and also to provide thermal energy fluctuations. It allows to overcome potential barriers met on the particle's pathway. Restricted to the classical world, without noise the Brownian particle would remain localized in a potential well, starting there with subthreshold energy and driven by a weak external drive. Therefore, in such a setup one generally expects that the rectification current response to subthreshold driving will increase with the noise intensity, which is proportional to temperature. However, for a very strong noise the potential barriers cease to matter and one expects that the rectification effect due to a spatial asymmetry of potential will vanish. Therefore, there should exist optimal thermal noise intensity and a corresponding temperature which typifies SR, at least in a broad sense [10].

The focus of this paper includes both the ratchet and SR effects in subdiffusive transport occurring in a viscoelastic environment, where both the mean displacement and the position variance grow sublinearly, $\langle \delta x(t)\rangle \propto t^{\alpha}$, and $\langle \delta x^2(t)\rangle \propto t^{\alpha}$, respectively, with $0 < \alpha < 1$. Such viscoelastic environments are typified by dense polymer solutions [11,12], colloidal suspensions [13,14], and molecularly crowded cytoplasm of biological cells [15–20]. The use of microrheology [14,21] allows the study of the frequency-dependent medium's viscosity, which causes an anomalously slow spread of the position variance of Brownian test particles. It relates these quantities to the frequency-dependent complex shear modulus $G^{*}(\omega)$ [11] using the model of the overdamped generalized Langevin equation (GLE) [22–24]. There are numerous experimental studies revealing power law scaling $G^{*}(\omega) \propto (i\omega)^{\alpha}$ (in a certain frequency range) in such viscoelastic media, as reviewed, for example, in Ref. [21]. They are consistent with a GLE description if we assume a power law decaying memory kernel, which is also used in this paper and yields the fractional Brownian motion description [25] in the overdamped potential-free case [26]. Within this description, the dielectric response of the particles trapped in the parabolic potentials is known to be of the Cole-Cole type [27,28], which indeed is typical for such media. However, the corresponding nonlinear

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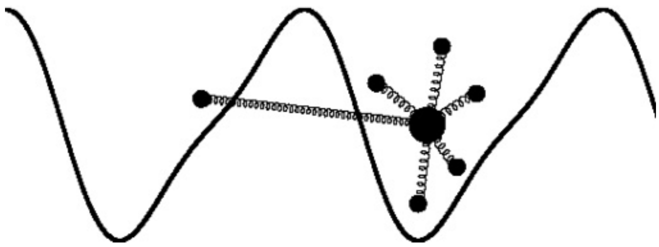

FIG. 1. Ratchet potential and central Brownian particle coupled to auxiliary Brownian particles modeling a viscoelastic environment.

dynamics is highly nontrivial [29–31] and remains largely unexplored. In particular, the operation of Brownian motors in such environments presents a very important unsolved problem of both general and applied interest. For example, recent experimental results [32] raise the question of how molecular motors efficiently operate in viscoelastic environments such as cytosol in biological cells. We do not attempt to solve this important biophysical problem in the present paper. However, we do study a toy model of general interest in statistical physics of nonequilibrium anomalous transport which is inspired by this general question.

Indeed, the very existence of such a nonlinear subdiffusive ratchet transport is not obvious and can be questioned. For example, within a continuous time random walk (CTRW) mechanism of subdiffusion featured by divergent mean residence times (MRTs) [1,3], the current response to external periodic driving is asymptotically zero [33]. This clearly prohibits any asymptotic rectification effect for such fluctuating tilt ratchets. However, a subdiffusive rocking ratchet based on the fractional Brownian motion (FBM) does exist [34], and the corresponding flashing-potential subdiffusive ratchet was introduced recently in Ref. [35]. In this paper, we explain the unusual properties of the rocking subdiffusive ratchets and show, in particular, that a resonance-like character of the emerging anomalous nonadiabatic ratchet effect is indeed of SR origin and that it is a genuine ratchet effect.

II. THE MODEL

Let us consider a GLE [22–24] for a Brownian particle with mass m,

$$m\ddot{x}+\int_0^t \eta(t-t')\dot{x}(t')dt' = f(x,t)+\xi(t), \tag{1}$$

where $f(x,t) = -\partial V(x,t)/\partial x$ is a deterministic force, $\xi(t)$ is zero-mean and Gaussian-distributed thermal noise, and $\eta(t)$ is the frictional memory kernel related to noise by the fluctuation-dissipation relation (FDR)

$$\langle \xi(t')\xi(t)\rangle = k_B T \eta(|t-t'|)\ . \tag{2}$$

The FBM emerges as solution of GLE (1) in the overdamped limit, $m \to 0$, of a force-free motion, $f \to 0$, for a power law frictional kernel $\eta(t) = \eta_\alpha t^{-\alpha}/\Gamma(1-\alpha)$, with $0 < \alpha < 1$ [$\Gamma(x)$ is standard gamma-function], and the FDR-related noise $\xi(t)$, which is termed the fractional Gaussian noise (FGN) [25]. The corresponding GLE is also termed the fractional Langevin equation (FLE) [24,36,37]. The GLE can be derived from a standard Hamiltonian model of the Brownian motion based on the coupling of Brownian particle to a thermal bath of harmonic oscillators modeling the environment [23]. The FLE model corresponds to subohmic thermal bath characterized by the spectral bath density $J(\omega) \propto \eta_\alpha \omega^\alpha$ [38]. Such a modeling requires a dense spectrum of the thermal bath or *quasi-infinite* number of oscillators. Alternatively, one can model the environment by a *finite* set of comoving auxiliary Brownian particles (cf. Fig. 1), with masses m_i coupled elastically with the spring constants k_i to the central Brownian particle and experiencing the viscous Stokes friction (with frictional constants η_i) and uncorrelated, $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$, white-noise thermal Gaussian forces $\sqrt{2\eta_i k_B T}\xi_i(t)$:

$$\begin{aligned} m\ddot{x} &= f(x,t) - \sum_{i=1}^N k_i(x-x_i),\\ m_i\ddot{x}_i &= k_i(x-x_i) - \eta_i\dot{x}_i + \sqrt{2\eta_i k_B T}\xi_i(t). \end{aligned} \tag{3}$$

These equations are similar to the starting equations of motion in the purely dynamical model [23,38] leading to the GLE description. These dynamical dissipationless equations can be obtained by setting $\eta_i \to 0$, that is, considering a purely dynamical evolution with a tremendously large number of oscillators $N \to \infty$ with a dense spectrum $\omega_i = \sqrt{k_i/m_i}$. The only nondynamical assumption in the dynamical model is that *initially* the thermal bath oscillators are canonically distributed at the temperature T, like in a standard molecular dynamics setup. We replace them by a *finite* set of auxiliary Brownian particles.

Considering the overdamped limit for auxiliary particles, $m_i \to 0$, this yields

$$\begin{aligned} m\ddot{x} &= f(x,t) + \sum_{i=1}^N u_i(t),\\ \dot{u}_i &= -k_i v - \nu_i u_i + \sqrt{2\nu_i k_i k_B T}\xi_i(t), \end{aligned} \tag{4}$$

upon introduction of fluctuating viscoelastic forces, $u_i = -k_i(x-x_i)$, where $\nu_i = k_i/\eta_i$ are the relaxation rates of viscoelastic forces. The last equation for u_i has the form of relaxation equation for elastic force introduced by Maxwell in his macroscopic theory of viscoelasticity [39], which is augmented by the corresponding Langevin force in accordance with FDR. Such a description was introduced in Refs. [29,34] to model anomalous Brownian motion in complex viscoelastic media within a generalized Maxwell model. Indeed, by choosing initial $u_i(0)$ as independent zero-mean Gaussian variables with variance $\langle u_i^2(0)\rangle = k_i k_B T$ and excluding the dynamics of auxiliary variables u_i, the GLE (1) with FDR (2) follows immediately with the memory kernel

$$\eta(t) = \sum_{i=1}^N k_i e^{-\nu_i t}. \tag{5}$$

For $N = 1$, an earlier Markovian embedding of the GLE with exponentially decaying memory kernel [40] is readily reproduced. Furthermore, by choosing $\nu_i = \nu_0/b^{i-1}$, $k_i = C_\alpha(b)\eta_\alpha \nu_i^\alpha/\Gamma(1-\alpha)$, where b is a scaling parameter and $C_\alpha(b)$ is a fitting dimensionless constant, the above power law kernel $\eta(t)$ can be well approximated over about $r = N\log_{10} b - 2$ temporal decades between two time cutoffs,

$\tau_l = \nu_0^{-1} < t < \tau_h = \tau_l b^{N-1}$. Similar scaling and approximation are well known in the anomalous relaxation theory [1,41]. Physically, ν_0 corresponds to a high-frequency cutoff in $J(\omega)$, or the largest medium's frequency, and ν_0/b^{N-1} corresponds to the slowest medium's mode, both of which are always present in reality. For the case $\alpha = 0.5$ studied numerically in this work, the choice $b = 10$, $C_{0.5} = 1.3$, $N = 12$, and $\nu_0 = 100$ provides an excellent approximation to the FLE dynamics over at least ten time decades, until $t_{\max} \sim 0.1\tau_h = 10^8$. This is checked [29,34,42] by comparison with the exact solution for the position variance obtained within both GLE and FLE [22,36] in the force-free case. The anomalous friction coefficient η_α and the memory kernel $\eta(t)$ are assumed to be temperature independent in accordance with the Hamiltonian model of the generalized Brownian motion [23,38], yielding a temperature-independent spectral bath density $J(\omega)$. This is a standard assumption done also in other toy ratchet models [8,9].

Stochastic dynamics is studied in a driven ratchet potential, $V(x,t) = U(x) - Ax\cos(\Omega t) + f_0 x$, where [7]

$$U(x) = -U_0[\sin(2\pi x/L) + 0.25\sin(4\pi x/L)] \tag{6}$$

is a spatially asymmetric periodic potential with amplitude U_0 and period L, A and Ω are the amplitude and frequency of the periodic forcing, and f_0 is a load. We scale time in units of the (anomalous) velocity relaxation constant $\tau_v = (m/\eta_\alpha)^{1/(2-\alpha)}$, distance in the units of L, energy in units of $\tilde{E} = mL^2/\tau_v^2$, force in $\tilde{F} = \tilde{E}/L$, and temperature in $\tilde{T} = \tilde{E}/k_B$. The role of the inertial effects can be characterized by the dimensionless parameter $r_v = 1/(\omega_b \tau_v)$, where $\omega_b = (2\pi/L)(3^{3/8}/2^{1/4})\sqrt{U_0/m}$ is the bottom and (imaginary) top frequency of the considered potential at $A = 0$, $f_0 = 0$. The inertial effects can only be negligible for $r_v \ll 1$ and not too small α [37]. The borderline value of U_0 corresponding to $r_v = 1$ is in the dimensionless units $U_0^* \approx 0.0157$. For the simulations done in this work the inertial effects are very essential. This might seem paradoxical since in the Markovian approximation $\dot{x}(t') \to \dot{x}(t)$ to Eq. (1) the effective Markovian friction $\eta_{\rm eff}(t) = \int_0^t \eta(t')dt' \propto \eta_\alpha t^{1-\alpha}$ increases to infinity in the course of time. Such a Markovian approximation is, however, not affordable for the considered viscoelastic dynamics. Figure 1 illustrates this point: A part of auxiliary particles is strongly coupled to the Brownian particle. They are comoving, being strongly correlated, similar to a polaron-like picture for quantum particles in polar media. However, particles that are more weakly coupled, more strongly damped, and much slower cannot follow immediately. They create an elastic force, pulling the central particle back and retarding its overall motion [35]. This prohibits any Markovian approximation on the level of the reduced (x,v) dynamics as the corresponding slow hidden dynamics cannot be adiabatically eliminated. Nevertheless, a highly dimensional Markovian approximation with N extra dimensions for overdamped auxiliary particles works remarkably well. Of course, given a finite N the truly asymptotical dynamics becomes normal for $t \gg \tau_h = b^{N-1}/\nu_0$. However, τ_h grows exponentially fast with N and therefore it can be totally irrelevant, as in our simulations, to figure out the correct asymptotic FLE dynamics. We take a physical limit of very large t to study subdiffusive dynamics with τ_h regarded as infinite on this time scale and define the subvelocity as

$$v_\alpha = \Gamma(1+\alpha)\lim_{t\to\infty}\frac{\langle x(t)\rangle}{t^\alpha}. \tag{7}$$

Practically, the corresponding values of v_α were calculated by fitting the dependence $\langle x(t)\rangle$ with the at^α dependence, extracting the corresponding prefactor a within a last time window of simulations done until $t_{\rm sim} = 10^6 \ll \tau_h$. In all simulations we have used $n = 10^4$ trajectories for ensemble averaging. The stochastic Heun algorithm has been implemented with double precision on a graphical processor unit (GPU) [43], which allowed us to parallelize and accelerate stochastic simulations by a factor of about 100 as compared with the standard computing. The driving strength was fixed at $A = 0.8$, whereas the temperature, potential amplitude, and the driving frequency varied.

III. RESULTS

First, we fixed the temperature at $T = 0.25$ and varied the driving frequency and the potential amplitude; see Fig. 2. Remarkably, the rectification response is absent in the adiabatic limit $\Omega \to 0$. A similar result was obtained earlier in Ref. [34], where the time scaling was different. This is in a sharp contrast with the case of normal diffusion where the adiabatic current is maximal [8,9]. Indeed, the subdiffusion and subtransport in periodic potentials turn out to be asymptotically insensitive to the potential amplitude U_0 [29–31]. This surprising feature is due to the influence of sluggish dynamics of the medium's degrees of freedom, which cannot immediately follow a faster moving Brownian particle. They cause ultraslow dynamics on the time scale which largely exceeds the mean time spent in a potential well. In this respect, the medium's dynamics is not influenced directly by an external force. Therefore, for an adiabatically slow driving a periodic potential does not play any role in the long time limit and the ratchet transport is absent. For a very fast driving, the transport is also obviously increasingly suppressed upon increasing the driving frequency. Therefore, there should exist an optimal value of driving frequency when the corresponding subvelocity

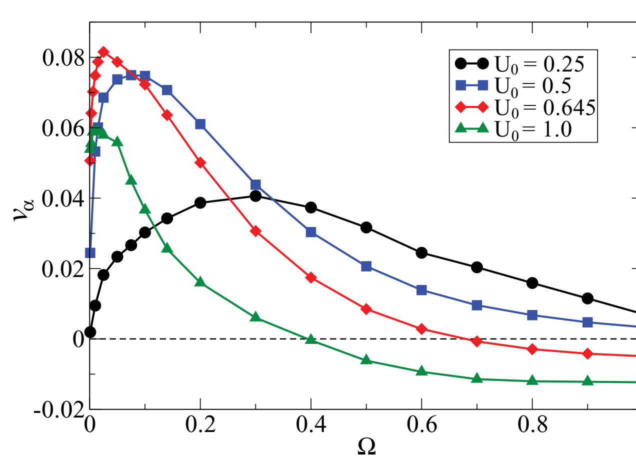

FIG. 2. (Color online) Anomalous current (subvelocity v_α) as a function of the driving frequency for different U_0 at $T = 0.25$, $A = 0.8$, for zero load $f_0 = 0$.

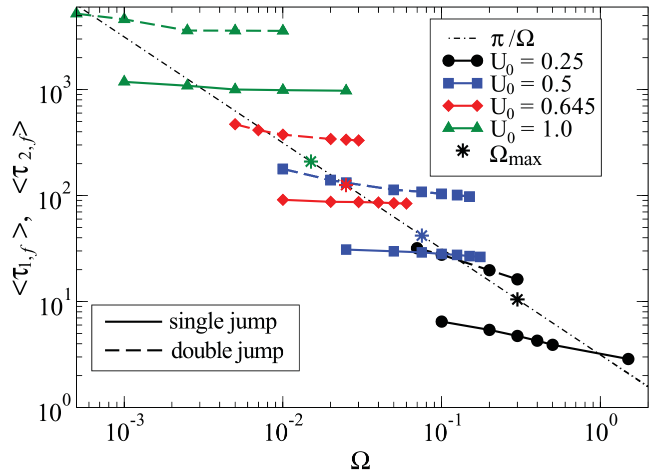

FIG. 3. (Color online) Mean times of transitions, $\langle\tau_{1,f}\rangle$ and $\langle\tau_{2,f}\rangle$, as functions of the driving frequency Ω for the same barrier heights, temperature, and driving amplitude as in Fig. 2. The values $\Omega_{\max}$ corresponding to the maxima of subvelocity in Fig. 2 are indicated by the symbol on the line (in the double logarithmic plot) $\mathcal{T}_{1/2} = \pi/\Omega$.

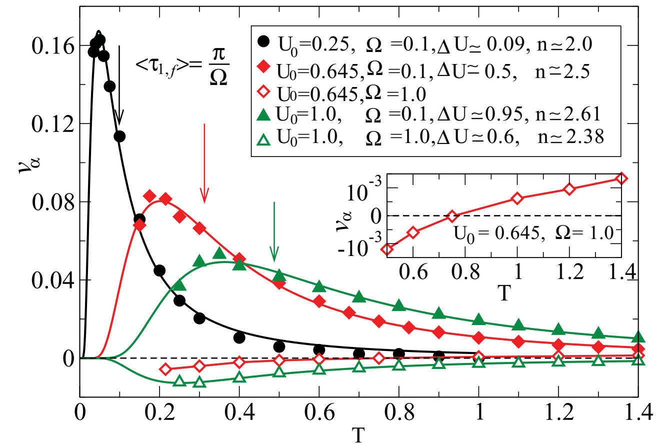

FIG. 4. (Color online) Anomalous current (subvelocity v_α) as a function of temperature for different values U_0 and Ω. Symbols correspond to numerical results and curves (except for one curve magnified in the insert, which displays the current inversion) to a typical SR dependence with fitting parameters ΔU and n as discussed in the text.

attains a maximum, $v_{\alpha,\max} = \max_\Omega v_\alpha(\Omega)$, which turns out to be a SR-related effect. This maximal value is optimized also with the potential amplitude. For example, $v_{\alpha,\max}$ is larger for $U_0 = 0.645$ than for $U_0 = 0.5$ and $U_0 = 1.0$; see Fig. 2.

In order to clarify the physical origin of maximal $v_{\alpha,\max}$ we have calculated the frequency-dependent mean times of the transitions to the neighboring potential well, $\langle\tau_{1,f}\rangle$, and to the second next potential well, $\langle\tau_{2,f}\rangle$, in the transport direction. If the maximum of v_α is due to a SR-related synchronization, then the transport should become optimal when the particles advance in the transport direction over one or two potential periods during the driving half-period $\mathcal{T}_{1/2} = \pi/\Omega$ [10]. This means that the condition $\langle\tau_{1,f}\rangle < \mathcal{T}_{1/2} < \langle\tau_{2,f}\rangle$ should be obeyed. Indeed, the numerical results displayed in Fig. 3 show that this is the case for not too high potential amplitude U_0. However, since $\langle\tau_{1,f}\rangle$ and $\langle\tau_{2,f}\rangle$ are only weakly frequency dependent, we are dealing with the phenomenon of SR rather than stochastic synchronization. Here, the response maximizes when an external driving frequency fits into an intrinsic frequency of stochastic dynamics. However, no frequency entrainment occurs.

Also noteworthy is the inversion of the current direction for sufficiently large U_0 and Ω. The subtransport then occurs in the counterintuitive direction, contrary to the direction predicted by the slow adiabatic tilt argumentation; see Fig. 2 for $U_0 = 0.645$ and $U_0 = 1.0$. It occurs only for a sufficiently fast driving with the period which becomes comparable with the time scale of velocity relaxation given by the corresponding anomalous relaxation constant $\tau_v = 1$. A similar inversion for high-frequency driving occurs also in the normal diffusion case [7,8]. However, in contrast to [7,8], where the dynamics is overdamped, the considered anomalous dynamics includes essential inertial effects.

A SR-related phenomenon occurs also in the dependence of subvelocity on temperature; see Fig. 4. An adiabatic driving argumentation predicts $v_\alpha(T) \propto T^{-n}\exp(-\Delta U/T)$ with $n = 2$ and $\Delta U = 2U_0$ in the lowest order of perturbation theory over A/T (quadratic response) for a weak driving, $AL \ll 2\pi U_0$. This is a typical SR dependence yielding the maximum versus temperature at $T_{\max} = \Delta U/n$. We are dealing with a strong nonadiabatic driving beyond this perturbative result and hence the validity of this approximation is not guaranteed. As a first educated-guess correction, one can assume that strong nonadiabatic driving somehow renormalizes the parameters entering this SR dependence. Therefore, we use ΔU and n in it and in Fig. 4 as some fitting parameters to describe the numerical results. The SR origin of the maxima for sufficiently small Ω is substantiated by a correspondence between the forward tilt half-periods and the numerical mean times $\langle\tau_{1,f}\rangle$ of the jump durations in the forward directions (i.e., $\langle\tau_{1,f}\rangle = \pi/\Omega$ [10]) at a temperature near to the transport optimization as indicated by arrows in Fig. 4. Even though the optimal transport does not correspond precisely to this matching condition, nevertheless the tendency is obvious. Therefore, we are dealing here with a genuine SR phenomenon.

Furthermore, an inversion of the subcurrent direction with temperature is detected in Fig. 4 (see also the insert therein) for $U_0 = 0.645$ and a high-frequency driving $\Omega = 1.0$. Similar temperature inversions occur also in the normal diffusion case [7]. Moreover, the negative current for a larger Ω is also optimized with temperature. This does not have any relation to a true SR or synchronization, as the driving here is very fast. Similar situations are commonly described in the literature as SR in a broad sense [10].

Not only anomalous transport but also its dispersion is of profound interest. In particular, a small dispersion means intuitively a high quality of transport. Viscoelastic subdiffusion exhibits highly surprising properties in periodic potentials. It turns out that such subdiffusion is asymptotically not affected by the presence of periodic potential even though the time course of the transient to this asymptotical regime does strongly depend on the potential amplitude and can last a very long time [29–31]. Asymptotically, in the considered units, $\langle\delta x^2(t)\rangle \sim 2D_\alpha t^\alpha/\Gamma(1+\alpha)$ with subdiffusion

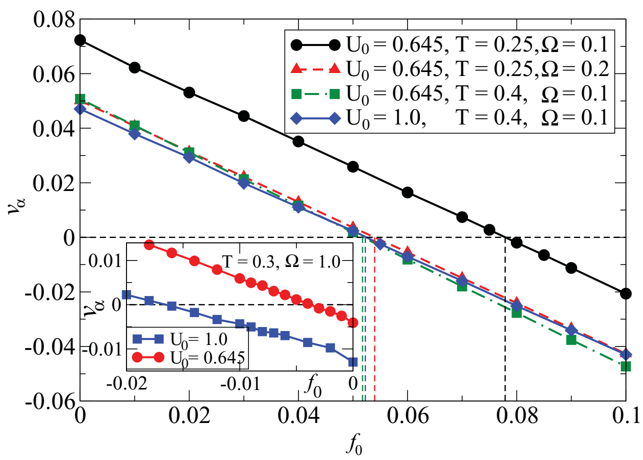

FIG. 5. (Color online) Anomalous current (subvelocity v_α) as a function of positive load f_0 at different U_0, T, and a small driving frequency $\Omega = 0.1$. Insert shows dependencies v_α vs negative load f_0 at $T = 0.3$ and some high $\Omega = 1.0$ at different U_0.

coefficient $D_\alpha = T$. A periodic driving can generally affect D_α. However, it is not changed strongly [31,34,35]. Therefore, the generalized Peclet number [34,35] $\mathrm{Pe}_\alpha := v_\alpha L/D_\alpha$, which measures the coherence of transport [44], can be appreciably large. Namely, with lowering the barrier height in Fig. 4 the maximum of subvelocity is shifted towards smaller temperatures and the diffusion coefficient becomes smaller. Therefore, the corresponding Pe_α can substantially exceed the value of one, signifying thereby that a highly coherent (low dispersive) subdiffusive ratchet transport is possible. This property of viscoelastic subdiffusive transport stays in sharp contrast to the alternative highly dispersive CTRW mechanism, where $\mathrm{Pe}_\alpha = 0$ (e.g., for a statically tilted periodic potential [31]) and the very existence of a similar ratchet transport is questionable overall.

Finally, the dependence of the directed subtransport on the load f_0 in the opposite direction is shown in Fig. 5. The existence of a stopping force shows clearly that we are dealing with a genuine ratchet effect. Given the asymptotic independence of the viscoelastic subtransport on the potential amplitude U_0 in static washboard potentials [29], one expects a very simple dependence $v_\alpha(f_0) = v_\alpha(0) - f_0$ (in dimensionless units) to hold for small driving frequencies Ω. Indeed, the numerical results are consistent (taking numerical errors into account) with this prediction. However, for larger Ω and the inverted transport, the deviations become more pronounced.

IV. CONCLUSION

In conclusion, in this work we elaborated on a toy yet fundamental model of viscoelastic subdiffusive ratchet transport, which is related to a fractional stochastic Langevin dynamics in driven periodic potentials, within a highly efficient Markovian embedding approach. We showed that such a transport is not only possible beyond the adiabatic driving regime but also displays a number of surprising properties which cannot be even expected within an alternative CTRW mechanism based on divergent mean residence times. In particular, the viscoelastic subdiffusive ratchet transport can be optimized by ambient thermal noise and/or frequency of the external driving due to a genuine SR effect. The very occurrence of SR in such a profoundly non-Markovian dynamics with long-lasting memory effects presents a highly nontrivial result. This kind of non-Markovian SR differs from the one described earlier in Ref. [45]. Furthermore, the subdiffusive transport can exhibit a surprisingly good quality due to a small dispersion (linearly diminishing subdiffusion coefficient) at low temperatures and small barrier heights. We also demonstrated that this is a genuine ratchet effect and our subdiffusive Brownian motors can sustain a substantial load and do a useful work. A number of questions, in particular one on the thermodynamical efficiency of such anomalous isothermal engines, remain open. They will be addressed in subsequent studies.

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