Persistence of a Brownian particle in a time-dependent potential

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We investigate the persistence probability of a Brownian particle in a harmonic potential, which decays to zero at long times, leading to an unbounded motion of the Brownian particle. We consider two functional forms for the decay of the confinement, an exponential decay and an algebraic decay. Analytical calculations and numerical simulations show that for the case of the exponential relaxation, the dynamics of Brownian particle at short and long times are independent of the parameters of the relaxation. On the contrary, for the algebraic decay of the confinement, the dynamics at long times is determined by the exponent of the decay. Finally, using the two-time correlation function for the position of the Brownian particle, we construct the persistence probability for the Brownian walker in such a scenario.

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I. INTRODUCTION

The phenomenon of persistence has been of continuing interest over the past decade. Persistence is quantified through the persistence probability $p(t)$ that a stochastic variable has not changed its sign over a time t. In a wide class of nonequilibrium systems this probability decays algebraically with an exponent θ, and the exponent has been studied in systems that include free random walk in homogeneous [1,2] and disordered media [3], critical dynamics [4], surface growth [5–11], polymer dynamics [12], diffusive processes with random initial conditions [13–15], advected diffusive processes [16], and finance [17,18]. A precise theoretical prediction for $p(t)$ can be worked out only for a select few cases [19], the simplest scenario being an exponentially decaying stationary correlator, as in the case of an overdamped Brownian motion. In general, for most Gaussian stochastic processes the decay of the stationary correlator, $C(T) \equiv \langle \overline{X}(T)\overline{X}(0)\rangle$, is nonexponential. The behavior of $C(T)$ in the neighborhood of zero characterizes the density of zero crossings for the underlying stochastic process [13,19]. When $C(T)$ near zero has a quadratic dependence on time in the first order, the number of zero crossings of the stochastic process is finite, and the exponent θ is extracted using the independent interval approximation (IIA) [13] or the sign time distribution of the stochastic variable [15]. Conversely, when $C(T) \sim 1 - O(T^\alpha)$, with $\alpha < 2$, the density of zero crossings is infinite, and perturbation expansion about a random walk correlator gives a good estimate of the persistence exponent [6].

The simplest of all these systems, which exhibit an algebraic decay of $p(t)$ with an exponent $1/2$, is the case of an overdamped Brownian particle. Occurring in the interface of science and engineering, Brownian motion is ubiquitous around us and plays a dominant role in the nanoscopic and mesoscopic world. Not only is the underlying principle of this stochastic process used for theoretical modeling of a wide range of complex phenomena [20], but Brownian motion in itself serves as an experimental tool for probing microscopic environments [20–23]. In the popular Langevin picture, the erratic motion of a Brownian particle is well described by Newton's equation of motion with a viscous drag and a δ-correlated stochastic force acting on the particle. While the non-Markovian nature of the phenomenon can be taken into consideration by using a generalized Langevin equation with a finite correlation time for the stochastic noise and a memory-dependent friction, in the following discussion we shall restrict ourselves to the Markovian scenario.

In this article, we investigate the persistence probability of a Brownian particle in a time-dependent potential, a scenario corresponding to the trapping of a tracer particle in some potential which eventually relaxes to zero. To keep the following discussions at an analytically tractable level, we choose a harmonic potential, given by $U(x,t) = \frac{1}{2} f(t)x^2$. The function $f(t)$ can be viewed as a time-dependent spring constant, with $f(t) \to 0$ as $t \to \infty$, so that the particle motion becomes unbounded in the long-time limit. The converse situation of a constant confinement strength has already been studied in Refs. [19,24,25]. In the Fokker-Planck description, the calculation of the persistence probability translates to solving the backward Fokker-Planck equation, with an absorbing wall at $x = 0$. An alternative approach to determine the survival probability, as outlined in Ref. [1,19,24], is from the two-time correlation function for the position of the stochastic variable x, exploiting the fact that for a Gaussian stationary process with a correlator decaying exponentially at all times, the persistence probability also decays exponentially.

The rest of this article is organized as follows: we introduce the dynamical equations of motion and construct the two-time correlation functions in Sec. II. A discussion on the mean-square displacement of the Brownian walker and the relevant time scales due to the time-dependent trap is also presented in Sec. II. We study two types of relaxation phenomena: an exponential and an algebraic relaxation of the confinement, discussed in Secs. II A and II B, respectively. Finally, the persistence probability is discussed in Sec. III.

II. BROWNIAN PARTICLE IN A TIME-DEPENDENT POTENTIAL

For simplicity, we take the overdamped limit for which the dynamics of a Brownian particle, with a unit mass, is governed

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by

$$\dot{x} = -f(t)x(t) + \eta(t), \quad (1)$$

where $\eta(t)$ is the stochastic velocity characterizing the solvent. The above equation is further supplemented by the moments of the stochastic noise,

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = 2D\delta(t-t'). \quad (2)$$

At this point, we assume that the stochastic noise is "internal," characterized by the viscosity and the temperature of the solvent, while the time-dependent confinement is "external" and does not change the δ correlation. An experimental realization of the model system would correspond to a laser trapping of a tracer, with the intensity of a laser decaying in time. In such a scenario, the transport parameters and the temperature T get renormalized [26,27]. The corresponding solution to Eq. (1) is given by

$$x(t) = e^{-\int_0^t f(t')dt'} \int_0^t dt_1 \eta(t_1) e^{\int_0^{t_1} f(t_1')dt_1'}, \quad (3)$$

with the initial condition $x(0) = 0$. Denoting $g(t) = \int_0^t f(t')dt'$, the two-time correlation function can be constructed from Eq. (3),

$$\langle x(t_1)x(t_2) \rangle = e^{-g(t_1)} e^{-g(t_2)} \int_0^{t_1} dt_1' \int_0^{t_2} dt_2' \langle \eta(t_1')\eta(t_2') \rangle e^{g(t_1')} e^{g(t_2')}. \quad (4)$$

A. Exponential relaxation

We first consider the case when the relaxation of the potential is given by an exponential decay, $f(t) = \lambda e^{-t/\tau}$. There are two time scales in the system, τ and λ^{-1}. The latter determines the time scale when the Brownian particle is confined in the potential, whereas the former determines the relaxation of the potential [Fig. 1(a)]. We further consider the situation when $\tau > \lambda^{-1}$; the relaxation time scale of the potential is larger than the entrapment time scale λ^{-1}. When $\tau < \lambda^{-1}$, the confinement decays even before the particle can be trapped, with the result that the Brownian particle undergoes free diffusion.

Consequently, the function $g(t)$ takes the form $g(t) = \lambda\tau(1 - e^{-t/\tau})$. Substituting for $g(t)$ in Eq. (4) and subsequently using (2), we arrive at

$$\langle x(t_1)x(t_2) \rangle = 2D e^{\lambda\tau e^{-t_1/\tau}} e^{\lambda\tau e^{-t_2/\tau}} \int_0^{t_2} dt_2' \, e^{-2\lambda\tau e^{-t_2'/\tau}}, \quad (5)$$

with the assumption that $t_1 > t_2$. Performing the integral over t_2' in Eq. (5), the two-time correlation function becomes

$$\langle x(t_1)x(t_2) \rangle = 2D\tau \, e^{\lambda\tau e^{-t_1/\tau}} e^{\lambda\tau e^{-t_2/\tau}} \times [\mathrm{Ei}(-2\lambda\tau) - \mathrm{Ei}(-2\lambda\tau e^{-t_2/\tau})], \quad (6)$$

where $\mathrm{Ei}(x)$ is the exponential integral, defined as

$$\mathrm{Ei}(t) = -\int_{-t}^{\infty} z^{-1} e^{-z} \, dz. \quad (7)$$

Using Eq. (6), the mean-square displacement $\langle x^2(t) \rangle$ reads

$$\langle x^2(t) \rangle = 2D\tau \, e^{2\lambda\tau e^{-t/\tau}} [\mathrm{Ei}(-2\lambda\tau) - \mathrm{Ei}(-2\lambda\tau e^{-t/\tau})]. \quad (8)$$

At this point, it is instructive to construct the limiting behaviors of the mean-square displacement: when $t < \lambda^{-1}$ and $t > \tau$. There are two scenarios we consider below, the first when τ is large so that the limit $\tau \to \infty$ is appropriate and the second when τ remains finite. In the limit of $\tau \to \infty$, the relaxation of the potential is slow, and the Brownian particle feels a constant confinement strength λ, and (8) reduces to

$$\langle x^2(t) \rangle = \frac{D}{\lambda}(1 - e^{-2t\lambda}) + O(\tau^{-1}). \quad (9)$$

To construct the corresponding two-time correlation function, we expand the exponentials in Eq. (5) and keep the terms which are independent of τ. The evaluation of the integral over t_2' then gives

$$\langle x(t_1)x(t_2) \rangle = \frac{D}{\lambda}[e^{-\lambda(t_1-t_2)} - e^{-\lambda(t_1+t_2)}], \quad (10)$$

which is exactly the correlation function for a nonstationary Ornstein-Uhlenbeck process [25]. Eventually, for $t \gg \tau$ the particle motion becomes unbounded and the mean-square displacement grows linearly with time. A formal quantitative result in this limit can be derived if we take the limit of $t \to \infty$ in Eq. (8) and expand the term within the brackets to get

$$\langle x^2(t) \rangle = 2D\tau \, e^{2\lambda\tau e^{-t/\tau}} \Big[-\gamma + 2\lambda\tau e^{-t/\tau} - \lambda^2\tau^2 e^{-2t/\tau} + \mathrm{Ei}(-2\lambda\tau) + \frac{t}{\tau} + \frac{1}{2}\ln(1/2\lambda^2\tau) \Big],$$

where γ is Euler's constant with a numerical value of $\sim$0.5772. Keeping in mind that $t \gg \tau$, the exponential functions in Eq. (11) can be ignored in comparison to the linearly growing term which survives, so that we recover the classic diffusion of the Brownian particle with $\langle x^2(t) \rangle = 2Dt$. In the opposite limit of $t \to 0$, a Taylor expansion of Eq. (8) yields

$$\langle x^2(t) \rangle = 2Dt + O(t^2). \quad (12)$$

The two limiting behaviors in Eqs. (11) and (12) are completely independent of the time scales and therefore do not contain any information about the confinement potential. On the contrary, when the relaxation is slow ($\tau \to \infty$), only the short-time dynamics is independent of λ or τ. The asymptotic mean-square displacement, in the limit of a slow relaxation, is constant in time and is determined by the ratio D/λ.

B. Algebraic relaxation

We now consider our second choice for the relaxation dynamics of the harmonic potential, an algebraic decay of the time-dependent spring constant,

$$U(x,t) = \frac{\lambda}{2}\left(\frac{\tau}{t}\right)^{\alpha} x^2, \quad (13)$$

with $\alpha \leqslant 1$. Using (4), the two-time correlation function is

$$\langle x(t_1)x(t_2) \rangle = \frac{2D}{1-\alpha} e^{-\lambda_1 t_1^{1-\alpha}} e^{-\lambda_1 t_2^{1-\alpha}} \int_0^{t_2^{1-\alpha}} du \, u^{\alpha/(1-\alpha)} e^{2\lambda_1 u}, \quad (14)$$

where $\lambda_1 = \lambda\tau^\alpha/(1-\alpha)$. The integral in Eq. (14) over u yields

$$\langle x(t_1)x(t_2)\rangle = \frac{2D}{1-\alpha}\, e^{-\lambda_1 t_1^{1-\alpha}} e^{-\lambda_1 t_2^{1-\alpha}} (-2\lambda_1)^{-1/(1-\alpha)} \gamma\left(\frac{1}{1-\alpha}, -2\lambda_1 t_2^{1-\alpha}\right), \quad (15)$$

where γ is the lower incomplete γ function, defined as

$$\gamma(a,z) = \int_0^z e^{-u} u^{a-1} du, \quad (16)$$

and $\Gamma(a,z)$ is the upper incomplete Γ function satisfying $\Gamma(a,z) + \gamma(a,z) = \Gamma(a)$. The numerical value of $\gamma(a,z)$ can be evaluated using Gauss's continued fraction, which converges for all values of z.

Substituting $t_1 = t_2 = t$, the mean-square displacement is given by

$$\langle x^2(t)\rangle = \frac{2D}{1-\alpha}\, e^{-2\lambda_1 t^{1-\alpha}} (-2\lambda_1)^{-1/(1-\alpha)} \gamma\left(\frac{1}{1-\alpha}, -2\lambda_1 t^{1-\alpha}\right) \quad (17)$$

Unlike the exponential relaxation of the potential, there is a single crossover time scale which emerges from Eq. (17),

$$\overline{\tau} = \left(\frac{1-\alpha}{\lambda\tau^\alpha}\right)^{1/(1-\alpha)}, \quad (18)$$

and it separates the regimes of normal diffusion and subdiffusion in the system [Fig. 1(b)]. A Taylor expansion of Eq. (17) for $t < \overline{\tau}$ gives

$$\langle x^2(t)\rangle = 2Dt + O(t^{2-\alpha}) \quad \text{for} \quad t < \overline{\tau}, \quad (19)$$

while the asymptotic expansion yields

$$\langle x^2(t)\rangle = \left(\frac{D}{\lambda\tau^\alpha}\right) t^\alpha + O(t^{-(1-2\alpha)}) \quad \text{for} \quad t > \overline{\tau}. \quad (20)$$

This counterintuitive result can be understood by considering the motion of a free Brownian particle. In the absence of the confinement potential the Brownian particle moves a distance $\sqrt{t}$ in time t. If we now switch on the potential, the strength of the potential becomes $\lambda(\tau/t)^\alpha t$, and for $\alpha < 1$ we see that the particle feels the "soft" walls all the time. Mathematically, this argument translates to the fact that the new time scale $\overline{\tau}$ in Eq. (18) diverges as $\alpha \to 1$ and is not defined in the real line for $\alpha > 1$. We note that for $\alpha = 0$, Eq. (17) reduces to that of the Ornstein-Uhlenbeck process,

$$\langle x^2(t)\rangle = \frac{D}{\lambda}[1 - e^{-2\lambda t}]. \quad (21)$$

In Fig. 1, we show the mean-square displacement of a Brownian particle whose dynamics is governed by Eq. (1) and (2), with $f(t)$ given by an exponential [Fig. 1(a)] and an algebraic [Fig. 1(b)] relaxation. The numerical integration of Eq. (1) was done using the Euler scheme with an integration time step of $dt = 0.001$. In the numerical solutions, the value of the diffusion coefficient D was taken as unity. For the exponential relaxation of the confinement, the measured mean-square displacement shows three distinct regimes: two diffusive regimes with a crossover in between. For very short ($t < \lambda^{-1}$) and long ($t > \tau$) times, the particle does not feel the trap, and its motion is purely diffusive, corresponding to Eqs. (11) and (12). In the intermediate times, we observe a plateau for $\lambda^{-1} < t < \tau$, corresponding to the trapping of the particle in the potential. To understand the origin of this plateau, we expand the exponential in Eq. (1), and retaining the zeroth order term then leads to a constant confinement, so that the mean-square displacement saturates to a value $\propto \lambda^{-1}$. This behavior can be observed in Fig. 1(a), which presents data for constant λ but different τ; the inset shows data for a constant τ but different λ. A comparison shows that the plateau is determined by λ^{-1}. On the contrary, for the algebraic relaxation, since a single time scale emerges from the dynamics, we observe only one crossover regime determined by $\overline{\tau}$, which separates the diffusive and the subdiffusive regimes [Fig. 1(b)].

III. PERSISTENCE PROBABILITY

To obtain the persistence probability, we take the route outlined in Refs. [1,19]: we map the nonstationary process $x(t)$ to a stationary Ornstein-Uhlenbeck process. This is usually achieved first by a normalization of $x(t)$ by $\sqrt{\langle x^2(t)\rangle}$, the root-mean-square distance the particle has traveled, and then using a suitable transformation in time. Once we have the stationary process $\overline{X}$, with the correlator $C(T)$, the persistence problem reduces to a calculation of no zero crossing of $\overline{X}$. When $C(T)$ is a purely exponential decay for all times, the persistence probability is the solution to the backward Fokker-Planck equation for an Ornstein-Uhlenbeck process, which can be shown to decay as $P(T) = \frac{2}{\pi}\sin^{-1}[C(T)]$ [1,24]. An application of this method therefore requires the transformation of the stochastic process $x(t)$ to a Gaussian stationary process. Since the correlation function in Eqs. (6) and (15) is nonstationary, we make the following transformations: we first define the normalized variable $\overline{X}(t) \equiv \frac{x(t)}{\sqrt{\langle x^2(t)\rangle}}$ and construct the two-time correlation function, following which we make a suitable transformation in time to make the correlator stationary, as well as an exponentially decaying function for all times.

Before we proceed to give a derivation of the persistence probability for the two models introduced above, we derive a general result applicable to the model system in Eq. (1). To transform the nonstationary process in Eq. (1), we consider the transformations $\overline{X} = x(t)/l(t)$ and $e^T = l^2(t)e^{2g(t)}$, where $l^2(t) = \langle x^2(t)\rangle$ and $g(t) = \int f(t')dt'$. Substituting these transformations in Eq. (1), we obtain a stationary Ornstein-Uhlenbeck process,

$$\frac{d\overline{X}}{dT} = -\frac{1}{2}\overline{X} + \overline{\eta}(T), \quad (22)$$

where $\overline{\eta}(T)$ is a Gaussian white noise with zero mean and unit variance. The relation between $\overline{\eta}$ and η can be determined from the transformation of the δ function and takes the form $\overline{\eta}(T) = \frac{l(t)}{2D_0}\eta(t)$. The stationary correlator for the process in Eq. (22) is then given by $C(T) = e^{-T/2}$. Accordingly, the persistence probability in real time decays as $p(t) \sim e^{-g(t)}/l(t)$. In the following, we illustrate this explicitly for the two specific cases presented in Secs. II A and II B.

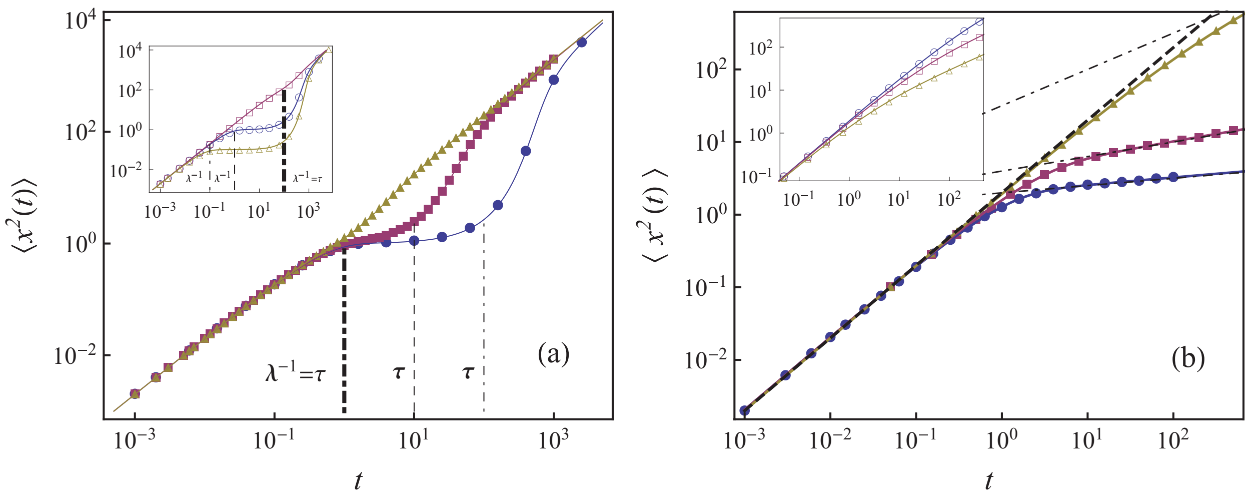

FIG. 1. (Color online) (a) Double logarithmic plot of the mean-square displacement $\langle x^2(t)\rangle$ for the exponential relaxation of the confinement with $\lambda = 1$ and $\tau = 1.0$ (triangles), 10.0 (squares), and 100.0 (circles). The solid lines are the plots of Eq. (8) for the corresponding values of τ and λ. The thick dot-dashed line in the main plot corresponds to $\lambda^{-1} = \tau = 1$, while the dashed and the thin dot-dashed lines corresponds to $\tau = 10$ and $\tau = 100$, respectively. The inset shows the variation of the mean-square displacement for different values of λ with $\tau = 100$ fixed and $\lambda = 0.01$ (squares), 1.0 (circles), and 10.0 (triangles). The solid lines are a plot of Eq. (8) for the corresponding values of λ and τ. The thick dot-dashed line corresponds to $\lambda^{-1} = \tau = 100$, while the dashed lines denote the values of $\lambda^{-1} = 0.1$ and $\lambda^{-1} = 1$. (b) Plot of mean-square displacement for $\lambda = 1.0, \tau = 0.001$, and $\alpha = 0.1$ (circles), 0.2 (squares), and 0.5 (triangles) and the corresponding plots of Eq. (17) for the three values of α. The thick dashed black line is the plot of $2Dt$, and the thin dot-dashed black lines are the plots of Eq. (20) for the corresponding values of α. The inset is a plot of the mean-square displacement for $\lambda = 1.0, \alpha = 0.5$, with $\tau = 0.001$ (circles), 0.01 (squares), and 0.1 (triangles).

A. Exponential relaxation

The two-time correlation function for $\overline{X}$ reads

$$\langle \overline{X}(t_1)\overline{X}(t_2)\rangle = \frac{\langle x(t_1)x(t_2)\rangle}{\sqrt{\langle x^2(t_1)\rangle\langle x^2(t_2)\rangle}}. \tag{23}$$

Using Eqs. (6) and (8) in the above equation, we have

$$\langle \overline{X}(t_1)\overline{X}(t_2)\rangle = \sqrt{\frac{\mathrm{Ei}(-2\lambda\tau) - \mathrm{Ei}(-2\lambda\tau e^{-t_2/\tau})}{\mathrm{Ei}(-2\lambda\tau) - \mathrm{Ei}(-2\lambda\tau e^{-t_1/\tau})}}. \tag{24}$$

The time transformation $e^T = l^2(t)e^{2g(t)}$ reads

$$e^T = \mathrm{Ei}(-2\lambda\tau) - \mathrm{Ei}(-2\lambda\tau e^{-t/\tau}), \tag{25}$$

which transforms (24) to

$$\langle \overline{X}(T_1)\overline{X}(T_2)\rangle = e^{-\frac{1}{2}(T_1 - T_2)}. \tag{26}$$

The correlator for the stochastic process $\overline{X}$ is stationary and exponentially decaying. The asymptotic behavior of the persistence probability for such a process is then given by $P(T) \sim e^{-T/2}$ [19]. Transforming back to real time, the persistence probability for the process $x(t)$ is then given by

$$p(t) \sim [\mathrm{Ei}(-2\lambda\tau) - \mathrm{Ei}(-2\lambda\tau e^{-t/\tau})]^{-1/2}. \tag{27}$$

For $t \ll \lambda$ and $t \gg \tau$, a Taylor expansion and an asymptotic expansion of the above equation give $p(t) \sim t^{-1/2}$. Finally, in the limit of $\tau \to \infty$, the persistence probability reads

$$p(t) \sim [(1 - e^{-2\lambda t})e^{2\lambda t} + O(\tau^{-2})]^{-1/2}, \tag{28}$$

which is identical to the result of Ref. [25]. To determine the persistence probability of the Brownian particle using a numerical integration, we chose an ensemble of random initial conditions in the neighborhood of zero [so that the sign of $x(0)$ is well defined] and followed the sign change of the position. The fraction of particles which did not change the sign of the coordinates in time t gives an estimate of the persistence probability. The results presented in Eq. (27) (the colored lines in Fig. 2) and (28) (the black dashed line in Fig. 2)

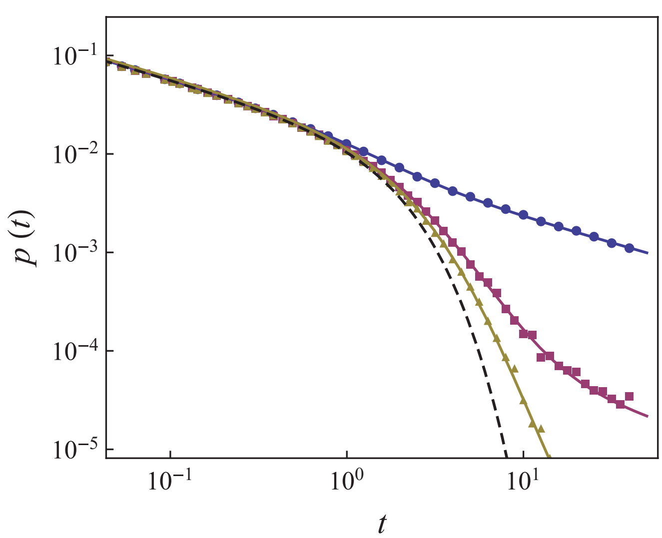

FIG. 2. (Color online) Double logarithmic plot of persistence probability where $f(t)$ decays exponentially for $\lambda = 1.0$ and $\tau = 1.0$ (circles), 5.0 (squares), and 10.0 (triangles). The solid and the dashed lines are the plots of Eq. (27) for the corresponding values of τ, and dashed line is the plot of Eq. (28).

are compared with the measured persistence probability (the solid symbols), using the numerical simulation of Eq. (1), in Fig. 2. For short and long times, the persistence probability $p(t) \sim t^{-1/2}$, a signature of purely diffusive motion presented in Eqs. (11) and (12).

B. Algebraic relaxation

To determine the survival probability, we proceed in a similar way and construct the two-time correlation function for the normalized variable $\overline{X}$.

$$\langle \overline{X}(t_1)\overline{X}(t_2)\rangle = \sqrt{\frac{h(t_2)}{h(t_1)}}, \tag{29}$$

where the function $h(t)$ is the bracketed term in Eq. (17),

$$h(t) = (-2\lambda_1)^{1/(1-\alpha)}\ \gamma[1/(1-\alpha), -2\lambda_1 t^{1-\alpha}].$$

Defining the time transformation $e^T \equiv l^2(t)e^{2g(t)} = h(t)$, the nonstationary correlator in Eq. (15) is transformed into a Gaussian stationary correlator which decays exponentially. Following [19], the persistence probability in real time decays as

$$p(t) \sim \{(-2\lambda_1)^{-1/(1-\alpha)}\ \gamma[1/(1-\alpha), -2\lambda_1 t^{1-\alpha}]\}^{-1/2}. \tag{30}$$

We next consider the limiting behaviors of the persistence probability given in Eq. (30). Substituting $\alpha = 0$ in Eq. (30), the probability reduces to the case of a harmonically confined Brownian particle with constant confinement strength [25],

$$p(t) \sim [e^{2\lambda t}(1 - e^{-2\lambda t})]^{-1/2}. \tag{31}$$

For a finite value of $\alpha < 1$, when $t < \overline{\tau}$, the persistence probability decays as $p(t) \sim t^{-1/2}$, while an asymptotic expansion of Eq. (30) gives

$$p(t) \sim \frac{1}{t^{\alpha/2}} e^{-(t/\tau)^{1-\alpha}}. \tag{32}$$

In Fig. 3, we compare the results of Eqs. (30)–(32) with the measured persistence probability from the numerical integration of Eq. (1). The colored lines in Fig. 3 correspond to Eq. (30), while the black dashed lines are plots of Eq. (32). At short times, the motion is purely diffusive, and therefore we observe a $t^{-1/2}$ decay of $p(t)$ (the solid line in the inset in Fig. 3).

We note that even though the mean-square displacement for $t \gg \tau$ is similar to that of fractional Brownian motion, the decay of the persistence probabilities in the two scenarios is entirely different. For a particle which performs a fractional Brownian motion, the corresponding steady state persistence probability decays purely algebraically with an exponent $1 - \alpha/2$ [6].

C. Effect of inertia

Finally, before concluding, we remark upon the divergence of $p(t)$ as $t \to 0$. This singularity is entirely the artifact of coarse graining in Eq. (1), where we have neglected the inertia term. Strictly speaking, at this level of coarse graining, we are not allowed to take the $t \to 0$ limit since the inertia of the particle plays an important role at such short times. The

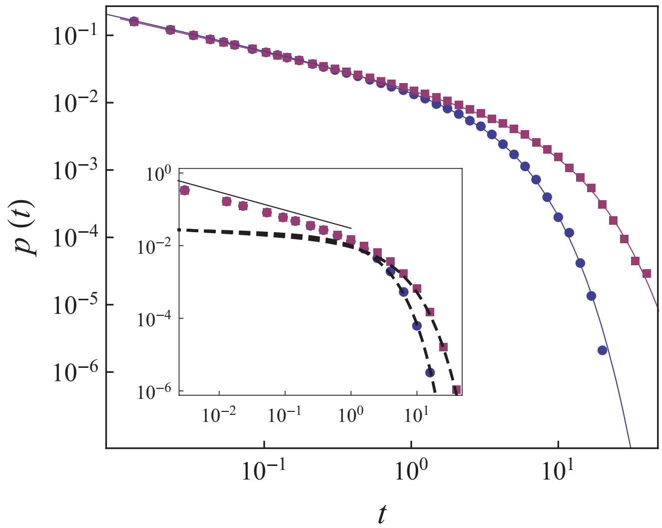

FIG. 3. (Color online) Plot of persistence probability for $\lambda = 1.0, \tau = 0.001$, and $\alpha = 0.1$ (circles) and 0.2 (squares). The solid lines in the main plot are plots of Eq. (30) for the corresponding values of α. The inset is a plot of the persistence probability for $\lambda = 1, \tau = 0.01$, with $\alpha = 0.1$ (circles) and 0.2 (squares). The solid black line is the plot of $t^{-1/2}$, and the dashed lines are the plots of the asymptotic expansion (32).

inclusion of the inertia term changes the short time dynamics of the particle to a deterministic one, as opposed to purely diffusive motion observed in the overdamped limit. Since the motion is now deterministic, the particle is persistently driven away from its initial position (the velocities remain strongly correlated), with the effect that the survival probability becomes constant. The purpose of this section is to demonstrate the fact that the inertia term removes the short-time singularity in the persistence probability. It is also motivated by the recent experimental observation of the ballistic regime of a Brownian particle and an expression for $p(t)$ in this regime would therefore be appropriate. An accurate analysis would correspond to solving Eq. (1) with the inertia term included, but it becomes difficult to extract any information from the resulting expressions. However, since we are looking at a time much shorter than λ^{-1}, exclusion of the confinement is justified as the particle does not feel the confinement at such small times.

To this end, we consider the complete Langevin equation for the momentum of a particle without any potential confinement,

$$\dot{p} = -\frac{\gamma}{m}p + \eta, \tag{33}$$

together with $\langle \eta(t)\rangle = 0$ and $\langle \eta(t)\eta(t')\rangle = 2k_B T\gamma\delta(t - t')$. Since, at short times, the dynamics of a Brownian particle in the model systems presented above is purely diffusive, it suffices to consider the Langevin equation for a free particle for our present discussion.

The two-time correlation function for the velocities decays exponentially as $\langle v(t_1)v(t_2)\rangle = \frac{k_B T}{m} e^{-|t_1-t_2|/\tau_0}$ [with an initial

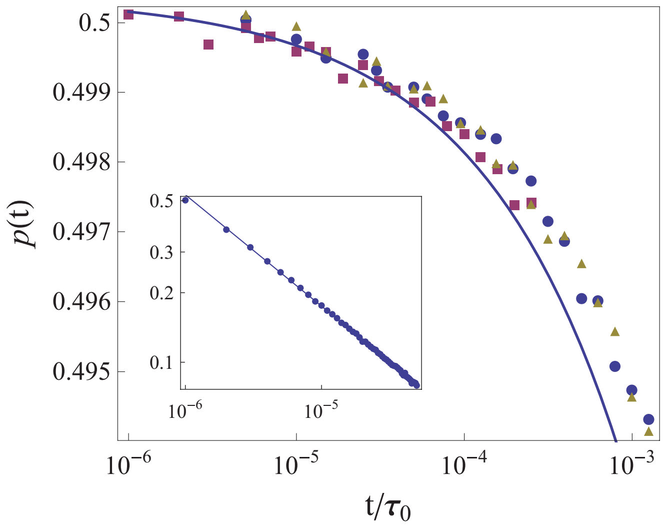

FIG. 4. (Color online) A double logarithmic plot of persistence probability of a Brownian walker with the inertia term included in the Langevin equation for the ratio of $\gamma/m = 0.2$ (squares), 1.0 (circles), and 5.0 (triangles). The solid line is a plot of the approximate result derived in Eq. (37). The inset shows a double logarithmic plot of the persistence probability of a Brownian particle in the overdamped limit. The solid line is power law fit to the data $t^{-\theta}$, with $\theta = 0.482422$. The analytical prediction for θ in this limit is 1/2.

condition $\langle v^2(0)\rangle = \frac{k_B T}{m}$] and the position correlation reads,

$$\langle x(t_1)x(t_2)\rangle = \frac{k_B T}{m}\left[\frac{2m}{\gamma}t_2 - \frac{m^2}{\gamma^2} + \frac{m^2}{\gamma^2}\left(e^{-t_1/\tau_0} + e^{-t_2/\tau_0} - e^{-(t_1-t_2)/\tau_0}\right)\right], \quad (34)$$

with $\tau_0 = m/\gamma$ and the assumption that $t_1 > t_2$. While the transformation of the correlator in Eq. (34) to a stationary process is nontrivial, we can still extract some information about the persistence probability in the limit of $t \to 0$. Keeping in mind this limit and that $t_1 > t_2$, a Taylor expansion of Eq. (34) yields

$$\langle x(t_1)x(t_2)\rangle = \frac{t_1 t_2}{m}\left(1 - \frac{1}{2}\frac{t_1}{\tau_0}\right). \quad (35)$$

The two-time correlation function $\langle \overline{X}(t_1)\overline{X}(t_2)\rangle$ reads

$$\langle \overline{X}(t_1)\overline{X}(t_2) = \sqrt{\frac{1 - t_1/2\tau_0}{1 - t_2/2\tau_0}}. \quad (36)$$

The transformation $e^T = (1 - t/2\tau_0)^{-1}$ transforms (36) into a stationary process with an exponentially decaying correlation function. The persistence probability $p(t)$ in real time the translates to

$$p(t) \sim \frac{2}{\pi}\sin^{-1}[\sqrt{1 - t/2\tau_0}] \quad (37)$$

The numerical integration of Eq. (33) was done with an implicit integration scheme based on the leapfrog algorithm for different values of the ratio γ/m and the value of T set to unity. The results of the simulations are presented in Fig. 4 and are compared to the approximate formula for $p(t)$ in Eq. (37).

IV. CONCLUSION

In conclusion, we have investigated the persistence probability of a harmonically confined Brownian particle in the overdamped limit, with the potential relaxing to zero at long times. We consider two functional forms of the relaxation: an exponential relaxation and an algebraic relaxation. The simple model system presented in this article is analogous to a moving wall [30], with a "hard" wall replaced by a "soft" wall. The external confinement can be realized using a laser-trapping experiment, with the intensity of the laser decaying in time. When the confining potential relaxes exponentially, we observe that the dynamics of the Brownian particle at short and long times is purely diffusive and independent of the relaxation time scales. On the other hand, for an algebraic relaxation, the motion at long times is determined by the exponent of the relaxation. Using the two-time correlation function for the position of the Brownian particle, we construct the persistence probability of the Brownian particle in the two scenarios.

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