Effective noise theory for the nonlinear Schrödinger equation with disorder

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For the nonlinear Shrödinger equation with disorder it was found numerically that in some regime of the parameters Anderson localization is destroyed and subdiffusion takes place for a long time interval. It was argued that the nonlinear term acts as random noise. In the present work, the properties of this effective noise are studied numerically. Some assumptions made in earlier work were verified, and fine details were obtained. The dependence of various quantities on the localization length of the linear problem were computed. A scenario for the possible breakdown of the theory for a very long time is outlined.

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I. INTRODUCTION

The nonlinear Schrödinger equation (NLSE) [1] in a random potential takes the form of

$$i\partial_t\psi = H_0\psi + \beta|\psi|^2\psi, \tag{1}$$

where H_0 is the linear part with a disordered potential, which on a lattice takes the form of

$$H_0\psi(x) = -[\psi(x+1) + \psi(x-1)] + \varepsilon(x)\psi(x). \tag{2}$$

In this work, it is assumed that $\varepsilon(x)$ are identical independent random variables (i.i.d.) uniformly distributed in the interval of $[\frac{-W}{2},\frac{W}{2}]$.

The NLSE was derived for a variety of physical systems under some approximations. It was derived in classical optics, where ψ is the electric field by expanding the index of refraction in powers of the electric field, keeping only the leading nonlinear term [2]. For Bose-Einstein condensates (BEC), the NLSE is a mean-field approximation where the term proportional to the density $\beta|\psi|^2$ approximates the interaction between the atoms. In this field, the NLSE is known as the Gross-Pitaevskii equation (GPE) [3–7]. It is well known that in 1D in the presence of a random potential with probability one, all the states are exponentially localized [8–10]. Consequently, diffusion is suppressed and in particular a wavepacket that is initially localized will not spread to infinity. This is the phenomenon of Anderson localization [11]. The problem defined by Eq. (1) is relevant for experiments in nonlinear optics, for example, disordered photonic lattices [12,13], where Anderson localization was found in the presence of nonlinear effects as well as experiments on BECs in disordered optical lattices [14–23]. The interplay between disorder and nonlinear effects leads to new interesting physics [20,21,24–27]. In spite of the extensive research, many fundamental problems are still open [28]. In particular, there is disagreement between the analytical and the numerical results [29–37].

A natural question is whether a wave packet that is initially localized in space will indefinitely spread for dynamics controlled by Eq. (1). A simple argument indicates that spreading will be suppressed by randomness. If unlimited spreading takes place, the amplitude of the wave function will decay since the l^2 norm is conserved. Consequently, the nonlinear term will become negligible and Anderson localization will take place as a result of the randomness as conjectured by Fröhlich *et al.* [38]. Contrary to this intuition, based on the smallness of the nonlinear term resulting from the spread of the wave function, it is claimed that for the kicked-rotor a nonlinear term leads to delocalization if it is strong enough [39]. It is also argued that the same mechanism results in delocalization for the model Eq. (1) with sufficiently large β, while, for weak nonlinearity, localization takes place [39,40]. Recently, it was rigorously shown that the initial wavepacket cannot spread so that its amplitude vanishes at infinite time, for large enough β [41]. It does not contradict spreading of a fraction of the wavefunction. Indeed, subdiffusion was found in numerical experiments [39,40,42–44]. It was also argued that nonlinearity may enhance discrete breathers [26,27]. In conclusion, it is *not* clear what is the long time behavior of a wave packet that is initially localized, if both nonlinearity and disorder are present [28]. The major difficulty in numerical resolution of this question is integration of Eq. (1) to long time. Most researchers who run numerical simulations use a split-step method for integration; however, it is impossible to achieve convergence for long times, and, therefore, some heuristic arguments assuming that the numerical errors do not affect the results qualitatively are utilized [39,43]. Moreover, the problem is chaotic; therefore, the trajectories that are found are not the actual trajectories and it is argued that it does not affect the statistical results.

Recent rigorous arguments [29,30] in the limit of strong disorder combined with perturbation theory [31,32,45] indicate that it is unlikely that subdiffusion persists forever and the asymptotic growth is at most logarithmic in time. Also other recent works based on a scaling theory [33] and phase space considerations [36,46] lead to similar indications. It is clear that there is a substantial regime in time and parameters where subdiffusion may hold, and the purpose of the present work is to analyze the dynamics in this regime.

Our analysis based on Refs. [43,44] is conveniently expressed, expanding the wavefunction

$$\psi(x,t) = \sum_n c_n(t)u_n(x)e^{-iE_nt}, \tag{3}$$

where u_n are the eigenfunctions of H_0 typically falling off exponentially:

$$u_n(x) \approx \frac{e^{-|x_n-x|/\xi}}{\sqrt{\xi}}\varphi(x), \tag{4}$$

where $\varphi(x)$ is a random function of order unity where ξ is the localization length. The localization center is x_n. The $c_n(t)$ satisfy

$$i\partial_t c_n(t) = \beta \sum_{m_1,m_2,m_3} V_n^{m_1,m_2,m_3} e^{i(E_n+E_{m_1}-E_{m_2}-E_{m_3})t} c_{m_1}^* c_{m_2} c_{m_3} \equiv F_n(t) \tag{5}$$

and

$$V_n^{m_1,m_2,m_3} = \sum_x u_n(x) u_{m_1}(x) u_{m_2}(x) u_{m_3}(x). \tag{6}$$

In Refs. [43,44], it is argued that $F_n(t)$ behaves as random noise with rapidly decaying correlation functions. A major purpose of the present work is to study the statistical properties of the F_n and to show how these are related to the spreading of the wavepacket found in previous works. The implications of this statistical behavior are analyzed in Sec. II and tested numerically in Sec. III. A scenario for the breakdown of the effective noise theory is outlined in Sec. IV. The results are summarized and open question are presented in Sec. V.

II. THE EFFECTIVE NOISE THEORY

In this section, a theory for the spreading of wavepackets for the NLSE will be developed. It follows the theory of SKFF (Skokos, Krimer, Komineas, and Flach [43,44]). It assumes spreading from the vicinity of the initial localized wave packet where the density is large to the regions where it is small. We denote by m_1,m_2,m_3 the states in the region where the amplitude of the states is typically large and by n a state where the amplitude is small, in particular–

$$|c_{m_1}|^2 \approx |c_{m_2}|^2 \approx |c_{m_3}|^2 \approx \rho, \tag{7}$$

where ρ is the density where it is large, while

$$|c_n|^2 \ll \rho. \tag{8}$$

It is assumed that the RHS of Eq. (5) is a random function denoted by $F_n(t)$. We turn to estimate its typical behavior. First, we note that the overlap sums of Eq. (6) are random functions. Within the scaling theory for localization, one expects that for sufficiently weak disorder their various moments are determined by the localization length. For the case where all indices (n,m_1,m_2,m_3) are identical, the average is just the inverse participation ratio that is proportional to $1/\xi$. For the general case, the scaling theory suggests it is a function only of ξ. Experience with scaling theories leads us to assume it is a power of ξ. Therefore, we try the form

$$\left\langle V_n^{m_1,m_2,m_3} \right\rangle = C_0^{(1)} \xi^{-\eta_1}, \tag{9}$$

and for the second moment, we try to fit to

$$\left\langle \left| V_n^{m_1,m_2,m_3} \right|^2 \right\rangle = C_0^{(2)} \xi^{-2\eta_2}. \tag{10}$$

Here, $C_0^{(1)}$ and $C_0^{(2)}$ are constants and $\langle .. \rangle$ is an average over realizations. We note that when the m_i and n are all different, the average of the overlap integrals vanishes. We should note that the localization length ξ is actually energy dependent. For weak disorder in the center of the band, $\xi \sim W^{-2}$ [47,48], this relation holds for most energies in the energy band [47]. In what follows, we will estimate the values of η_1 and η_2 for various disorder strengths and for various sites $(x_n,x_{m_1},x_{m_2},x_{m_3})$, which are within the localization length. Otherwise, the sum of Eq. (6) is negligible. It is not obvious that both Eqs. (9) and (10) will scale in this way, although it is expected from the scaling theory of localization that this is the case for sufficiently weak disorder, namely large ξ. We demonstrate that this is indeed the case and there is a typical magnitude of the value of the overlap sum of Eq. (6), and it scales as

$$V = C_1 \xi^{-\eta}, \tag{11}$$

where C_1 is a constant. Here and in what follows, we denote by ξ the localization length in the center of the band. In Sec. III B, it will be demostrated that $\eta \approx 1$.

Because of localization, the $V_n^{m_1,m_2,m_3}$ of Eq. (6) take appreciable values only if the localization centers of the states n,m_1,m_2,m_3 are within length scale of order ξ. Therefore, the sum on the right-hand side of Eq. (5) consists of the order of ξ^3 terms, at least for weak disorder. These are rapidly oscillating in time, and it is a nonlinear function of the $c_{m_i}(t)$. Therefore, it is suggestive that it can be considered random. This assumption will be tested in detail in Sec. III A. The right-hand side of Eq. (5) is assumed to take the form [44]

$$F_n = V\mathcal{P}\beta\rho^{3/2} f_n(t) = \frac{C_1}{\xi^\eta}\mathcal{P}\beta\rho^{3/2} f_n(t), \tag{12}$$

where C_1 is a constant and

$$\mathcal{P} = A_0 \beta^\gamma \xi^\alpha \rho \tag{13}$$

is proportional to the number of "resonant modes," namely ones that strongly affect the dynamics of the state n. Although it is reasonable to assume that the number of resonant modes is proportional to the density ρ, a strong argument for it is missing; nevertheless, it is consistent with all numerical results [43,44]. We assume here the form of Eq. (13), where A_0 is a constant independent of β and ξ. In the end of this section, we argue that within these assumptions $\gamma = 1$ in agreement with the assumption of [43,44]. The value of α is estimated numerically (see Sec. III C). Under these assumptions, Eq. (5) reduces to

$$i\partial_t c_n(t) = F_n(t). \tag{14}$$

With $F_n(t)$ being a random variable rather than the explicit sum in Eq. (5), and assuming $F_n(t)$ can be considered random with rapidly decaying correlations, in particular, we assume that the distribution function of $f_n(t)$ is stationary, the assumption tested in Sec. III A, and the integral of correlation function $C(t') = \langle f(0)f(t')\rangle$, where $\langle .. \rangle$ is the average over the random potential, converges. Integration results in

$$c_n(t) = -i\frac{C_1}{\xi^\eta}\mathcal{P}\beta\rho^{3/2}\int_0^t dt' f_n(t'). \tag{15}$$

Integrating over a time interval that is sufficiently large yields

$$\langle |c_n(t)|^2 \rangle = \frac{A_1}{\xi^{2\eta}}\mathcal{P}^2\beta^2\rho^3 t = A_1 A_0^2 \beta^{2(\gamma+1)}\rho^5\xi^{2\alpha-2\eta} t, \tag{16}$$

where A_1 is a constant. The value of $\langle |c_n(t)|^2 \rangle$ increases with time and equilibrium is achieved when it takes the value ρ. Transitions between states of the type of n (states with small amplitude) are ignored in this model. The required time for equilibration is

$$T = \frac{1}{B\xi^{-2}\rho^4}, \tag{17}$$

where we define

$$B = A_1 A_0^2 \beta^{2(1+\gamma)} \xi^{2\alpha-2\eta+2}. \tag{18}$$

The equilibration time T varies slowly compared to t [see discussion after (24)]. In other words, there is a separation of time scales. On the time scale T, the system seems to reach equilibrium by a diffusion process, and the density becomes constant in a region that includes the site n. Hence, on this time scale it seems to equilibrate. On longer time scales, there is an even longer equilibration time scale, and the resulting diffusion is even weaker. The consistency of the argument results in the fact that $\frac{dT}{dt} \to 0$ for $t \to \infty$. Therefore, it is assumed that the variations of ρ and T are slow on the scale of t. This assumption is checked in the end of this section. The resulting diffusion coefficient is

$$D = C\frac{\xi^2}{T} = CB\rho^4, \tag{19}$$

where C is a constant. The assumption is that the nonlinear term generates a random walk with the characteristic steps T and ξ in time and space. At time scales $t \gg T$, there is diffusion and

$$M_2 = Dt, \tag{20}$$

where $M_1 = \sum x|\psi(x,t)|^2$ and the variance $M_2 = \sum (x - M_1)^2 |\psi(x,t)|^2$ are the first and second moments. Since the second moment M_2 is inversely proportional to ρ^2, one finds

$$\frac{1}{\rho^2} = A_2 C B \rho^4 t, \tag{21}$$

where A_2 is a constant. Therefore,

$$\frac{1}{\rho^2} = (A_2 C B t)^{1/3}. \tag{22}$$

The second moment satisfies

$$M_2 = \frac{1}{A_2^{2/3}} (CBt)^{1/3} \tag{23}$$

and

$$T = \frac{1}{B\xi^{-2}\rho^4} = \frac{C^{2/3} A_2^{2/3} \xi^2 t^{2/3}}{B^{1/3}} = \frac{C\xi^2}{M_2} t. \tag{24}$$

The density ρ and the equilibration time T change with time as $\rho \sim t^{-\frac{1}{3}}$ and $T \sim t^{\frac{2}{3}}$. Therefore, for $\frac{d\rho}{dt} \sim t^{-\frac{4}{3}}$ and $\frac{dT}{dt} \sim t^{-\frac{1}{3}}$. First, note that in the long time limit $t \to \infty$, both derivatives vanish and $\frac{d\rho}{dt} \ll \frac{dT}{dt}$. Therefore, for the derivation of the equilibration time ρ can be considered constant and on long scales of spreading T and D can be considered constant. Therefore, the theory is consistent for large t. Since in the NLSE β appears only via the combination $\beta|\psi(x)|^2$, it can appear in Eqs. (18) and (19) only in the power 4 (that is in the combination $\beta^4\rho^4$); therefore, $\gamma = 1$.

In the next section this theory will be tested numerically.

III. NUMERICAL TESTS FOR THE EFFECTIVE NOISE THEORY

In this section, the theory presented in Sec. II is tested numerically. In Sec. III A the distribution of the $F_n(t)$ is computed, in Sec. III B the first moments of the overlap sums are calculated while in Sec. III C the dependence of the second moment M_2 of Eq. (23) on ξ is evaluated.

A. Statistical properties of $F_n(t)$

In this subsection, the statistical distribution of $F_n(t)$ is explored. For this purpose, the time-dependent NLSE Eq. (1) was solved numerically for a finite lattice of N sites, for N_R realizations of the random potential $\varepsilon(x)$ and for $W = 4$. The wavefunction $\psi(x,t)$ at time t was calculated for a single site excitation, namely the initial condition $\psi(x,0) = \delta_{x,0}$ using the split step method [44,49]. The details of the numerical calculation are presented in the Appendix. The expansion Eq. (3) of ψ in terms of eigenfunctions of the linear problem Eq. (2) yields

$$i\partial_t c_n(t) = \sum_x \beta|\psi(x,t)|^2 \psi(x,t) u_n(x) e^{itE_n} \equiv F_n(t). \tag{25}$$

This equation was used to calculate $F_n(t)$ numerically for a lattice of N sites. We sampled $F_n(t)$ for various times and verified that their distribution is stationary. In order to check whether $F_n(t)$ can be considered as noise, we calculated its power spectrum and autocorrelation function. First, we present results obtained for times up to $t = 10^5$ for $\beta = 1$, $W = 4$ ($\xi \approx 6.4$), $N = 1024$ for a single site excitation at $t = 0$. The calculation was preformed for $N_R = 50$ realizations. For nearly all these realizations it was found that the second moment grows as $M_2 \propto t^{1/3}$ in agreement with the results of Refs. [40,43,44]. We focus first on such realizations and present the results for a specific realization in Fig. 1.

The power spectrum is

$$S_n(\omega) = |\hat{F}_n(\omega)|^2, \tag{26}$$

where

$$\hat{F}_n(\omega) = \lim_{\tilde{t}\to\infty} \frac{1}{\sqrt{\tilde{t}}} \int_0^{\tilde{t}} F_n(t) e^{(-i\omega t)} dt. \tag{27}$$

It is plotted for some realization in Fig. 1(a) for $n = 0$. It exhibits a peak around $|\omega_0| \approx 1.72$, and its width is $\Delta\omega \approx 0.1$. The finite width is characteristic of noise. Also, the Fourier transform of

$$\tilde{F}_n(t) = F_n(t) e^{-i\omega_0 t} \tag{28}$$

will exhibit a wide power spectrum near $\omega = 0$, with the width of $\Delta\omega$ that is characteristic of noise. The autocorrelation function of $F_n(t)$ is

$$C_n(\tau) = \overline{F_n(t) F_n^*(t+\tau)}, \tag{29}$$

where bar denotes time average $\overline{g(t)} \equiv \lim_{\tilde{t}\to\infty} \frac{1}{\tilde{t}} \int_0^{\tilde{t}} g(t)dt$.

For $\tilde{F}_n(t)$, we define the autocorrelation function $\tilde{C}_n(\tau)$ that is just Eq. (29) with $F_n(t)$ replaced by $\tilde{F}_n(t)$. In Fig. 1(b) we plot $C_n^{(R)} = \mathrm{Re}[C_n(\tau)]$ for $n = 0$ while in Fig. 1(c) the zoomed version is plotted. Note an oscillation of frequency of the order $|\omega_0| \approx 1.72$ that is superimposed on the function.

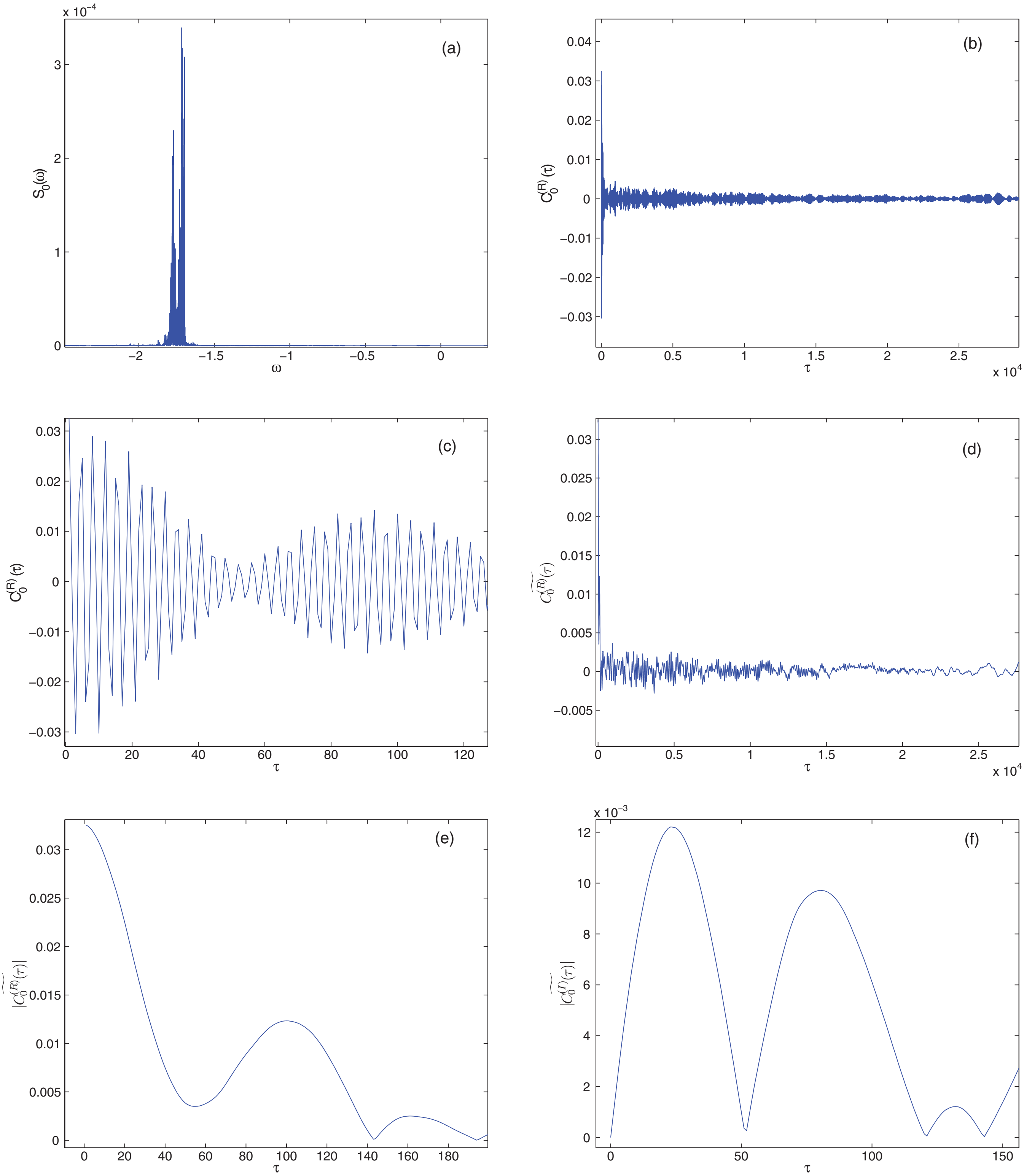

FIG. 1. (Color online) The correlation $C_n(t)$ and power spectrum $S_n(\omega)$ of $F_n(t)$ for $W=4$, $\beta=1$, $N=1024$, $t=10^5$,$n=0$. (a) The Power Spectrum $S_0(\omega)$, (b) The autocorrelation function $C_0^{(\mathrm{R})}(\tau)$, (c) The zoomed $C_0^{(\mathrm{R})}(\tau)$, (d) The autocorrelation function $C_0^{\tilde{(R)}}(\tau)$, (e) the zoomed $C_0^{\tilde{(R)}}(\tau)$, (f) the zoomed $C_0^{\tilde{(I)}}(\tau)$ [see text].

In the corresponding plots of $\tilde{C}_n^{(R)} = \mathrm{Re}[\tilde{C}_n(\tau)]$, presented in Figs. 1(d) and 1(e), one does not find this oscillation. Behavior of the imaginary part of the autocorrelation function $\widetilde{C}_n^{(I)} = \mathrm{Im}[\widetilde{C}_n(\tau)]$ is similar [see Fig. 1(f)]. All results presented in Fig. 1 are for $n=0$. Similar results were found also for $n=3$ and $n=15$. We see that the autocorrelation function decays by

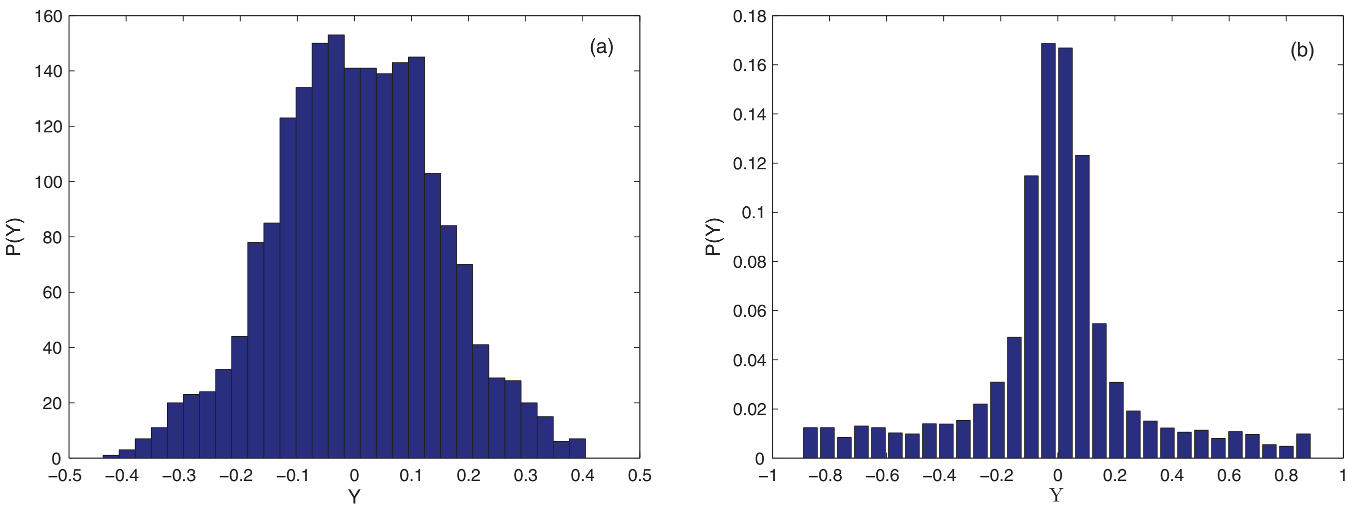

FIG. 2. (Color online) The distribution of $Y = \tilde{F}_n^{(R)}(kt_a)$ where $k = (1,2,\ldots K)$, $K = 500$, $t_a = 200$, $t = 10^5$ and the bin size 0.0596. (a) For the same realization described in the legend of Fig. 1. (b) The distribution of values found for all $N_R = 50$ realizations.

2 orders of magnitude on the scale of $\triangle\tau \approx 140$ (of the order of $2\pi/\triangle\omega \sim 65$). Therefore, the correlation of $\tilde{F}_n(t)$ behaves as the one of noise with short time correlations. For realizations where the growth of the second moment $M_2 \sim t^{1/3}$ was not found, the power spectrum was found to be substantially narrower by 2 orders of magnitude. The calculations were repeated for $\beta = 2$, where similar results were found, and for $\beta = 0.5$. For the latter case, the number of realizations where it was found that the second moment grows like $t^{1/3}$ is substantially smaller than for $\beta = 1$ or $\beta = 2$. In all cases where the width of the power spectrum was small the typical growth of the second moment $M_2 \sim t^{1/3}$ was not found and vice versa. *This demonstrates the strong relation between the effective noise behavior and the diffusive growth of the second moment. It also demonstrates the different behavior of various realizations of the randomness.*

We turn now to test the distribution of $\tilde{F}_n(t)$. For this purpose we sample $\tilde{F}_n(t)$ for a sequence of points separated by $t_a > \triangle\tau$, that is for points where the values of $\tilde{F}_n(t)$ are uncorrelated, and compute the distribution of $\tilde{F}_n(kt_a)$ for $k = (1,2,\ldots K)$. The results are presented in Fig. 2 for $t = 10^5$, $t_a = 200$, $K = 500$.

B. Estimate of scaling of the matrix elements $V_n^{m_1,m_2,m_3}$ with ξ

The overlap sum $V_n^{m_1,m_2,m_3}$ is a random function. In this subsection the scaling of its typical values with the maximal localization length [47]

$$\xi \approx \frac{96}{W^2} \tag{30}$$

is evaluated. This relation holds in the limit of weak disorder. In the numerical calculations presented in this paper, we vary W as the control parameter and the localization length is calculated from Eq. (30). The estimate of Eq. (30) is a reasonable approximation for $W < 5.5$ or $\xi > 3.15$ as was checked explicitly (and used) in this subsection. We note that the $V_n^{m_1,m_2,m_3}$ take values of substantial magnitude when all the centers of localization of the states $u_n, u_{m_1}, u_{m_2}, u_{m_3}$ are within a distance ξ. Only such overlap sums are considered. The average of the overlap sums over realizations vanishes unless (n,m_1,m_2,m_3) consists of two pairs of identical values, $n = m_1$ and $m_2 = m_3$ and all permutations. We calculated $\langle |V_n^{m_1,m_2,m_3}|^2\rangle$ and $\langle V_n^{m_1,m_2,m_3}\rangle$ (where $\langle\cdot\rangle$ denotes average over $N_R = 5000$ realizations), while $x_n, x_{m_1}, x_{m_2}, x_{m_3}$ are fixed fractions of ξ and ξ (and W) are varied. Assuming $\langle V_n^{m_1,m_2,m_3}\rangle \sim \xi^{-\eta_1}$ and $\langle |V_n^{m_1,m_2,m_3}|^2\rangle \sim \xi^{-2\eta_2}$ while the variance $\langle (V_n^{m_1,m_2,m_3})^2\rangle - \langle V_n^{m_1,m_2,m_3}\rangle^2$ scales as $\xi^{-2\eta_3}$, we estimate these exponents from figures like Fig. 3. We conclude that $\eta_1 \approx \eta_2 \approx \eta_3 \approx 1$. Therefore, the typical magnitude of the random variable $V_n^{m_1,m_2,m_3}$ scales as Eq. (11) with $\eta = 1$. Although this result is expected from the scaling theory of

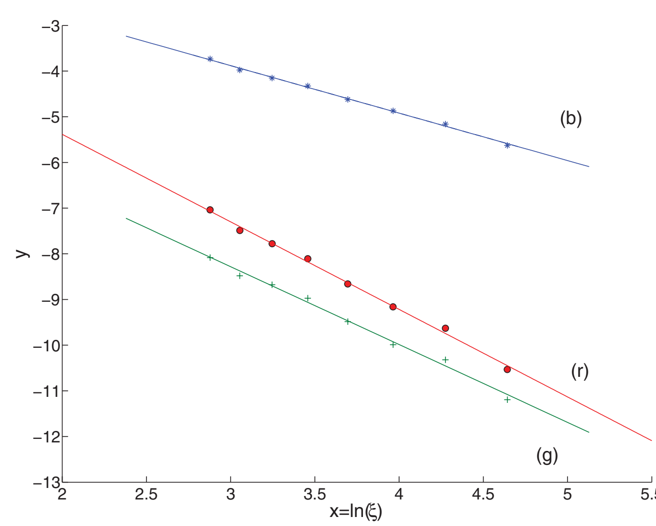

FIG. 3. (Color online) A log-log plot of (b) $y = ln\langle V_0^{0,\frac{\xi}{3},\frac{\xi}{3}}\rangle$, (r) $y = ln\langle (V_0^{0,\frac{\xi}{3},\frac{\xi}{3}})^2\rangle$ and (g) $y = ln(\langle (V_0^{0,\frac{\xi}{3},\frac{\xi}{3}})^2\rangle - \langle V_0^{0,\frac{\xi}{3},\frac{\xi}{3}}\rangle^2)$ as a function of $x = ln(\xi)$, for the parameters $N = 512$, $N_R = 5000$. The localization length varies in the interval $11 < \xi < 103$. The least square fit leads to $\eta_1 = 1.039$,$\eta_2 = 0.958$ and $\eta_3 = 0.853$ respectively. The symbols denote the numerical results and the lines the least square fit.

localization, it is not obvious *a priori*. In particular, it is not clear what is the effect of cancellations of various terms resulting of opposite signs.

For $\xi \ll 11$ we could not obtain smooth curves of $V_n^{m_1,m_2,m_3}$. The reason is that the centers of localization x_{m_i} are equal to the integer part of ξ/a, where a is fixed and ξ varies. For small ξ, the jumps in $V_n^{m_1,m_2,m_3}$ are significant, since ξ does not cover many integers. The results obtained indicate that scaling of the overlap sums as ξ^{-1} holds also for values $\xi < 11$. In summary, for a crude evaluation one can assume Eq. (11) holds with $\eta = 1$.

C. The scaling of the second moment M_2 with ξ

In this subsection, we will estimate the exponent α defined in Eq. (13). For this purpose we write Eq. (23) in the form

$$M_2 = At^{\frac{1}{3}} \tag{31}$$

with

$$A = A_4\xi^{\nu}, \tag{32}$$

where $\nu = \frac{2}{3}(\alpha - \eta + 1)$ [see Eq. (18)], while A_4 is a constant independent of ξ. We used the split-step method to obtain $\psi(x,t)$ for different realizations ($N_R = 30$) and computed ψ until $t = 10^6$. Only realizations that satisfied $M_2 \sim t^{\frac{1}{3}}$ at some stage of the calculation were taken into account. This was the case for nearly all the N_R realizations for $\xi > 7$ and $\beta < 4$. In the other regimes, it was not satisfied for a significant number of realizations. Fixing β, we estimate ν from plots like Fig. 4. For $1 < \beta < 3.5$ using the fact that $\eta \approx 1$, we find that for $1.235 < \nu < 1.71$ for various values of β. The exponent α of Eq. (13) takes the values $1.85 < \alpha < 2.56$. We note the strong uncertainty of ν and α. These results indicate that $A \sim \xi^{\nu}$. It is an estimate of the order of magnitude but not a verification of this power law.

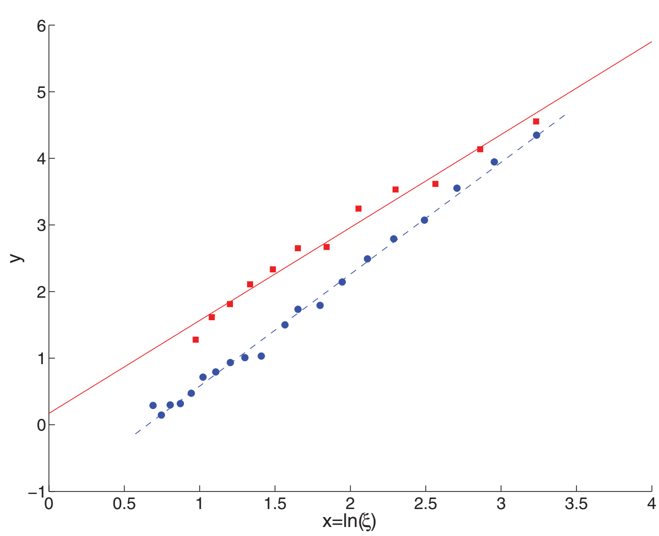

FIG. 4. (Color online) The dependence of A defined by Eqs. (31) and (32) for $\beta = 1$ (blue circles) and for $\beta = 3$ (red squares) on ξ. We denote $y = ln(A)$ and $x = ln(\xi)$. From the least-square fit we find $\nu = 1.684$ for $\beta = 1$ (blue) and $\nu = 1.395$ for $\beta = 3$ (red).

IV. POSSIBILITY FOR THE BREAKDOWN OF THE EFFECTIVE NOISE THEORY

For the effective noise theory, it is essential that $F_n(t)$ can be considered random. For this the number of terms in the sum of Eq. (5) that resonate with n should be large; namely, $\mathcal{P}$ should not be too small. The density ρ and, therefore, $\mathcal{P}$ decrease with time. If $\mathcal{P}$ is very small, there may be a situation that as a result of fluctuations, the sum of Eq. (5) is dominated just by one term and therefore it is effectively quasiperiodic. If spreading is a result of the randomness of F_n, it will stop then. Let us first estimate the time scale required to spread so that $\mathcal{P} \approx 1$. For this purpose, let us write Eq. (13) in the form

$$\mathcal{P} \approx \overline{A}\xi^{\alpha}\rho, \tag{33}$$

where $\overline{A} = A_0\beta$. Since ρ decreases with time t, there is a time scale when $\mathcal{P}$ will become very small. Assuming the constants are of the order of unity, using Eqs. (18) and (21), the time t^* when $\mathcal{P} \approx 1$ satisfies

$$\xi^{2\alpha}\frac{1}{[\xi^{2(\alpha-\eta+1)}t^*]^{\frac{1}{3}}} \approx 1 \tag{34}$$

or

$$\xi^{(\frac{4}{3}\alpha+\frac{2}{3}(\eta-1))} \approx t^{*\frac{1}{3}}, \tag{35}$$

resulting in

$$t^* \approx \xi^{[4\alpha+2(\eta-1)]} \tag{36}$$

for $1.85 < \alpha < 2.56$ and $\eta = 1$,

$$t^* \approx \xi^{\delta}, \tag{37}$$

where $7.4 < \delta < 10.24$

The time required for $\mathcal{P} \ll 1$, when the effective noise theory may fail, is even larger.

V. SUMMARY AND CONCLUSIONS

The effective noise theory was introduced in Ref. [39] and was further developed in Refs. [40,43,44]. It was found to be consistent with the numerical results in some regimes. In Sec. II our interpretation of this theory was presented. In Sec. III the details of this theory were tested numerically. In particular, the distribution of the effective driving F_n defined in Eq. (5) was studied. The correlation function was calculated as well and was found to be characterized by a wide power spectrum and rapid decay with time. These were found only for realizations where subdiffusion with the second moment growing as $t^{1/3}$ is found, indicating the relation between this spreading and the approximation of F_n as effective noise. These results are purely numerical and support the effective noise theory. An obvious challenge is to obtain these results analytically. We determined that the behavior $A \approx \xi^{\nu}$ [see Eq. (32)], with $1.235 < \nu < 1.71$ is a reasonable approximation. From this, we conclude that the dependence of $\mathcal{P}$ on ξ in Eq. (13) is controlled by the exponent $1.85 < \alpha < 2.56$. Although ξ varied over one decade and the evaluation of the exponent is crude, we believe it may give the correct order of magnitude.

We turn to speculate how the effective noise theory may break down for a long time scale. Assuming the effective noise theory holds for long time, $\mathcal{P}$ of Eq. (13) becomes extremely small, consequently the number of terms in the sum of Eq. (5) that contribute significantly may become of order unity and F_n may turn to be quasiperiodic rather than random. Therefore, there is a time scale t^* given by the estimate of Eq. (35) so that for $t > t^*$ the effective noise theory is invalid. For such long time, a sequence of peaks may replace the continuous region of the power spectrum in Fig. 1(a). If localization is destroyed by the effective noise F_n, it is reasonable to expect localization or spreading slower than subdiffusion (say logarithmic in time) on time scale t^* and larger. Existence of such a time scale is consistent with Refs. [29–31,33,36,45]. The scaling arguments used here should improve when the localization length ξ becomes large but then t^* becomes extremely large and it is impossible to explore numerically the scenario for the breakdown of the effective noise theory outlined in Sec. IV. Such a scenario may enable us to reconcile the numerical results where subdiffusion is found [28,40–44] with the analytical results predicting asymptotic spreading that is at most logarithmic [28–31]. These points should be subject of future research.

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APPENDIX: SOME DETAILS OF THE NUMERICAL CALCULATIONS

We used the split-step method [44,49] to obtain the time evolution starting from the initial wavefunction. The lattice size N used is 512 or 1024. The reason we used the relativity large lattice is because we wanted to avoid boundary effects, namely we required the wavefunction amplitude to be smaller than 10^{-12} on the boundary. The time step used in the split-step method is $dt = 0.1$. We used this time step because it is small enough relative to the time scales in the system at hand and large enough in order to complete the numerical calculation in reasonable time. It is the smallest time step used in Refs. [43,44]. The initial condition used is a single-site excitation in the middle of the lattice denoted by $x_n = 0$; namely, $\psi(x,t = 0) = \delta_{x,0}$.

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