Stochastic resonance in an ensemble of bistable systems under stable distribution noises and nonhomogeneous coupling

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In this paper, stochastic resonance of an ensemble of coupled bistable systems driven by noise having an α-stable distribution and nonhomogeneous coupling is investigated. The α-stable distribution considered here is characterized by four intrinsic parameters: $\alpha \in (0,2]$ is called the stability parameter for describing the asymptotic behavior of stable densities; $\beta \in [-1,1]$ is a skewness parameter for measuring asymmetry; $\gamma \in (0,\infty)$ is a scale parameter for measuring the width of the distribution; and $\delta \in (-\infty,\infty)$ is a location parameter for representing the mean value. It is demonstrated that the resonant behavior is optimized by an intermediate value of the diversity in coupling strengths. We show that the stability parameter α and the scale parameter γ can be well selected to generate resonant effects in response to external signals. In addition, the interplay between the skewness parameter β and the location parameter δ on the resonance effects is also studied. We further show that the asymmetry of a Lévy α-stable distribution resulting from the skewness parameter β and the location parameter δ can enhance the resonance effects. Both theoretical analysis and simulation are presented to verify the results of this paper.

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I. INTRODUCTION

Stochastic resonance (SR) is a noise-induced effect demonstrating the phenomenon of signal amplification, which has been extensively investigated in the past two decades [1–5]. The mechanism is such that an external forcing injected on a nonlinear system can be amplified under a proper dose of noise. Further noise-induced resonance phenomena including doubly stochastic resonance [6], SR on bone loss [7], SR in excitable systems [8], coherence resonance [9] and array-enhanced coherence resonance [10] have been studied to manifest the constructive role of noises.

While initially the studies focused on resonance of dynamical systems with a simple form of Gaussian noise, more recent works [11–14] have considered the significant role of Lévy stable distribution noises in a single system. In [13], the effect of the stability parameter $\alpha \in [1.5,2]$ on a subdiffusive bistable system was investigated. Additionally, growing experimental evidence also indicates that there is a stringent need to consider a more general kind of noise than a Gaussian type of noise [15–17]. As a more general heavy-tailed fluctuation, stable distribution has been observed and manifested in various situations ranging from economics [15], stochastic resonance [11], forging [16], finance market [17], and game theory [18], etc.

Recently, resonance phenomena have been extended to complex networks composed of an array of coupled units under Gaussian type noises and generated considerable interest [19]. For the sake of simplicity, the unit networks are usually coupled with a global coupling strength. Similar to the stochastic resonance phenomenon [20], it was revealed that when a small periodic force acts on coupled networks, the system has a maximal linear response at a certain system size. SR phenomena have also been found in globally coupled networks [21], globally coupled networks with time delays [22], scale-free networks [23], diffusively coupled networks [24], complex networks with time delays [25], diffusively coupled FitzHugh-Nagumo model [26], and coupled networks with both attractive and repulsive couplings [27]. In [21], analytical and numerical results revealed that when bistable or excitable systems are subjected to an external subthreshold signal, their response achieves their maximum for an intermediate value of diversity. In [28], a system of globally coupled active rotators near the excitable regime was considered and it displays a transition to a state of collective firing induced by disorder. However, it is well known that the connections in real-world networks are normally weighted, such as cortical networks, biological networks, transportation networks, and communication networks [19]. Therefore it is very interesting to study the dynamics of networks with nonhomogeneous coupling.

The above discussions show the importance of SR driven by noise having an α-stable distribution and the related mechanisms are now gradually uncovered. Up to now, all the previous studies regarding stable distribution noise induced resonance were confined to a single bistable or excitable system [11–13] instead of complex networks. More importantly, the importance of the stability parameter α, the skewness parameter β, the scale parameter γ, and the location parameter δ have not been systematically investigated in these contributions. To the best of the authors' knowledge, the results of SR in complex networks focus on fixed coupling strength instead of nonhomogeneous networks, despite the importance of nonhomogeneous coupling strengths in modeling a realistic network.

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In this paper, we therefore explore SR of globally coupled networks perturbed by noise having an α-stable distribution and nonhomogeneous coupling. It is revealed that a resonance collective behavior is observed in the presence of diversity in coupling strengths. The effects of the stability parameter α and the scale parameter γ on the resonance behaviors are shown. It is found that α plays a similar role as γ to show the diversity of noises and thus it can induce SR by tuning α. The interplay impacts of the skewness parameter β and the location parameter δ on SR are also presented. We find that the asymmetry of Lévy α-stable distributions resulting from the skewness parameter β can facilitate to enhance resonance effects when the location parameter δ is not zero.

The paper is organized as follows. In Sec. II, some preliminaries of noise having an α-stable distribution are presented and then the main results of SR in an ensemble of coupled bistable systems with Lévy α-stable distribution noises and nonhomogeneous coupling are provided. Finally, Sec. III draws the conclusion and the discussions.

II. METHOD AND RESULTS

A. Model

The following ensemble of globally coupled bistable systems is considered:

$$\dot{x}_i = x_i - x_i^3 + \xi_i + \frac{1}{N}\sum_{j=1}^{N} c_{ij}(x_j - x_i) + A\sin(\Phi t), \quad (1)$$

where $x_i(t), i = 1,2,\ldots,N$, is the state of the ith unit at time t. The system is subjected to an external periodic forcing with amplitude A and frequency $\Phi = \frac{2\pi}{T}$. $c_{ij} > 0$ denotes the nonhomogeneous coupling strength among the ith and jth unit. Here, we assume that c_{ij} takes independent values according to a normal distribution $f(a)$, which satisfies $\langle c_{ij}\rangle = D$ ($\langle\cdot\rangle$ represents mean value) and $\mathrm{var}\{c_{ij}\} = 2D_\sigma^2$. In this paper, we assume that all the coupling strengths are attractive, i.e., $c_{ij} > 0$, [27] and if the generated $c_{ij} < 0$, the values of c_{ij} are fixed at 0.01. The location and relative stability of the fixed points of an isolated ith unit are disturbed by the noise parameter ξ_i. The noise ξ_i follows a Lévy α-stable distribution whose characteristic function is given as follows [17,29]:

$$\phi(t,\alpha,\beta,\gamma,\delta) = \exp\left[it\delta - |\gamma t|^\alpha\left(1 - i\beta\,\mathrm{sgn}(t)\tan\frac{\pi\alpha}{2}\right)\right],$$
$$\text{for} \quad \alpha \neq 1, \quad (2)$$

and

$$\phi(t,\alpha,\beta,\gamma,\delta) = \exp\left[it\delta - \gamma|t|\left(1 + i\beta\frac{2}{\pi}\mathrm{sgn}(t)\ln|t|\right)\right],$$
$$\text{for} \quad \alpha = 1. \quad (3)$$

Note that any probability distribution is determined by its characteristic function $\phi(t)$ by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(t)e^{-ixt}dt. \quad (4)$$

$\alpha \in (0,2]$ is a stability parameter; $\beta \in [-1,1]$ is called the skewness parameter for measuring asymmetry; $\gamma \in (0,\infty)$ is a scale parameter and $\delta \in (-\infty,\infty)$ is a location parameter.

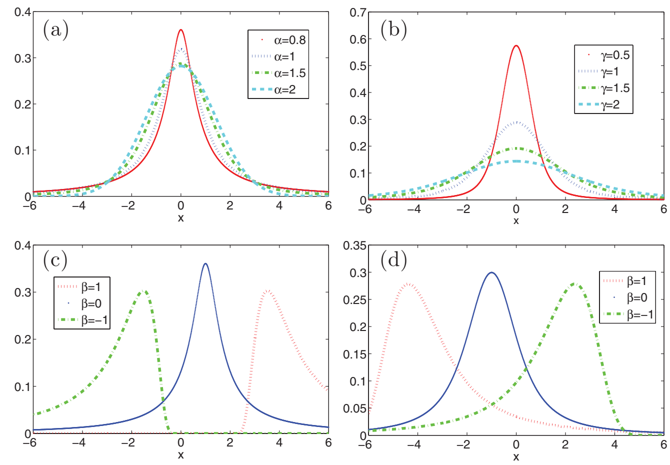

FIG. 1. (Color online) Probability density function of stable distribution noises. (a) Varying α when $\beta = 0$, $\gamma = 1$, $\delta = 0$; (b) varying γ when $\alpha = 1.5$, $\beta = 0$, $\delta = 0$; (c) varying β when $\alpha = 0.8$, $\gamma = 1$, $\delta = 1$; (d) varying β when $\alpha = 1.2$, $\gamma = 1$, $\delta = -1$.

When $\beta = 0$, the distribution is symmetric around δ and is referred to a (Lévy) symmetric α-stable distribution. The scale parameter γ is a measure of the width of the distribution and α is the exponent or index of the distribution and specifies the asymptotic behavior of the distribution when $\alpha < 2$.

In probability theory, a random variable is said to be stable or to have a stable distribution if it has the property that a linear combination of two independent copies of the variable has the same distribution, up to location and scale parameters. The stable distribution family is also named as the Lévy α-stable distribution. Note that the normal distribution, the Cauchy distribution, and the Lévy distribution all have the above properties, which follows that they are special cases of stable distributions [17,29]. When $\alpha = 2$, the distribution reduces to a Gaussian distribution with variance $\sigma^2 = 2\gamma^2$ and the skewness parameter β has no effect on the distribution. In such a case, δ describes the mean value. When $\alpha = 1$ and $\beta = 0$, the distribution becomes Cauchy distribution. When $\alpha = \frac{1}{2}$ and $\beta = 1$, the distribution turns to be a Lévy distribution. Figure 1 depicts the typical probability density function of a Lévy α-stable distribution. From Fig. 1, as α decreases, three changes occur to the density: the peak turns higher, the region flanking the peak turns lower, and the tails turn heavier. Hence we find that the stability parameter α can also be used to determine the diversity, which is similar to the scale parameter γ when $\alpha = 2$ [21]. One can further observe from Fig. 1, as α increases, that the peak of $\beta = 1$ moves from the right to the left one. If $\beta > 0$, then the distribution is skewed with the right tail of the distribution heavier than the left. When $\beta = 1$, the stable distribution is totally skewed to the right. Therefore the SR problem of bistable systems under a stable distribution considered in this paper extends the diversity-induced resonance [21] into a more general noise distribution case.

In the following, in order to avoid such a case that some random variables can be extremely large, ξ_i is generated according to the Lévy α-stable distribution with a predefined bound $[-\epsilon_1,\epsilon_2]$. Different from recent works in studying Lévy α-stable distribution induced noise phenomena [11–13], under this assumption, we can also show the effect of small α. In addition, such a bound has also been widely adopted in characterizing random time delays with a normal distribution [30].

To study the response of the periodically driven system, the spectral amplification factor $R = 4A^{-2}|\langle e^{i\Phi t}X(t)\rangle|$ is quantified [31], where $X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ is the average state of the units at time t. The spectral amplification factor R characterizes the amount of information in the signal transmission with a particular external forcing. In the following, the parameters are fixed as $T = 100, A = 0.2, \epsilon_1 = \epsilon_2 = \epsilon = 500$, and $N = 200$.

B. Main results

In Fig. 2, the amplification factor R versus D and D_σ is plotted, where D and D_σ are provided below (1). We find that there exists an optimum area for the maximum of amplification factor R. When D_σ is properly chosen and D increases, there exists a peak in R. Meanwhile, when D is not very large, it is found that, as increasing D_δ, R first increases until achieving its peak, and then it decreases. This observation is a typical resonance behavior. Therefore a proper diversity of the coupling strength is conducive to the enhancement of resonance effects even when D is small. It is also found that as D decreases, the area of varying D_σ for a maximum of amplification factor R becomes narrower gradually. This phenomenon can be understood as follows. When D is not very large and the diversity in the coupling strength D_σ between the units increases, a unit can be excited by its neighbors, although it is unable to respond to the external forcing. When D is too large, the system will become globally synchronized and D_σ cannot induce a resonance behavior. As D decreases, more c_{ij} will reach the lower bound 0.01 and hence less units cannot be excited by their neighboring units for response to the external stimulus. Therefore the area of varying D_σ for maximum of amplification factor R turns narrower gradually.

In Fig. 3, the amplification factor R as a function of the coupling strength D, the stability parameter α, and the scale parameter γ is plotted. From Fig. 3(a), it is seen that there exists a peak by varying D under a certain stability parameter α. If α decreases, a larger D is required to excite the best resonance effect. Hence the location of optimal value of R in terms of D increases as α decreases. From Fig. 3(b), the stability parameter α plays a similar role as the scale parameter γ and the effect of γ on resonant effects was also studied in [21,22], in which only the Gaussian type of noise ($\alpha = 2$) was considered and γ represents diversity. In summary, the stability parameter α can well characterize diversity like the scale parameter γ.

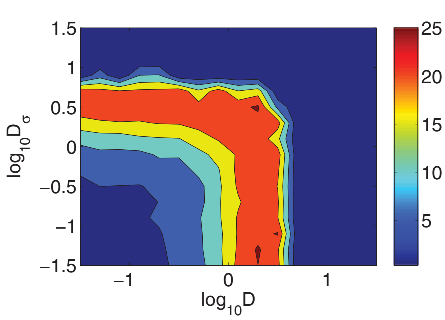

FIG. 2. (Color online) The spectral amplification factor R of the coupled bistable systems is plotted as a function of D and its variance $2D_\sigma^2$. ξ_i is generated from a stable distribution with $\alpha = 2$, $\beta = 0$, $\gamma = \frac{\sqrt{2}}{2}$, and $\delta = 0$.

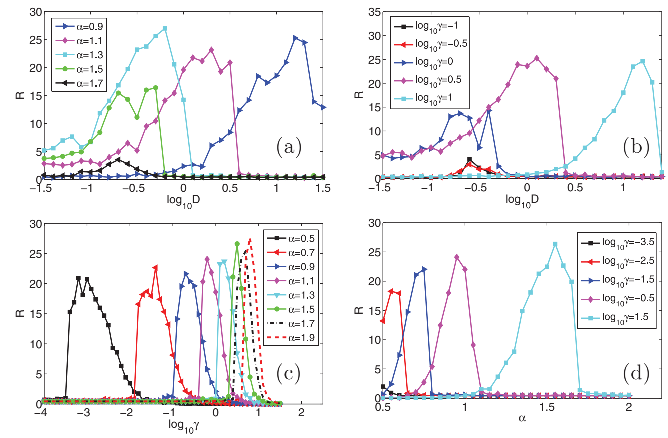

FIG. 3. (Color online) The spectral amplification factor R of the coupled bistable systems is plotted as a function of D, α, and γ. (a) R as a function of D and α when $\beta = 0$, $\gamma = \frac{\sqrt{2}}{2}$, $\delta = 0$, and $D_\sigma = 0$; (b) R as a function of γ and α when $\beta = 0$, $\delta = 0$, $D = 1$, and $D_\sigma = 0$; (c) R as a function of γ and D when $\alpha = 1.5$, $\beta = 0$, $\delta = 0$, and $D_\sigma = 0$; (d) R as a function of α and γ when $\beta = 0$, $\delta = 0$, $D = 1$, and $D_\sigma = 0$.

In Fig. 3(c), the amplification factor R as a function of the stability parameter α and the scale parameter γ is plotted. From Fig. 3(c), we observe that there exists a peak by varying the scale parameter γ under different α. When the scale parameter γ increases, a larger stability parameter α is needed to achieve the best resonance behavior of the coupled bistable systems. In addition, the peaks become higher for larger α and γ. The observed phenomenon can be explained as follows. As α increases, the stable distribution approaches the normal distribution. Although the stability parameter α can induce diversity to the element, it will also lead to several large random variables which might destroy the resonance behavior. As shown in Fig. 1, the diversity is maintained by properly choosing the scale parameter γ to produce resonance behaviors and enlarging the stability parameter α could reduce the probability of generating large random variables. Therefore there exists a tradeoff between decreasing α to induce diversity and increasing α to avoid extremely large noises. This phenomenon is further manifested by Fig. 3(d), where it is seen that there exist the best coherent states as a function of the stability parameter α. The value of R increases to reach a peak and then decreases when the stability parameter α further increases, which shows that an overlarge α or a small α cannot invoke resonance effects. In summary, decreasing α can induce diversity, which is helpful for resonance effects and could also do harm to resonance effect due to the existence of overlarge random variables at the same time. Therefore a better way to induce diversity to maximize resonance behaviors is by increasing α and properly choosing γ.

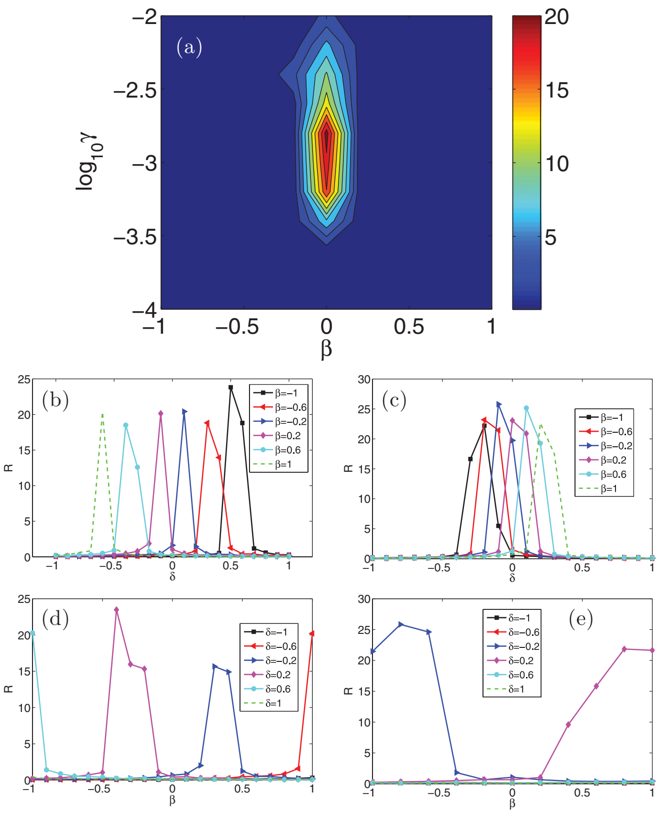

FIG. 4. (Color online) The spectral amplification factor R of the coupled bistable systems is plotted as a function of β, γ, and δ. (a) R as a function of β and γ when $\alpha = 0.5$, $\delta = 0$, $D = 1$, $D_\sigma = 0$; (b) R as a function of β and δ when $\alpha = 0.5$, $\gamma = 10^{-3}$, $D = 1$, $D_\sigma = 0$; (c) R as a function of β and δ when $\alpha = 1.5$, $\gamma = 10^{0.5}$, $D = 1$, $D_\sigma = 0$; (d) R as a function of δ and β when $\alpha = 0.5$, $\gamma = 10^{-3}$, $D = 1$, $D_\sigma = 0$; (e) R as a function of δ and β when $\alpha = 1.5$, $\gamma = 10^{0.5}$, $D = 1$, $D_\sigma = 0$.

In Fig. 4, the amplification factor R as a function of the skewness parameter β, the scale parameter γ, and the location parameter δ is plotted. The amplification factor R as a function of the skewness parameter β and the scale parameter γ is shown in Fig. 4(a). One observes that there exists a peak by varying the skewness parameter β under a certain γ with the location parameter $\delta = 0$, which indicates that the resonance effect is maximum if the distribution of the stable distribution noises is symmetric ($\beta = 0$) when $\delta = 0$. From Figs. 4(b) and 4(c), it is observed that the location parameter δ can help to enhance resonance effects when $\alpha = 0.5$ and $\alpha = 1.5$. Figures 4(d) and 4(e) show that tuning the asymmetry of stable distribution noises, i.e., adjusting the skewness parameter β, will be conducive to enhancing resonance effects. From Figs. 4(b) and 4(c), there exists a best resonance effect as a function of the location δ. From Fig. 4(b), we find that increasing the skewness parameter β makes the locations of the peaks move from right to left when the stability parameter $\alpha < 1$. That is, a smaller location parameter γ is required for the best resonance behavior of the coupled systems. For example, in Fig. 4(b), when $\delta \approx -0.6$, $\alpha = 0.5$, and $\beta = 1$ maximizes R and when $\delta \approx -0.4$, $\alpha = 0.5$, and $\beta = 0.6$ is necessary. Conversely, from Fig. 4(c), if the stability parameter $\alpha > 1$ and the skewness parameter β increases, a larger location parameter γ is needed for the best resonance effect of the coupled systems. Moreover, we find from Figs. 4(b)–4(e) that when α decreases, tuning β is becoming more effective to produce resonance behaviors. For instance, in Figs. 4(b) and 4(d), when $\alpha = 0.5$, tuning β can enable the systems to response to the external forcing when $\delta \in (-0.6,0.6)$. Nevertheless, when $\alpha = 1.5$, the area reduces around $\delta \in (-0.4,0.4)$ which could produce resonance effects by tuning β. This is due to the fact that as α decreases, the effect of β becomes more pronounced [29]. In summary, we find that the interplay between the skewness parameter β and the location parameter δ play constructive roles in enhancing resonance effects.

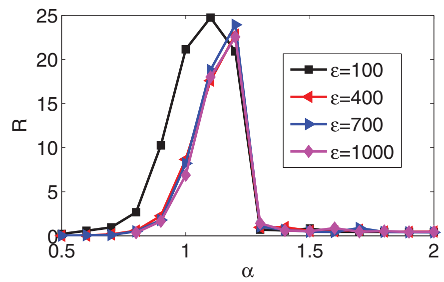

FIG. 5. (Color online) The spectral amplification factor R of the coupled bistable systems is plotted as a function of the bound ϵ and the stability parameter α. ξ_i is generated from a stable distribution with $\beta = 0$, $\gamma = \frac{\sqrt{2}}{2}$, $\delta = 0$, $D = 1$, $D_\sigma = 0$.

The dependence of the predefined bound ϵ on R is shown in Fig. 5. Large value of ϵ indicates that the random variables generated by a stable distribution under boundary restriction reflects the real stable distribution better. From Fig. 5, it is observed that the difference between $\epsilon = 100$ and $\epsilon = 400$ is obvious, but the discrepancy for larger values of ϵ ($\epsilon = 400$, 700, and 1000) is almost negligible. Therefore the boundary assumption $\epsilon = 500$ adopted here is reasonable to reflect a stable distribution in a wide area.

C. Analytical analysis

In order to quantify the response of the coupled bistable systems to the external stimulus, the approximate theory [21] is employed to conduct an analytical analysis. Since the model considered in this paper is rather complicated, we reduce the model into a simpler one with $D_\sigma = 0, \alpha > 1$.

Denote $X(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$ as the average state of the units. In the globally coupled case studied here, (1) can be rewritten in the following macroscopic way:

$$\dot{x}_i = DX + (1 - D)x_i - x_i^3 + \xi_i + A\sin(\Phi t). \qquad (5)$$

By averaging (5), we get

$$\dot{X} = X - \frac{1}{N}\sum_{i=1}^{N} x_i^3 + \sum_{i=1}^{N} \xi_i + A\sin(\Phi t). \qquad (6)$$

Then, one can introduce θ_i such that $x_i = X + \theta_i$. Denote $\frac{1}{N}\sum_{i=1}^{N}\theta_i^2 = \mathcal{V}(t) \geqslant 0$. Under the assumption of θ_i being

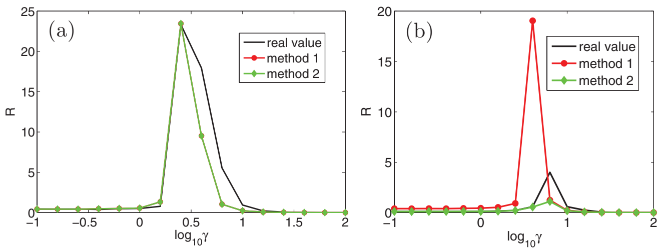

FIG. 6. (Color online) The approximate theory for the spectral amplification factor R. (a) R as a function of γ when $\alpha = 1.5$, $\beta = 0$, $\delta = 0$, $D = 1$, $D_\sigma = 0$; (b) R as a function of γ when $\alpha = 1.5$, $\beta = 0.6$, $\delta = 0.5$, $D = 1$, $D_\sigma = 0$.

evenly distributed, we yield from (6)

$$\dot{X} = X[1 - 3\mathcal{V}(t)] - X^3 + \sum_{i=1}^{N} \xi_i + A\sin(\Phi t). \quad (7)$$

For simplicity, let $\sum_{i=1}^{N} \xi_i = 0$, as in [21], and one gets the following.

Method 1:

$$\dot{X} = X[1 - 3\mathcal{V}(t)] - X^3 + A\sin(\Phi t). \quad (8)$$

If the mean value of ξ_i is not neglected due to the asymmetry of the noise distribution and $\alpha > 1$, the expectation $\mathbb{E}(Z)$ of the random variables Z generated from a stable distribution is δ according to [29]. Then, (7) can also be written as the following.

Method 2:

$$\dot{X} = X[1 - 3\mathcal{V}(t)] - X^3 + \delta + A\sin(\Phi t). \quad (9)$$

By employing (8) and (9), we can perform approximating results. Two cases are taken into account: the first case is systems driven by a symmetrical distribution noise and the other one is systems driven by an asymmetrical distribution noise. From Fig. 6(a), for the symmetrical distribution noise, both methods are in good agreement with the results obtained from a direct numerical simulation of the complete system (1). However, for the asymmetrical distribution noise, method 1 is too poor to well approximate the real value of R. Method 2 can well predict the optimal value γ for R much better due to the introduction of the term of δ. The predicted amplitude is lower than the real R. Compared with method 1, method 2 attains much closer to the real R.

III. CONCLUSION AND DISCUSSIONS

In conclusion, systems disturbed by Lévy α-stable distribution noises display richer dynamical behaviors than systems perturbed by only white Gaussian noises, since the Lévy α-stable distribution includes many well-known distributions such as the normal distribution and the Cauchy distribution as special cases. We have shown that a Lévy α-stable distribution, in the For of quenched noise, can result in and enhance a resonance behavior for the response of an ensemble of coupled bistable systems to an external periodic stimulus. In addition, the SR in coupled bistable systems with nonhomogeneous coupling is also studied. It is shown that the resonance effect can also be induced by properly choosing the diversity in the coupling strength. An approximating theory is employed and extended to analyze globally coupled bistable systems under Lévy α-stable distribution noises.

In [21], it is shown that γ represents diversity to induce SR when $\alpha = 2$. Here, our results show that the stability parameter α plays a similar role as the scale parameter γ to induce diversity to the coupled systems and thus result in a resonance effect in response to external forcing. One difference is that decreasing α will not only induce diversity to enhance the resonance behavior but also lead to overlarge random variables to reduce the resonance behavior. Hence a better way to induce diversity to enhance resonance behaviors is to increase α and properly choosing γ. In addition, an appropriate skewness parameter β, which represents the symmetry of noise distribution, is able to facilitate the resonance phenomenon. Similarly, an appropriate amount of the location parameter δ is helpful to make the systems optimally respond to external signal. It is found that for increasing β with $\alpha < 1$, a smaller δ is required for the best resonance effect. The situation is contrary when $\alpha > 1$.

We expect that all the results presented in this paper are of great significance for the investigation of SR in various coupled systems with nonhomogeneous coupling and different sources of noise, and can establish a valuable guideline for broader theoretical and experimental researches. Since nonhomogeneous coupling and various noises are omnipresent in natural systems, the presented results in this paper may be beneficial to the development of potential applications in areas of system biology, ecology, signal processing, and neuroscience [32–34]. One direction for future research is to study stochastic resonance of coupled systems with nonhomogeneous coupling, where both repulsive and attractive couplings are considered [27]. Another future research direction is to utilize Lévy α-stable distribution noises to investigate noise-induced synchronization and noise-induced coherence in coupled systems [10,35,36].

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