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# Simplicity of the spherical spin-glass model

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We revisit an approach to the replica-based analysis of the spherical spin-glass model that makes use of a mapping of the problem onto a one-dimensional interacting charge system. A saddle point approximation leads to the conclusion that the interaction between charges is irrelevant in the thermodynamic limit, and as a consequence, that there is no nontrivial correlation between replicas for this model. This allows us to show that quenched and annealed disorder averages agree for the spherical spin glass. We demonstrate this result within two different mathematical frameworks, and we also relate our analysis to the conclusions that follow from the replica symmetry ansatz.

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## I. INTRODUCTION

The spherical spin-glass model was originally introduced and solved by Kosterlitz, Thouless, and Jones (KTJ) in 1976 [1]. Perhaps the most important feature of the model is that it can be solved exactly, which facilitates its use as a convenient test bed for new analytic approaches to the spin glasses and other disorder-dominated problems. KTJ originally solved the model via an analysis making use of the asymptotic properties of a large random matrix [2]. In addition, they solved the model via a replica symmetry (RS) ansatz, originally introduced by Edwards and Anderson [3] and utilized by Sherrington and Kirkpatrick in their attempt to solve the "solvable" infinite-ranged Ising spin-glass model [4]. It was later shown by Almeida and Thouless that the replica symmetry ansatz is unstable [5], which leads to the conclusion that a replica symmetry breaking (RSB) solution, such as the one introduced by Bray and Moore [6] or, most notably, by Parisi [7], is required to obtain physical results for the infinite-ranged Ising spin glass. Nevertheless, in the case of the spherical model spin glass, the RS solution was found by KTJ to provide results equivalent to those given by their exact, random matrix based analysis. This equivalence served at the time to bolster confidence in ansatz based replica theory.

Despite the long history and importance of the spherical spin glass (which remains one of the few known examples of an exactly solvable spin-glass model), one of its fundamental features appears to have gone previously unnoticed: Quenched and annealed disorder averages agree for this model. It is the purpose of this paper to demonstrate this result. We employ an approach, previously discussed in Refs. [8,9], that takes advantage of a mapping of the problem onto the finite-temperature partition function of a set of logarithmically interacting charges. In this mapping, there is one charge for each system replica, and correlations between the charges carry information regarding the overlap between the different replicas. Our analysis here differs from that in the previous works [8,9], in that we apply a saddle point approximation that allows us to demonstrate the irrelevance of the interactions in the thermodynamic limit. This leads to a rigorous, ansatz-free evaluation of the replicated partition sum that reveals the full simplicity of the spherical spin-glass thermodynamics.

The paper is organized as follows. The charge mapping procedure is reviewed in the following section; in Sec. III the saddle point "Mehta" approximation is applied that allows for the isolation of the interaction terms; Sec. IV contains a short discussion of our results. Finally, two appendixes are included. Appendix A contains an evaluation relating to the interaction portion of the partition function, and Appendix B relates our work to the predictions of the RS ansatz.

## II. MAPPING TO ONE-DIMENSIONAL INTERACTING CHARGE SYSTEM

We briefly review the charge mapping approach to the replicated spherical spin-glass partition function in this section. For background material we refer the reader to [1,8,9] and to the text by De Dominicis and Giardina [10]. Here, we shall take as our starting point the replicated Hamiltonian for the mean-spherical spin-glass model,

$$H = \sum_{\alpha} \left\{ -\sum_{ij} J_{ij} s_i^{\alpha} s_j^{\alpha} + \Lambda \left( \sum s_i^{\alpha 2} - nN \right) \right\}. \qquad (1)$$

The spins $s_i^{\alpha}$ above are continuous variables to be integrated over, with $i$ and $j$ currently acting as site indices, and $\alpha$ as a replica index. The coefficients of interaction $J_{ij}$ for each pair of spins are random variables independently distributed according to the probability density function

$$P(J_{ij}) \propto \exp\left[ -\frac{N J_{ij}^2}{2\bar{J}^2} \right]. \qquad (2)$$

These coefficients are to be integrated over in order to effect a disorder averaging. Finally, $n$ is the number of replicas, $N$ is the number of spins, and $\Lambda$ is a Lagrange multiplier that is used to enforce the mean-spherical constraint $\langle \sum_i s_i^2 \rangle = N$. After

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integrating over the coefficients of interaction, one obtains

$$\langle Z^n\rangle_J = \int_s \exp\left[\frac{\beta^2 \bar{J}^2}{2N}\sum_{\alpha,\beta}\left(\sum_i s_i^\alpha s_i^\beta\right)^2 - \beta\Lambda\left(\sum_{i,\alpha} s_i^{\alpha 2} - nN\right)\right]. \quad (3)$$

An auxiliary field is next introduced that decouples the different lattice sites. This takes the form of a real, symmetric matrix $Q_{\alpha\beta}$ that we refer to in the following as the overlap matrix:

$$\langle Z^n\rangle_J \propto \int_{Q,s} \exp\left[\frac{-N}{2\beta^2\bar{J}^2}Q_{\alpha\beta}^2 - Q_{\alpha\beta}\sum_i s_i^\alpha s_i^\beta - \beta\Lambda\sum_{i,\alpha} s_i^{\alpha 2}\right]$$
$$= \int_Q \exp\left[-\frac{N}{2\beta^2\bar{J}^2}Q_{\alpha\beta}^2 - \frac{N}{2}\mathrm{Tr}\ln(\beta\Lambda + Q)\right]. \quad (4)$$

Notice that the exponent is now a function of the eigenvalues of $Q$ alone. This key observation was exploited in Refs. [8,9] by switching from an integration over the elements of $Q$ to an integration over its eigenvalues and eigenbasis. The Jacobian appropriate for real, symmetric matrices separates into the form [2]

$$J(\lambda_1,\ldots,p_1,\ldots) = f(\lambda_i)g(p_j), \quad (5)$$

where $f = \prod_{(ij)}|\lambda_i - \lambda_j|$, with the product over all unordered pairs of distinct indices, and the measure $g(p_j)$ indicates an average over all eigenvector orientations. Thus the replicated partition function reduces to

$$\langle Z^n\rangle_J = \int_{\lambda_i} \exp\left[-\sum_i V(\lambda_i) + n\beta\Lambda N\right]\prod_{(ij)}|\lambda_i - \lambda_j|, \quad (6)$$

where the potential is given by

$$V(\lambda) = \frac{N}{2}\left(\ln(\beta\Lambda + \lambda) + \frac{\lambda^2}{\beta^2\bar{J}^2}\right). \quad (7)$$

The eigenvalues $\lambda_i$ may now be reinterpreted as logarithmically interacting charges that are confined by the one-dimensional potential $V$. The logarithmic interaction was dropped in Ref. [8], but was taken into account in Ref. [9], where the evaluation of (6) was carried out through the analytic continuation of a result previously discussed by Forrester and Witte [11]. This approach relied upon the assumption that the logarithmic interaction in Eq. (6) could be replaced by $2\ln|\lambda_i - \lambda_j|$. This assumption was motivated, but not completely justified in Ref. [9]. We will see below that this assumption follows from the fact that this interaction does not affect the free energy whatsoever.

### III. MEHTA APPROXIMATION

The important observation required to simplify the partition function (6) is that the potential $V(\lambda)$ is proportional to $N$. Thus in the thermodynamic limit the potential will be very steep, and all charges might be expected to sit near a global minimum. However, due to the logarithmic term in the potential, the energy is unbounded from below near $\lambda = -\beta\Lambda$. If the charges

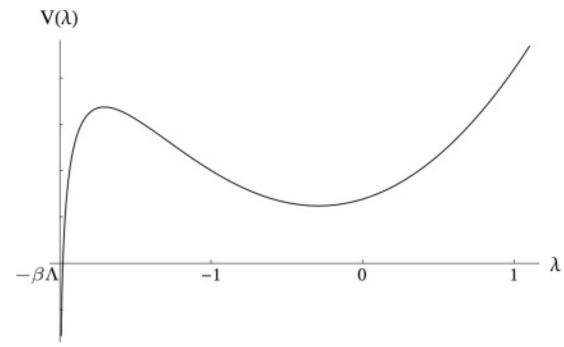


FIG. 1. Qualitative shape of the potential felt by the charges. The potential is proportional to $N$, and the partition function may be expanded about the local minimum near the origin.

are allowed to access all states, they will be bound to the region near $-\beta\Lambda$, and the partition function will diverge—a consequence of the $\Lambda s^2$ term present in Eq. (1), which results in instability for certain realizations of the $J_{ij}$ when $\Lambda$ is held fixed. In the corresponding analysis for the hard-spherical model considered in Ref. [1], the charges are required to always sit far from this divergence, however, and we can conclude that the physically appropriate analytic continuation of (6) is that given by an expansion about the local minimum near the origin, as depicted in Fig. 1. We will see below that this simplification gives free energy results identical to those obtained previously.

Applying a quadratic saddle point approximation about the local minimum $\lambda_0 = \frac{\beta}{2}[-\Lambda + \sqrt{\Lambda^2 - 2\bar{J}^2}]$ gives

$$\langle Z^n\rangle_J \sim e^{n\beta\Lambda N - nV(\lambda_0)}\int_{\lambda_i} e^{-\alpha\sum(\lambda_i-\lambda_0)^2}\prod_{(ij)}|\lambda_i - \lambda_j|$$
$$= e^{n\beta\Lambda N - nV(\lambda_0)}\alpha^{-n/4}f(n), \quad (8)$$

where the quadratic coefficient $\alpha$ is given by

$$\alpha = \frac{N}{\beta^2\bar{J}^2}\frac{\sqrt{\Lambda^2 - 2\bar{J}^2}}{\Lambda + \sqrt{\Lambda^2 - 2\bar{J}^2}}. \quad (9)$$

All of the physical parameters of the model have been extracted in the second line of (8) through a rescaling of the integration variables. What is left is the function $f(n)$, the $k = 1/2$ Mehta integral [2]. This retains the logarithmic interactions, but now carries only an $n$ dependence. As shown in Appendix A, this function is well behaved as $n \to 0$, and it does not affect the free energy at large $N$. Thus, despite the mutual proximity of the charges at the saddle point, the interactions are irrelevant in the thermodynamic limit, and $f(n)$ can be dropped in Eq. (8).

The final step of the replica technique may now be applied. We differentiate $\langle Z^n\rangle_J$ with respect to $n$ to obtain the disorder-averaged free energy,

$$\langle \ln Z\rangle_J = \lim_{n\to 0}\frac{\partial}{\partial n}\langle Z^n\rangle_J = -\left(V(\lambda_0)+\frac{1}{4}\ln(\alpha)-\beta\Lambda N\right). \tag{10}$$

The mean-spherical constraint, $\frac{\partial}{\partial\Lambda}\langle \ln Z\rangle_J = 0$, is then

$$N\beta = \frac{1}{2}\frac{\bar{J}^2}{(\Lambda^2-2\bar{J}^2)(\Lambda+\sqrt{\Lambda^2-2\bar{J}^2})} + \frac{N}{2\bar{J}^2}\{\Lambda-[\Lambda^2-2\bar{J}^2]^{1/2}\}. \tag{11}$$

In the low temperature regime, with $T < T_C = \sqrt{2}\bar{J}$ [12],

$$\Lambda = \sqrt{2}\bar{J} - \frac{\sqrt{2}\bar{J}}{8N(1-\sqrt{2}\bar{J}\beta)} + O(N^{-3/2}), \tag{12}$$

so that all three terms contribute and the constraint equation is satisfied to $O(N^1)$. In the high temperature regime, with $T > T_C$, the second term in the constraint equation is subdominant and we have

$$\frac{1}{2\bar{J}^2}\{\Lambda-[\Lambda^2-2\bar{J}^2]^{1/2}\} = \beta. \tag{13}$$

This has one solution,

$$\Lambda = \beta\bar{J}^2 + \frac{1}{2\beta}. \tag{14}$$

The final disorder-averaged free energy expression, obtained by plugging these solutions back into (10), is then given by

$$f = \begin{cases} -\sqrt{2}\bar{J} + \frac{T}{4} + \frac{T}{2}\ln\frac{\bar{J}}{\sqrt{2}T}, & \text{for} \quad T \leqslant T_C, \\ -\frac{\bar{J}^2}{2T} - \frac{T}{2}(1+\ln 2), & \text{for} \quad T \geqslant T_C. \end{cases} \tag{15}$$

The result (15) is equivalent to those of [1,8–10]. The interesting feature of the present solution is that it is characterized by a diagonal overlap saddle point, implying that there are no inherent correlations introduced between the different replicas of the system upon disorder averaging. This is apparent in the saddle point exponent in Eq. (8), which scales exactly as $n$ for each positive integer $n$. Because of this feature, Carleman's condition [13] $\sum_{n=1}^{\infty}\langle Z^n\rangle_J^{-1/2n} = \infty$ holds, which guarantees that there is at most one distribution $P(Z)$ (representing the probability of obtaining a physical sample with partition sum $Z$) that can generate the disorder-averaged, positive integer moments of $Z$, which are given in Eq. (8). A $\delta$ function distribution can generate these moments, and we thus conclude that the partition sum must be $\delta$ distributed. That is,

$$P(Z) = \delta(Z - e^{-\beta N f}), \tag{16}$$

with $f$ given by (15). This result can be heuristically understood to follow from the self-averaging nature of the eigenvalue distribution of the coupling matrix: Because, in the thermodynamic limit, the eigenvalue distribution of $J_{ij}$ approaches the semicircle distribution for nearly every realization of the disorder [2], the partition sum is expected to be, in turn, sharply distributed. The above analysis represents a rigorous demonstration that this is so. The validity of (10) and the equivalence of quenched and annealed averages for this model [14] ($\langle Z^n\rangle_J = \langle Z\rangle_J^n$) both follow immediately from (16), which is our main result. We show in Appendix B that similar conclusions also hold for the hard-spherical model of Ref. [1].

## IV. DISCUSSION

While we have shown here that the thermodynamics of the spherical spin-glass model are quite simple (in the sense that quenched and annealed disorder averages agree), the model exhibits many of the challenging characteristics that are associated with glassy systems (random couplings, a phase transition, and also off-equilibrium dynamical behavior [10,15]). Thus, although the model is Gaussian, it would be wrong to consider it trivial, and it may yet provide new insight into the properties of disordered systems. In particular, we note that the charge-mapping approach might be applied to other models, and then compared back to the solution presented here, perhaps providing a new perspective of the mechanisms that cause RSB. We feel that this prospect is a promising one, given the cleanness of the exact, ansatz-free solution that has resulted here.

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## APPENDIX A: EVALUATION OF MEHTA INTEGRAL IN $n \to 0$ LIMIT

In this section, we evaluate the $n \to 0$ limit of the Mehta integral that appears in Eq. (8). We use the fact that the Mehta integral reduces to [2]

$$\frac{1}{(2\pi)^{n/2}}\int_{\lambda_i} e^{-\sum_i \lambda_i^2/2}\prod_{(ij)}|\lambda_i-\lambda_j|^{2k} = \prod_{j=1}^{n}\frac{\Gamma(1+jk)}{\Gamma(1+k)} = \frac{1}{\Gamma(1+k)}\exp\left[\sum_{j=1}^{n}\ln\Gamma(1+jk)\right]. \tag{A1}$$

The sum in the exponent may be simplified via the formula [16]

$$\ln\Gamma(z) = \int_0^\infty \left\{(z-1)e^{-t} - \frac{e^{-t}-e^{-zt}}{1-e^{-t}}\right\}\frac{dt}{t}. \tag{A2}$$

For $k = 1/2$, the sum becomes

$$\sum_{j=1}^{n}\ln\Gamma\left(1+\frac{j}{2}\right) \sim n\int_0^\infty\left\{\frac{1}{4}e^{-t} - \frac{1}{(e^t-1)} + \frac{t}{2(e^t-1)(e^{t/2}-1)}\right\}\frac{dt}{t}. \tag{A3}$$

Each of these integrals diverge at small $t$. To evaluate them, the lower limit is replaced by a constant $\epsilon$, which we will later take to zero at the end of the calculation. After making this replacement, the first and third integrals are easily evaluated.

To evaluate the second, the integrand is multiplied by a damping factor so that one can subtract out the divergence at small $t$ without then obtaining a divergence at large $t$. We write

$$\int_{\epsilon}^{\infty} \frac{1}{(e^t-1)}\frac{dt}{t} = \lim_{\alpha\to 0}\int_0^{\infty}\left(\frac{1}{(e^t-1)} - \frac{1}{t} + \frac{1}{2}\right)\frac{e^{-\alpha t}dt}{t} + \int_{\epsilon}^{\infty}\left(\frac{1}{t} - \frac{1}{2}\right)\frac{e^{-\alpha t}dt}{t} - \int_0^{\epsilon}\left(\frac{1}{(e^t-1)} - \frac{1}{t} + \frac{1}{2}\right)\frac{dt}{t}. \quad \text{(A4)}$$

The second integral is easily evaluated and then expanded. The third is evaluated by expanding the integrand in a power series. The first is evaluated via the first Binet $\ln\Gamma$ expression [17],

$$\ln\Gamma(z) = (z - 1/2)\ln z - z + \frac{1}{2}\ln 2\pi + \int_0^{\infty}\left(\frac{1}{(e^t-1)} - \frac{1}{t} + \frac{1}{2}\right)\frac{e^{-tz}}{t}dt. \quad \text{(A5)}$$

Putting all this together and taking $\alpha \to 0$ gives

$$\int_{\epsilon}^{\infty} \frac{1}{(e^t-1)}\frac{dt}{t} \sim \frac{\gamma}{2} - \frac{1}{2}\ln\pi + \frac{1}{2\epsilon} + \frac{1}{2}\ln\epsilon. \quad \text{(A6)}$$

Finally, combining with the other terms in Eq. (A3) gives

$$\sum_{j=1}^{n}\ln\Gamma\left(1 + \frac{j}{2}\right) \sim n\left(\frac{1}{2}\ln\pi - \frac{3}{4}\gamma\right). \quad \text{(A7)}$$

Therefore the $n \to 0$, $k = 1/2$ Mehta integral goes to

$$\frac{1}{(2\pi)^{n/2}}\int_{\lambda_i} e^{-\sum_i \lambda_i^2/2}\prod_{(ij)}|\lambda_i - \lambda_j|^{2k} \sim \frac{1}{\Gamma(3/2)}\left\{1 + n\left(\frac{1}{2}\ln\pi - \frac{3}{4}\gamma\right)\right\} + O(n^2), \quad \text{(A8)}$$

which is well behaved.

## APPENDIX B: HARD-SPHERICAL MODEL ANALYSIS

The absence of replica correlations in the mean-spherical spin glass (as seen above) runs counter to the intuition one obtains from the hard-spherical model, which is characterized by a strictly off-diagonal overlap. One might wonder if it is the presence of variable diagonal elements in the mean-spherical system that results in its simple thermodynamics, and whether the hard-spherical model, being further constrained, might exhibit a more complicated behavior under disorder averaging. We show here that this is not the case: In contrast to a naive interpretation of the RS solution, quenched and annealed averages also agree within the hard-spherical system.

Following [1], the analysis for the hard-spherical spin glass begins similarly to that above. We write

$$\langle Z^n\rangle_J = \int_{s,\Lambda}\exp\left[\frac{\beta^2\bar{J}^2}{2N}\sum_{\alpha\neq\beta}\left(\sum_i s_i^{\alpha}s_i^{\beta}\right)^2 - \beta\Lambda\left(\sum_{i,\alpha}s_i^{\alpha 2} - nN\right) + \frac{nN\beta^2\bar{J}^2}{2}\right], \quad \text{(B1)}$$

where $\Lambda$ is now formally to be integrated over in order to enforce the hard-spherical constraint. Applying a Hubbard-Stratonovich transformation once again results in an exponent that depends only on the eigenvalues of $Q_{\alpha\beta}$, but this is now constrained to be off-diagonal,

$$\langle Z^n\rangle_J = \int_{Q,\Lambda}\exp\left[\frac{nN\beta^2\bar{J}^2}{2} + nN\beta\Lambda - \sum_i V(\lambda_i)\right]. \quad \text{(B2)}$$

In the thermodynamic limit, the partition sum (B2) will be dominated by those configurations that minimize the exponent. To evaluate the sum, we again switch to an integration over the eigenvalues and eigenvectors of $Q_{\alpha\beta}$. The integration over eigenvalues is now subject to the traceless constraint, and the orientation averages are further restricted so as to maintain an off-diagonal coupling matrix. The orientation averages do not affect the free energy here [18]. Generalizing slightly, we consider the partition function subject to the constraint that the eigenvalues sum to $s$, writing (with terms independent of the $\lambda_i$ temporarily suppressed)

$$\langle Z^n(s)\rangle \equiv \int_{\lambda_i}\delta\left(\sum_i\lambda_i - s\right)\exp\left[-\sum_i V(\lambda_i)\right]. \quad \text{(B3)}$$

The inverse Laplace transform of $\langle Z^n(s)\rangle$ is given by

$$\langle Y^n(t)\rangle = \int_{\lambda_i,s}\langle Z^n(s)\rangle e^{st} = \int_{\lambda_i}\exp\left[-\sum_i\{V(\lambda_i) - t\lambda_i\}\right]. \quad \text{(B4)}$$

The eigenvalue integrations are now independent, and they are free to each sit at the same optimal saddle point location where

$$\partial_\lambda V|_{\lambda_0(t)} = t. \quad \text{(B5)}$$

This effectively sends the $V$ to its Legendre transform. The partition sum is obtained by inverting the Laplace transform with $s$ taken to zero,

$$\langle Z^n\rangle_J = \int_{\lambda_i,\Lambda,t}\langle Y^n(t)\rangle e^{t(s\to 0)} = \min_{\Lambda,\lambda}\exp\left[n\left\{\frac{N\beta^2\bar{J}^2}{2} + N\beta\Lambda + V - \lambda\partial_\lambda V\right\}\right]. \quad \text{(B6)}$$

It is important to note that the exponent is actually to be minimized here, in contrast to most spin-glass problems where the physical solutions are obtained by maximizing the free energy in the $n \to 0$ limit. The $n$ dependence is trivial in this model, and consequently, there can be no peculiarities in the analysis for any value of $n$. Carrying out the minimization procedure is straightforward, and the free energy expressions (15) are once again obtained. Importantly, the exponent in Eq. (B6) scales linearly with $n$: Both the mean and hard-spherical systems satisfy equivalence of quenched and annealed disorder averages.

We can now relate the exact solution to that obtained via the RS ansatz. The RS ansatz sends each off-diagonal element of $Q_{\alpha\beta}$ to $Q$. The eigenvalues of the resulting matrix are then given by

$$\begin{array}{ll} -Q & (n-1)\text{-fold degenerate,} \\ (n-1)Q & 1\text{-fold degenerate.} \end{array} \quad \text{(B7)}$$

Plugging into (B2) gives

$$\sum_i V(\lambda_i) = (n-1)V(-Q) + V[(n-1)Q] = n\{V(-Q) + QV'(-Q)\} + O(n^3), \quad \text{(B8)}$$

which agrees to $O(n)$ with the exact saddle point upon sending $Q \to -Q$. However, the corrections at higher orders in $n$ in Eq. (B8) are not consistent with our exact analysis. These corrections relate to higher order cumulant averages of the free energy with respect to the disorder averaging. To resolve this discrepancy, we have evaluated the second order fluctuations of the free energy about the replica symmetric saddle point, and have found that the RS saddle is stable only in the $n \to 0$ limit [19]: The RS ansatz fails at finite $n$, and it can only be used to evaluate the first cumulant average of $\ln Z$.


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