

**Nonlocal product rules for percolation**Saulo D. S. Reis,<sup>\*</sup> André A. Moreira,<sup>†</sup> and José S. Andrade Jr.<sup>‡</sup>*Departamento de Física, Universidade Federal do Ceará, 60451-970 Fortaleza, Ceará, Brazil*

(Received 2 December 2011; published 11 April 2012)

Despite original claims of a first-order transition in the product rule model proposed by Achlioptas *et al.* [*Science* **323**, 1453 (2009)], recent studies indicate that this percolation model, in fact, displays a continuous transition. The distinctive scaling properties of the model at criticality, however, strongly suggest that it should belong to a different universality class than ordinary percolation. Here we introduce a generalization of the product rule that reveals the effect of nonlocality on the critical behavior of the percolation process. Precisely, pairs of unoccupied bonds are chosen according to a probability that decays as a power law of their Manhattan distance, and only that bond connecting clusters whose product of their sizes is the smallest becomes occupied. Interestingly, our results for two-dimensional lattices at criticality shows that the power-law exponent of the product rule has a significant influence on the finite-size scaling exponents for the spanning cluster, the conducting backbone, and the cutting bonds of the system. In all three cases, we observe a clear transition from ordinary to (nonlocal) explosive percolation exponents.

DOI: [10.1103/PhysRevE.85.041112](https://doi.org/10.1103/PhysRevE.85.041112)

PACS number(s): 64.60.ah, 64.60.al, 89.75.Da

**I. INTRODUCTION**

The percolation paradigm represents a formidable example where a simple geometrical construction leads to profound concepts in statistical physics, with special emphasis on phase transitions, and real applications in science and technology [1–6]. Standard percolation processes are based on local rules, since they are accomplished through random allocation of sites or bonds, therefore disregarding any spatial correlation or global information involved in the occupation of other elements on the lattice. However, in the case of long-range spatially correlated percolation [7–11], the probability for a site to be occupied depends on the occupancy of other sites. Moreover, it has been shown that spatial long-range correlations in site occupancy can give rise to important changes on the structural characteristics of the spanning cluster as well as its corresponding conducting backbone [9]. These changes are strong enough to modify the scaling exponents of traditional (local) percolation.

Recently, a different percolation model has been proposed, the so-called product rule (PR) percolation, in terms of a bond occupation process that is essentially nonlocal [12]. In this model, at each step, two unoccupied bonds are randomly chosen and associated with weights given by the product of the cluster sizes they would potentially connect. Only that bond which has the smallest weight becomes occupied. By comparison with the traditional percolation model [1], the PR model presents a more abrupt transition when applied to different network topologies [13–28]. As potential applications, the PR model has been recently associated to the growth dynamics of Protein Homology Networks [29] as well as to the formation of bundles of single-walled nanotubes [30].

Regardless of initial claims of a first-order transition in the PR model [12], however, recent analytical and numerical works [31–34] have demonstrated that the alleged “explosive

percolation” process actually displays a continuous, i.e., a second-order phase transition. This apparent drawback of the PR model has been somehow overstated, in the sense that the model proposed by Achlioptas *et al.* [12] certainly represents an original and interesting contribution to the field. For instance, much less importance has been given to the nonlocal attributes of the PR algorithm. As a consequence of this nonlocality, the model exhibits a percolation transition that, although continuous in nature, seems to belong to a different universality class than ordinary percolation [15,22,28]. Under this framework, the adjacent edge rule (AER) model [35] represents a particular case of the PR process that is analytically tractable, since the selection is restricted to two adjacent bonds. When applied to random graphs, this model still displays a more abrupt transition and different scaling properties than ordinary percolation.

In this work we introduce a generalization of the PR model in which the range of its nonlocal features can be systematically controlled. This is carried out imposing that pairs of bonds for selection are randomly chosen according to a probability that decays as a power law of their Manhattan distance, namely the distance measured as the number of connections separating the sites in a regular lattice. This physically plausible assumption is inspired on a geographical model for complex networks where long-ranged shortcuts are incorporated to regular lattices. Such a conceptual construction has been extensively used as a way to explain the emergence of optimal navigation and efficient transport in small-world systems [36–41]. As a consequence of the selection rule adopted here, we show that the scaling properties of the system become dependent on the specific value of the corresponding power-law exponent. A transition is then revealed from the traditional to the PR percolation behavior. Moreover, the results of our extensive numerical simulations provide strong evidence for the fact that the AER model, when applied to regular lattices in two dimensions, falls in the same universality class as ordinary percolation. This paper is organized as follows. In Sec. II we introduce the generalized PR model. The results from computational simulations are presented and discussed in Sec. III. Conclusions are drawn in Sec. IV.

<sup>\*</sup> saulo@fisica.ufc.br<sup>†</sup> auto@fisica.ufc.br<sup>‡</sup> soares@fisica.ufc.br

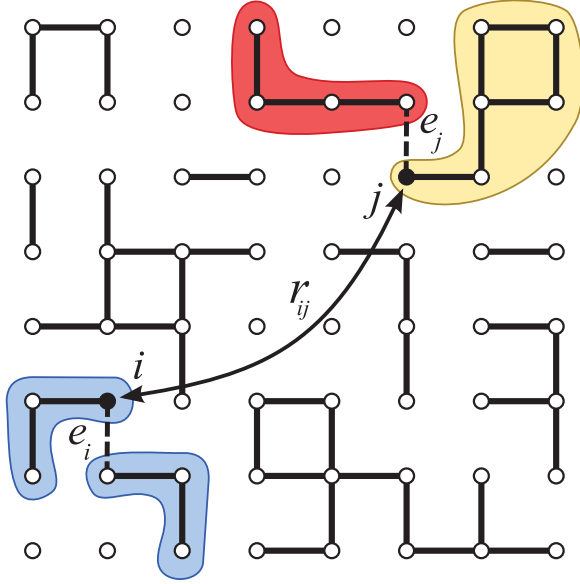


FIG. 1. (Color online) Pair of unoccupied bonds  $e_i$  and  $e_j$  (dashed lines) randomly selected for the application of the product rule, according to the probability  $P(r_{ij}) \sim r_{ij}^{-\alpha}$ , where  $r_{ij}$  is the Manhattan distance between sites  $i$  and  $j$  (black circles), and  $\alpha$  is a variable exponent. Following the PR model, the bond  $e_i$ , merging the two clusters in blue (with three sites each), becomes occupied. The bond  $e_j$  would merge the clusters in red (four sites) and yellow (six sites), but remains unoccupied.

## II. MODEL FORMULATION

As shown in Fig. 1, our bond percolation process takes place on a square lattice of size  $L$ . At each step, two sites  $i$  and  $j$  are randomly selected with probability  $P(r_{ij}) \sim r_{ij}^{-\alpha}$ , where  $r_{ij}$  is the Manhattan distance between  $i$  and  $j$ , measured as the number of connections separating these sites in the underlying regular lattice [42]. From each site  $i$  and  $j$ , one bond is then selected among its four adjacent edges, namely  $e_i$  and  $e_j$ , respectively. If at least one of these two bonds is already occupied, the entire process of selection is restarted. If not, following the product rule, weights are assigned to each of these bonds, in proportionality to the product of the size (number of sites) of the clusters they would potentially connect. In the case a bond connects two sites in the same cluster, the weight is equal to the square of the cluster size. The bond associated with the smallest weight becomes occupied, while the other stays unoccupied, but can be selected again in later steps. The percolation process stops when one among all clusters, namely the spanning cluster, connects the lattice from top to bottom [1]. At that point, the fraction  $p$  of occupied bonds corresponds to the percolation threshold  $p_c$ .

Our model displays two distinct limiting behaviors, depending on the exponent  $\alpha$ . For  $\alpha = 0$ , we recover the usual PR, for which the preliminary random selection of the bonds  $e_1$  and  $e_2$  constitutes a highly nonlocal process [14,31]. In the limit of  $\alpha \rightarrow \infty$ , the bonds  $e_1$  and  $e_2$  are always adjacent, which corresponds to the AER process proposed in Ref. [35], but applied here to regular lattices. Although more spatially restricted than the PR process, the AER is still nonlocal, since it requires information about the masses of the joining

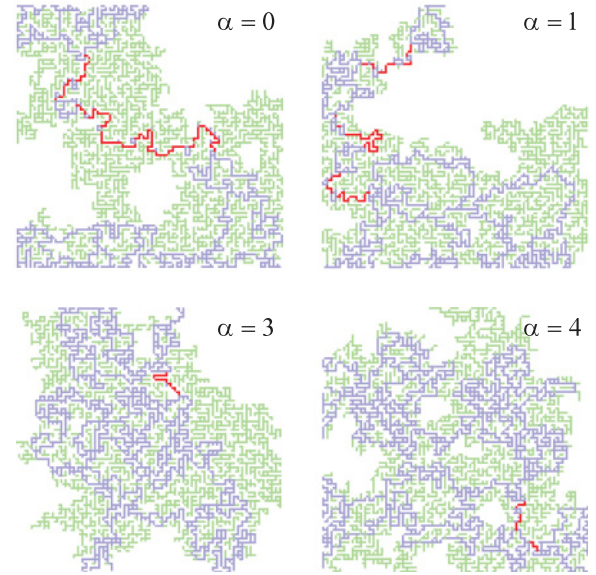


FIG. 2. (Color online) Snapshots of the largest cluster at  $p_c$  for different values of the exponent  $\alpha$ , and a lattice size  $L = 64$ . The bonds forming the conducting backbone are in blue, the cutting bonds are in red, and the remaining bonds of the largest cluster are presented in green. Although no major difference can be observed on the mass  $M_{\text{clus}}$  of the largest cluster, one can notice that the conducting backbone occupies a larger fraction of the largest cluster as  $\alpha$  increases, leading to a substantial decrease on the number of cutting bonds  $M_{\text{cut}}$ .

clusters [31]. As we show later, the finite low dimensionality of the square lattice employed here attenuates even further the already weaker nonlocal features of the AER process.

## III. RESULTS AND DISCUSSION

Figure 2 shows typical realizations of the largest cluster at the critical point  $p_c$  for different values of  $\alpha$ . For  $\alpha = 0$ , we obtain  $p_c = 0.527 \pm 0.001$ , which is in good agreement with previous simulation results of the PR on the square lattice [21,22]. Moreover, we find that  $p_c$  decreases smoothly and monotonically with  $\alpha$  from this value to  $0.522 \pm 0.001$  at  $\alpha = 4$  (not shown). Next we apply the burning algorithm [43] to compute the mass of the spanning cluster  $M_{\text{clus}}$ , the mass of its conducting backbone  $M_{\text{back}}$ , and the mass (number)  $M_{\text{cut}}$  of cutting bonds. The last ones, also called red bonds, if removed, would break the spanning cluster in two, therefore destroying the global connectivity of the system.

Our results show that, regardless of the value of  $\alpha$ , all these critical quantities scale with the system size  $L$  as typical power laws,  $M_{\text{back}} \sim L^{d_{\text{back}}}$ ,  $M_{\text{cut}} \sim L^{d_{\text{cut}}}$  (see Fig. 3), and  $M_{\text{clus}} \sim L^{d_{\text{clus}}}$  (not shown), where  $d_{\text{back}}$ ,  $d_{\text{cut}}$ , and  $d_{\text{clus}}$  are the fractal dimensions of the conducting backbone, the cutting bonds, and the largest cluster, respectively. In Figs. 4(a)–4(c) we show that all these exponents exhibit a monotonic variation with  $\alpha$ , going from a saturation regime of (nonlocal) explosive percolation at  $\alpha = 0$  to another compatible with ordinary bond percolation (BP) at sufficiently large values of  $\alpha$ . Accordingly, for  $\alpha = 0$ , we recover the previously numerically calculated values of  $d_{\text{clus}} = 1.96 \pm 0.01$  [15,16],  $d_{\text{back}} = 1.52 \pm 0.03$ , and  $d_{\text{cut}} = 1.02$  [28]. In all three cases, by increasing  $\alpha$ , a

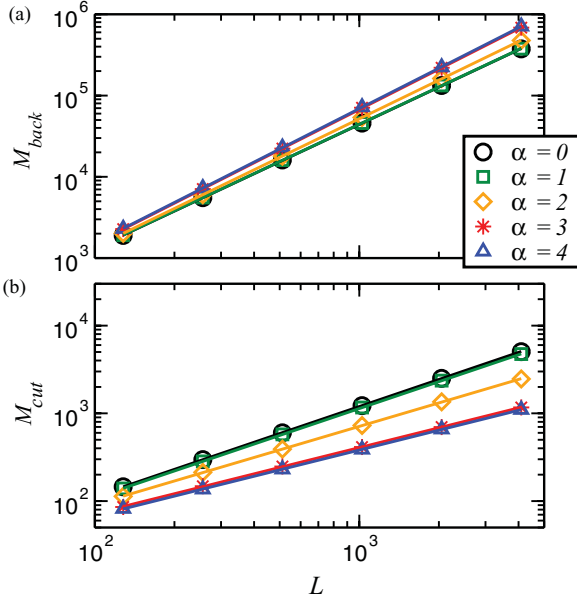


FIG. 3. (Color online) (a) Log-log dependence of the mass of the conducting backbone  $M_{\text{back}}$  on the system size for different values of the exponent  $\alpha$ . (b) The same as in (a), but for the number of cutting bonds  $M_{\text{cut}}$ . In both cases and for all values of  $\alpha$ , the evidence of scaling behavior substantiates the calculation of the fractal dimensions  $d_{\text{back}}$  and  $d_{\text{cut}}$  as the slopes of the corresponding straight lines that are best-fitted to the simulation data. All quantities are averaged over at least 2500 realizations precisely at the point in which the largest cluster appears.

crossover from PR to BP takes place in the interval  $1 < \alpha < 3$ . More precisely,  $d_{\text{clus}}$  decreases in this interval and starts fluctuating around 1.89 for  $\alpha > 3$  [see Fig. 4(a)], in agreement with the classical two-dimensional (2D) value of  $91/48$  [1]. After increasing in the interval  $1 < \alpha < 3$ , the exponents  $d_{\text{back}}$  and  $d_{\text{cut}}$  remain practically constant for  $\alpha > 3$ , around 1.64 [2] [see Fig. 4(b)] and 0.75 [1] [see Fig. 4(c)], respectively. These values are fully compatible with previously reported numerical calculations for ordinary (local) percolation in 2D. The variations of the exponents  $d_{\text{clus}}$  and  $d_{\text{back}}$  within  $1 < \alpha < 3$  reflect relevant changes in compactness of the spanning cluster and its conducting backbone. Although the spanning cluster becomes less compact as  $\alpha$  increases ( $d_{\text{clus}}$  decreases), the mass of the backbone  $M_{\text{back}}$  tends to occupy a larger fraction of  $M_{\text{clus}}$ , since the dimension  $d_{\text{back}}$  increases in the same interval of  $\alpha$  values, as shown in Fig. 3. In these circumstances, a more compact conducting backbone implies a smaller number of cutting bones (see Fig. 3), therefore explaining the decrease in the exponent  $d_{\text{cut}}$ .

Next we provide some analytical arguments that indicate how nonlocal features are introduced in our percolation model through the power-law probability  $P(r_{ij})$ . We first consider the average distance between all pairs of sites in an empty lattice,  $\langle r \rangle = \sum_{r=1}^L r N_r r^{-\alpha} / \sum_{r=1}^L N_r r^{-\alpha}$ , where  $N_r = 4r$  is the number of sites that are at a Manhattan distance  $r$  from a given site in the square lattice. Approximating the sum by an integral, we obtain that  $\langle r \rangle \sim \int_1^L r^{2-\alpha} dr$ . It follows that, for  $\alpha \geq 3$  and sufficiently large lattice sizes,  $\langle r \rangle$  is always finite, and the scaling exponents should be those of standard

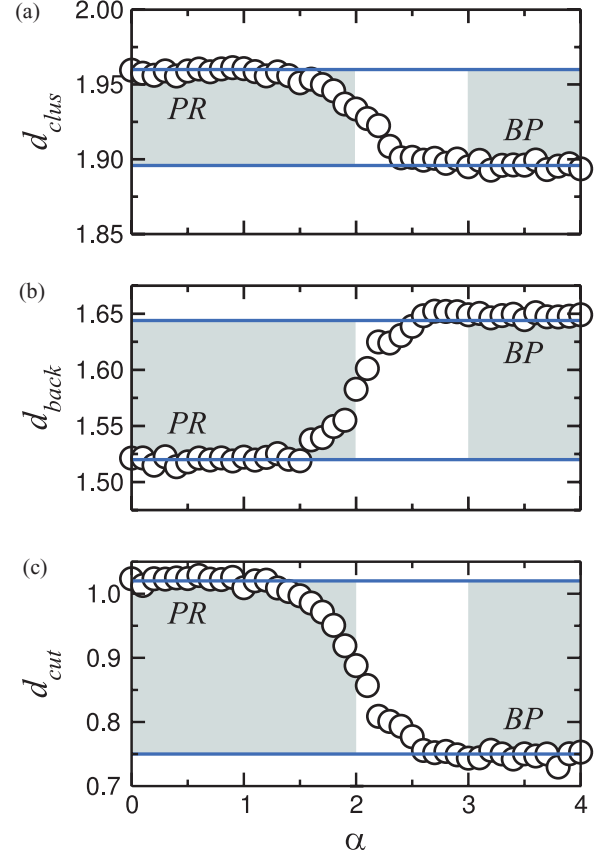


FIG. 4. (Color online) Dependence on the exponent  $\alpha$  of the size-scaling exponents for (a) the mass of the spanning cluster  $d_{\text{clus}}$ , (b) the mass of the conducting backbone  $d_{\text{back}}$ , and (c) the number of cutting bonds  $d_{\text{cut}}$ . In all cases, a crossover can be observed in the interval  $1 < \alpha < 3$  from a regime of nonlocal explosive percolation at  $\alpha = 0$ , to a regime that is compatible with ordinary bond percolation (BP), at sufficiently large values of  $\alpha$ . The solid blue lines correspond to  $d_{\text{clus}} = 1.96$  [15,16] and  $91/48$  [1] in (a),  $d_{\text{back}} = 1.52$  [28] and  $1.64$  [2] in (b), and  $d_{\text{cut}} = 1.02$  [28] and  $0.75$  [1] in (c).

2D percolation. As a consequence, the effect of nonlocality on the scaling properties of the system would only play a role for  $\alpha < 3$ . In addition, distinct nonlocal behaviors should be expected for the intervals  $\alpha < 2$  and  $2 \leq \alpha < 3$ . For  $\alpha < 2$ ,  $\langle r \rangle$  is limited by the network size, leading to  $\langle r \rangle \sim L$ , implying that the scaling exponents of PR in 2D should be recovered. The mismatch observed in Fig. 4 between expected and numerically calculated exponents for this interval are certainly due to strong finite-size effects for  $\alpha \approx 2$ . The results displayed in Fig. 4 suggest that the scaling exponents vary continuously in the interval  $2 \leq \alpha < 3$ . The fact that the average distance scales as  $\langle r \rangle \sim L^{3-\alpha}$  provides support to the conjecture that nonlocal effects are continuously vanishing in this range.

In order to better confirm the role of nonlocality on the PR process, we perform additional simulations in the two limits of the model at higher dimensions, namely for  $\alpha = 0$ , which corresponds to the original PR process, and for the AER process,  $\alpha \rightarrow \infty$ . In these cases, improved performance can be achieved by adopting the so-called jump method to analyze the behavior of the order parameter  $M_{\text{clus}}$  [27,44,45]. For each realization, we compute the average fraction  $p$  of

TABLE I. Estimated values of the percolation threshold  $p_c$  and the scaling exponent  $d_{\text{clus}}$  for hypercubic lattices of dimension  $D$  calculated using the jump method [27,44,45]. The presented values correspond to averages over a minimum of 2500 realizations of systems with sizes up to  $L = 4096$  ( $D = 2$ ), 256 ( $D = 3$ ), 64 ( $D = 4$ ), 28 ( $D = 5$ ), and 16 ( $D = 6$ ).

$D$	$p_{c,AP}$	$p_{c,AER}$	$d_{\text{clus-AP}}$	$d_{\text{clus-AER}}$	$d_{\text{clus-Classical}}$ [1]
2	$0.526550 \pm 0.000005$	$0.52007 \pm 0.00001$	$1.955 \pm 0.002$	$1.899 \pm 0.001$	91/48
3	$0.322096 \pm 0.000001$	$0.285360 \pm 0.000008$	$2.788 \pm 0.003$	$2.530 \pm 0.003$	2.53
4	$0.234160 \pm 0.000003$	$0.202163 \pm 0.000004$	$3.665 \pm 0.009$	$3.079 \pm 0.005$	3.06
5	$0.184656 \pm 0.000006$	$0.160454 \pm 0.000004$	$4.61 \pm 0.01$	$3.59 \pm 0.04$	3.54
6	$0.152642 \pm 0.000005$	$0.134113 \pm 0.000002$	$5.558 \pm 0.005$	$4.46 \pm 0.01$	4

occupied bonds at which a jump takes place, defined as the maximum change on  $M_{\text{clus}}$  from the occupation of a single bond. This value of  $p$  corresponds to the percolation threshold  $p_c$ . The results for  $p_c$  and  $d_{\text{clus}}$  in both limits and different dimensions are summarized in Table I. Interestingly, the discrepancy between the fractal dimensions calculated for PR and AER models increases substantially with lattice dimensionality. Moreover, our calculations suggest that the resemblance between regular BP and the limiting case  $\alpha \rightarrow \infty$  stands only up to five dimensions, when compared with previous results from the literature [1,2]. The difference found between the exponents  $d_{\text{clus}}$  of BP and AER for  $D > 5$  can be explained in terms of the nonlocal features involved in the PR process. As reported in Refs. [31,35], this model displays an unusual scaling behavior. As a consequence, an upper critical dimension, above which the scaling exponents become equal to mean-field values, may not be observed in this case.

#### IV. CONCLUSION

In summary, we have proposed a generalization of the PR model where the range of nonlocality in the percolation process can be explicitly tuned. Our results show that this model displays a rich variety of scaling behaviors, going from ordinary to nonlocal explosive percolation. We expect our PR model, since it is based on a geographical choice of bond pairs, to provide relevant physical insights into the role of nonlocality on the critical properties of percolation.

#### ACKNOWLEDGMENTS

We thank the Brazilian Agencies CNPq, CAPES, FUNCAP and FINEP, the FUNCAP/CNPq Pronex grant, and the National Institute of Science and Technology for Complex Systems in Brazil for financial support.

- 
- [1] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1992).
- [2] M. Sahimi, *Applications of Percolation Theory* (Taylor & Francis, London, 1994).
- [3] M. Sahimi and S. Arbabi, *Phys. Rev. Lett.* **68**, 608 (1992).
- [4] P. Grassberger and Y.-C. Zhang, *Physica A* **224**, 169 (1999).
- [5] J. S. Andrade, L. S. Lucena, A. M. Alencar, and J. E. Freitas, *Physica A* **238**, 163 (1997).
- [6] H. E. Stanley, J. S. Andrade, S. Havlin, H. A. Makse, and B. Suki, *Physica A* **266**, 5 (1999).
- [7] S. Havlin, R. Blumberg Selinger, M. Schwartz, H. E. Stanley, and A. Bunde, *Phys. Rev. Lett.* **61**, 1438 (1988).
- [8] S. Havlin, M. Schwartz, R. Blumberg Selinger, A. Bunde, and H. E. Stanley, *Phys. Rev. A* **40**, 1717 (1989).
- [9] S. Prakash, S. Havlin, M. Schwartz, and H. E. Stanley, *Phys. Rev. A* **46**, R1724 (1992).
- [10] H. A. Makse, S. Havlin, M. Schwartz, and H. E. Stanley, *Phys. Rev. E* **53**, 5445 (1996).
- [11] H. A. Makse, G. W. Davies, S. Havlin, P. C. Ivanov, P. R. King, and H. E. Stanley, *Phys. Rev. E* **54**, 3129 (1996).
- [12] D. Achlioptas, R. M. D'Souza, and J. Spencer, *Science* **323**, 1453 (2009).
- [13] Y. S. Cho, J. S. Kim, J. Park, B. Kahng, and D. Kim, *Phys. Rev. Lett.* **103**, 135702 (2009).
- [14] E. J. Friedman and A. S. Landsberg, *Phys. Rev. Lett.* **103**, 255701 (2009).
- [15] R. M. Ziff, *Phys. Rev. Lett.* **103**, 045701 (2009).
- [16] F. Radicchi and S. Fortunato, *Phys. Rev. Lett.* **103**, 168701 (2009).
- [17] Y. S. Cho, B. Kahng, and D. Kim, *Phys. Rev. E* **81**, R030103 (2010).
- [18] Y. S. Cho, S.-W. Kim, J. D. Noh, B. Kahng, and D. Kim, *Phys. Rev. E* **82**, 042102 (2010).
- [19] A. A. Moreira, E. A. Oliveira, S. D. S. Reis, H. J. Herrmann, and J. S. Andrade, *Phys. Rev. E* **81**, 040101(R) (2010).
- [20] N. A. M. Araújo and H. J. Herrmann, *Phys. Rev. Lett.* **105**, 035701 (2010).
- [21] F. Radicchi and S. Fortunato, *Phys. Rev. E* **81**, 036110 (2010).
- [22] R. M. Ziff, *Phys. Rev. E* **82**, 051105 (2010).
- [23] N. A. M. Araújo, J. S. Andrade, R. M. Ziff, and H. J. Herrmann, *Phys. Rev. Lett.* **106**, 095703 (2011).
- [24] W. Chen and R. M. D'Souza, *Phys. Rev. Lett.* **106**, 115701 (2011).
- [25] J. Gómez-Gardeñes, S. Gómez, A. Arenas, and Y. Moreno, *Phys. Rev. Lett.* **106**, 128701 (2011).
- [26] H. Hooyberghs and B. Van Schaeybroeck, *Phys. Rev. E* **83**, 032101 (2011).
- [27] S. S. Manna and A. Chatterjee, *Physica A* **390**, 177 (2011).
- [28] J. S. Andrade, H. J. Herrmann, A. A. Moreira, and C. L. N. Oliveira, *Phys. Rev. E* **83**, 031133 (2011).



- [29] H. D. Rozenfeld, L. K. Gallos, and H. A. Makse, *Eur. Phys. J. B* **75**, 305 (2010).
- [30] Y. Kim, Y. K. Yun, and S. H. Yook, *Phys. Rev. E* **82**, 061105 (2010).
- [31] P. Grassberger, C. Christensen, G. Bizhani, S.-W. Son, and M. Paczuski, *Phys. Rev. Lett.* **106**, 225701 (2011).
- [32] R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Phys. Rev. Lett.* **105**, 255701 (2010).
- [33] H. K. Lee, B. J. Kim, and H. Park, *Phys. Rev. E* **84**, 020101(R) (2011).
- [34] O. Riordan and L. Warnke, *Science* **333**, 322 (2011).
- [35] R. M. D'Souza and M. Mitzenmacher, *Phys. Rev. Lett.* **104**, 195702 (2010).
- [36] J. Kleinberg, *Nature (London)* **406**, 845 (2000).
- [37] M. Barthelemy, *Phys. Rep.* **499**, 1 (2011).
- [38] G. Li, S. D. S. Reis, A. A. Moreira, S. Havlin, H. E. Stanley, and J. S. Andrade, *Phys. Rev. Lett.* **104**, 018701 (2010).
- [39] Y. Li, D. Zhou, Y. Hu, J. Zhang, and Z. Di, *Europhys. Lett.* **92**, 58002 (2010).
- [40] M. P. Viana and L. d. F. Costa, *Phys. Lett. A* **375**, 1626 (2011).
- [41] L. K. Gallos, H. A. Makse, and M. Sigman, *Proc. Natl. Acad. Sci. USA* **109**, 2825 (2012).
- [42] In fact, to account for the number  $N_r$  of neighbors that are at a Manhattan distance  $r$  from a particular site in a square lattice ( $N_r = 4r$ ), the random distance  $r$  must be generated following a power-law distribution proportional to  $r^{(1-\alpha)}$ .
- [43] H. J. Herrmann, D. C. Hong, and H. E. Stanley, *J. Phys. A* **17**, L261 (1984).
- [44] J. Nagler, A. Levina, and M. Timme, *Nature Phys.* **7**, 265 (2011).
- [45] K. J. Schrenk, N. A. M. Araújo, and H. J. Herrmann, *Phys. Rev. E* **84**, 041136 (2011).