Spin glasses in the nonextensive regime

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Spin systems with long-range interactions are "nonextensive" if the strength of the interactions falls off sufficiently slowly with distance. It has been conjectured for ferromagnets and, more recently, for spin glasses that, everywhere in the nonextensive regime, the free energy is exactly equal to that for the infinite range model in which the characteristic strength of the interaction is independent of distance. In this paper we present the results of Monte Carlo simulations of the one-dimensional long-range spin glasses in the nonextensive regime. Using finite-size scaling, our results for the transition temperatures are consistent with this prediction. We also propose and provide numerical evidence for an analogous result for *diluted* long-range spin glasses in which the coordination number is finite, namely, that the transition temperature throughout the nonextensive regime is equal to that of the infinite-range model known as the Viana-Bray model.

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I. INTRODUCTION

In the theory of phase transitions, it is often helpful to study models in a range of dimensions from above the "upper critical dimension" d_u, where mean-field critical behavior is valid, to below the "lower critical dimension" d_l, where fluctuations destroy the transition. For Ising spin glasses $d_l \simeq 2.5$ [1] and $d_u = 6$. However, it has been difficult to cover this broad range numerically for spin glasses since d_u is quite large, and slow dynamics prevents equilibration at low temperatures when the number of spins N $(=L^d)$ is greater than a few thousand. It follows that at and above d_u, one cannot study a sufficient range of linear sizes L to perform the necessary finite-size scaling (FSS) analysis.

As a result, there has been a lot of recent attention on long-range models in one dimension, in which the interactions fall off with a power of the distance. Such models have a long history going back to Dyson [2,3], who considered a ferromagnet with interactions J_{ij} falling off like $1/r^\sigma$ and found a paramagnet-ferromagnet transition for $\sigma \leqslant 2$. Kotliar *et al.* [4] studied the spin glass version of this model, which has received a lot of attention numerically in the last few years [5–9].

Varying the power σ in the long-range spin glass model, one has a range of behavior similar that obtained by varying the dimension in short-range models; namely, there is a "lower critical value" $\sigma_l = 1$ [10], *above* which there is no transition at finite temperature, and an "upper critical value" $\sigma_u = 2/3$ [4], *below* which the transition has mean-field critical exponents. Note that *increasing* σ makes the interactions more short range and so corresponds to *decreasing* d.

A precise connection between d for short-range models and σ for long-range models can be made in the mean-field region ($d > 6$ or $1/2 < \sigma < 2/3$), namely [11],

$$d = \frac{2}{2\sigma - 1} \quad (d > 6,\ 1/2 < \sigma < 2/3). \tag{1}$$

This mapping shows that $d \to \infty$ for $\sigma \to 1/2$. Since the transition temperature in mean-field theory is given by

$$\left(T_c^{MF}\right)^2 = \sum_j \left[J_{ij}^2\right]_{\rm av}, \tag{2}$$

we see that for smaller values of σ, i.e., $0 \leqslant \sigma \leqslant 1/2$, the strength of the interactions has to be scaled with an inverse power of the system size to obtain a sensible thermodynamic limit. We call this regime "nonextensive." The extreme limit of this region, $\sigma = 0$, is the Sherrington-Kirkpatrick (SK) model [12], which is "infinite range." To complete the picture of the one-dimensional long-range spin glass model, in this paper we study the nonextensive regime ($0 \leqslant \sigma < 1/2$).

The nonextensive regime for ferromagnets has already been investigated [13,14]. This work shows that the behavior in the whole nonextensive regime is the same, with a suitable rescaling of the interactions, as that of the infinite-range ferromagnet in which every spin interacts equally with every other spin, i.e., $\sigma = 0$. We give intuitive arguments for this in the Appendix.

It is interesting to ask if the same is true for spin glasses. In a recent paper Mori [15] has claimed that this is so; i.e., for all $0 \leqslant \sigma < 1/2$ the behavior is the same as that of the SK model ($\sigma = 0$), provided the interactions are scaled with system size so that $\sum_{j\neq i}[J_{ij}^2]_{\rm av}$ is set to the same value for all σ. However, this argument is just at the level of replicating the Hamiltonian, so it becomes translationally invariant, and then arguing that the earlier work for ferromagnets can be taken over directly. While plausible, this result is by no means rigorous, and so we test it here by Monte Carlo simulations.

One of the models we simulate here is the usual one in which every spin interacts with every other spin. However, it is also interesting to carry out the same study for a diluted model [7] with a fixed average coordination number z. This model has received a lot of attention recently because the computer time per sweep only varies as Nz (rather than N^2 for the undiluted model), so it can be simulated much more efficiently than the undiluted model for large N. The diluted model with $\sigma = 0$ is called the Viana-Bray [16] model. It corresponds to a spin glass on a random graph, the exact solution of which is expected to be the Bethe-Peierls approximation. By analogy with Mori's proposal, we suggest here that the behavior of the diluted spin glass model is identical to that of the Viana-Bray model $\sigma = 0$ everywhere in the nonextensive region ($0 \leqslant \sigma < 1/2$). We shall also provide numerical evidence for this.

We should emphasize that *universal* quantities, such as critical exponents, are expected to be the same everywhere in both the mean-field ($1/2 < \sigma < 2/3$) and nonextensive ($0 \leqslant \sigma < 1/2$) regimes. The claim that we test is that *all* the behaviors of these models (not just the critical behavior) are *identical* for all σ in the nonextensive regime, at least in the thermodynamic limit. We therefore need to look at *nonuniversal* quantities and focus here on one particularly convenient quantity, the value of the transition temperature T_c.

This paper is organized as follows: In Sec. II we describe the models used in the simulations and give their corresponding mean-field transition temperatures. In Sec. III we give the details of the Monte Carlo simulations and FSS analysis. The results are given in Sec. IV, and our conclusions are summarized in Sec. V. The Appendix provides an intuitive explanation of why the behavior of the ferromagnet is independent of σ in the nonextensive regime.

II. MODELS

The Hamiltonian that we study is

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij} S_i S_j, \tag{3}$$

where S_i ($i = 1,2,\ldots,L$) are Ising spins which take values ± 1 and J_{ij} are statistically independent, quenched, random variables. The mean is taken to be zero, and the variance satisfies

$$\left[J_{ij}^2\right]_{\text{av}} \propto r_{ij}^{-2\sigma}, \tag{4}$$

where, for the distance r_{ij}, we put the sites on a ring and take the chord distance between sites i and j [17], i.e.,

$$r_{ij} = \frac{L}{\pi} \sin\left(\frac{\pi |i-j|}{L}\right). \tag{5}$$

The form of the distribution of J_{ij} is different for the undiluted and diluted models. For the undiluted case the distribution of J_{ij} is Gaussian,

$$P(J_{ij}) = \frac{1}{\sqrt{2\pi}\,\Delta J_{ij}} \exp\left(\frac{-J_{ij}^2}{2(\Delta J_{ij})^2}\right) \quad \text{(undiluted)}, \tag{6}$$

where the variance is given by

$$(\Delta J_{ij})^2 = \frac{C^2}{r_{ij}^{2\sigma}}, \tag{7}$$

in which C is a constant to be determined below.

In order to compare models with different values of σ, for each σ and L, we scale the variance so that

$$\sum_j \left[J_{ij}^2\right]_{\text{av}} = 1 \quad \text{(undiluted)}, \tag{8}$$

where the sum is for fixed i and we have $J_{ii} = 0$. Equation (8) determines the value of C in Eq. (7). Because we consider the nonextensive regime, C must vanish for $L \to \infty$ like $L^{\sigma - \frac{1}{2}}$.

The expression for the mean-field transition temperature in Eq. (2) is the exact result for the SK model, $\sigma = 0$. Hence, from Eq. (8), we have

$$T_c(\sigma = 0) = 1 \quad \text{(undiluted)}. \tag{9}$$

For the diluted model, rather than the *strength* of the interaction falling off like $1/r_{ij}^{\sigma}$, most bonds are absent, and it is the *probability* of there being a nonzero bond which falls of with distance (asymptotically like $1/r_{ij}^{2\sigma}$). If a bond is present, it is chosen from a Gaussian distribution with mean zero and variance unity (i.e., independent of r_{ij}). In other words

$$P(J_{ij}) = (1 - p_{ij})\,\delta(J_{ij}) + p_{ij}\,\frac{1}{\sqrt{2\pi}}\, e^{-J_{ij}^2/2} \quad \text{(diluted)}, \tag{10}$$

where $p_{ij} \propto 1/r_{ij}^{2\sigma}$ at large distance.

It is convenient to fix the mean number of neighbors z. The pairs of sites with nonzero bonds are then generated as follows. Pick a site i at random. Then pick a site j with probability $\widetilde{p}_{ij} = A/r_{ij}^{2\sigma}$, where A is determined by normalization. If there is already a bond between i and j, repeat until a pair i,j is selected which does not already have a bond.[1] At that point set J_{ij} equal to a Gaussian random variable with zero mean and variance unity. This process is repeated $Nz/2$ times so the number of sites connected to a given site has a Poisson distribution with mean z. Because each site has, on average, z neighbors and the variance of each interaction is unity, we have

$$\sum_j \left[J_{ij}^2\right]_{\text{av}} = z \quad \text{(diluted)}. \tag{11}$$

The transition temperature for the diluted model with $\sigma = 0$ was shown by Viana and Bray [16] to be given by the solution of

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-x^2/2} \tanh^2\left(\frac{x}{T_c}\right) = \frac{1}{z}. \tag{12}$$

We choose $z = 6$. for which we find

$$T_c(\sigma = 0) = 2.0564 \quad \text{(diluted)}. \tag{13}$$

III. METHOD

We perform Monte Carlo simulations on the models described in Sec. II. To speed up equilibration we use the parallel tempering (exchange) Monte Carlo method [18]. In this approach one simulates N_T copies of the spins with the same interactions, each at a different temperature between a minimum value $T_{\min}$ and a maximum value $T_{\max}$. In addition to the usual single spin-flip moves for each copy, we perform global moves in which we interchange the temperatures of two copies at neighboring temperatures with a probability which satisfies the detailed balance condition. In this way, the temperature of a particular copy performs a random walk between $T_{\min}$ and $T_{\max}$, thus helping to overcome the free energy barriers found in the simulation of glassy systems.

For the simulations of the undiluted model to be in equilibrium the following equality must be satisfied [17]:

$$U = -\frac{\left(T_c^{MF}\right)^2}{2T}\,(1 - q_l) \quad \text{(undiluted)}, \tag{14}$$

[1] Note that if $z\widetilde{p}_{ij} \ll 1$, then p_{ij} in Eq. (10) is given by $p_{ij} = z\widetilde{p}_{ij}$, but otherwise, there are corrections due to rejection of pairs i,j when there is already a bond between them.

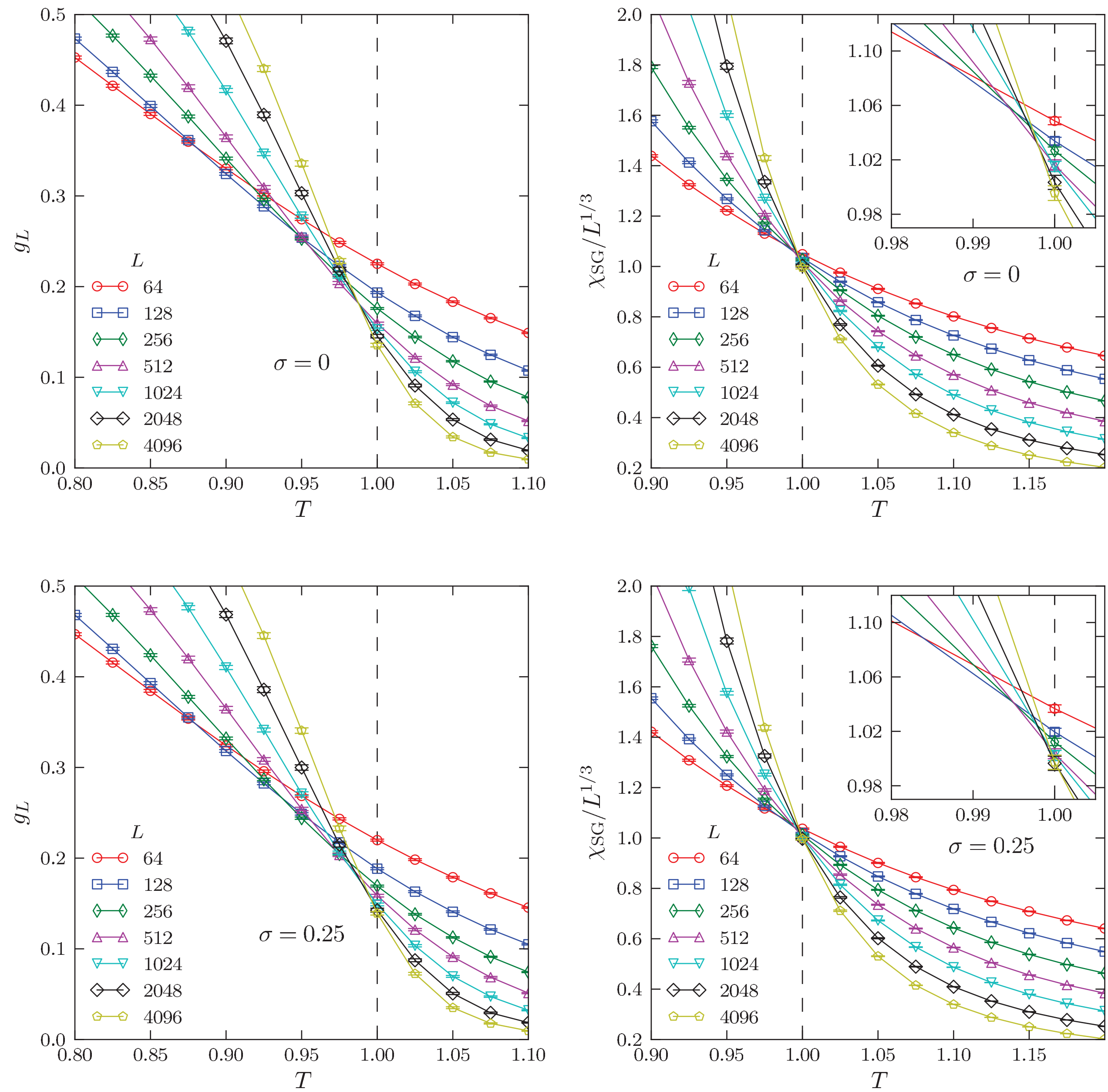

FIG. 1. (Color online) Data for the the scaled spin glass susceptibility and the Binder ratio for (top) the undiluted model with $\sigma = 0$, the SK model, and (bottom) the undiluted model with $\sigma = 0.25$. The dashed vertical line shows the exact value of the transition temperature ($T_c = 1$) for the SK model.

where

$$U = -\sum_{\langle i,j\rangle}[J_{ij}\langle S_i S_j\rangle]_{\rm av} \tag{15}$$

is the average energy and q_l is the "link overlap" defined by

$$q_l = \frac{2}{N}\sum_{\langle i,j\rangle}\frac{[J_{ij}^2]_{\rm av}}{\left(T_c^{MF}\right)^2}[\langle S_i S_j\rangle^2]_{\rm av} \quad \text{(undiluted)}, \tag{16}$$

in which T_c^{MF} is given by Eq. (2) [and here set equal to unity by the scaling of the interactions; see Eq. (8)]. Equation (14) is obtained by integrating by parts with respect to J_{ij} the expression for the average energy and noting that the distribution is Gaussian. This equation is useful because, very plausibly, the two sides approach their common value from *opposite* directions [17], so if the two sides agree, the system has reached equilibrium (at least for the energy and link overlap).

For the diluted model, the equilibration test takes the form [5]

$$U = -\frac{z}{2T}(1 - q_l) \quad \text{(diluted)}, \tag{17}$$

where the link overlap is now defined by

$$q_l = \frac{2}{Nz}\sum_{\langle i,j\rangle}[\epsilon_{ij}\langle S_i S_j\rangle^2]_{\rm av} \quad \text{(diluted)}, \tag{18}$$

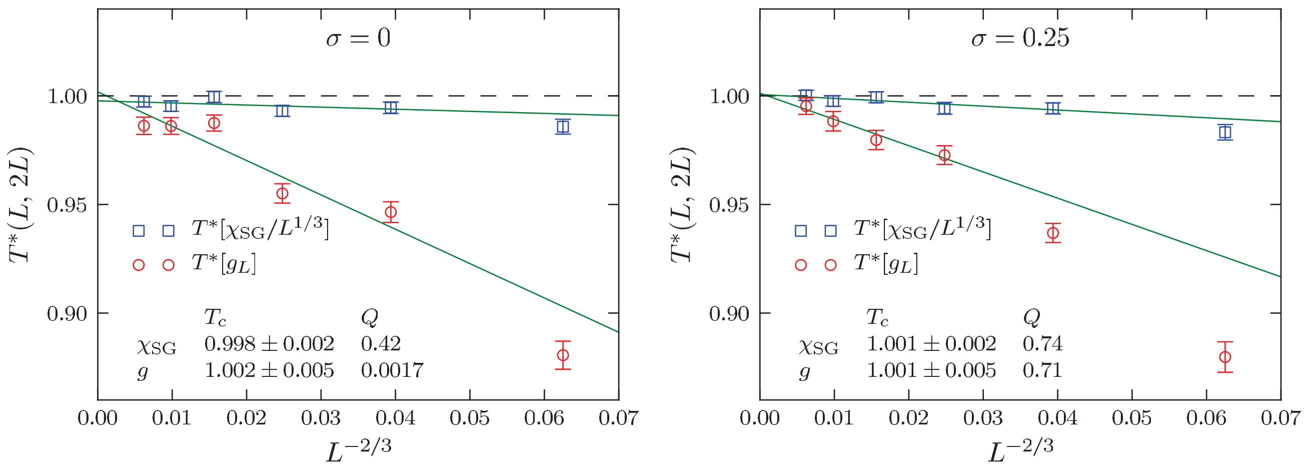

FIG. 2. (Color online) Results for the intersection temperatures for (left) the SK model and (right) the undiluted $\sigma = 0.25$ model.

in which $\epsilon_{ij} = 1$ if there is a bond between i and j and zero otherwise. As with Eq. (14), we expect that the two sides of Eq. (17) approach each other from opposite directions as equilibrium is approached.

We consider results obtained by successively doubling the number of sweeps, in each case averaging over the last half of the sweeps, and we accept the data as being in equilibrium if the last three data points agree with each other within the error bars. The total number of sweeps used in this check is shown as N_{equil} in Tables I and II. We then do "production" runs where, in addition to N_{equil} sweeps for equilibration, we do 10 to 20 times as many sweeps, N_{meas}, during which measurements are performed. All the parameters used in the simulations are given in Tables I and II. To avoid bias, each distinct thermal average, for example, in Eq. (16), is evaluated in a separate copy (replica) of the system with the same interactions.

TABLE I. Simulation parameters for the undiluted models. N_{samp} is the number of samples, and N_{equil} and N_{meas} are the number of sweeps for equilibration and for the measurement phase, respectively. We simulate N_T temperatures between T_{min} and T_{max}.

σ	L	N_{samp}	N_{equil}	N_{meas}	T_{min}	T_{max}	N_T
0	64	16000	1000	10000	0.5	1.65	47
0	128	16000	1000	10000	0.5	1.6	45
0	256	16000	1000	10000	0.5	1.6	45
0	512	8000	1000	10000	0.75	1.55	33
0	1024	8000	1000	10000	0.75	1.5	31
0	2048	4000	1000	10000	0.75	1.5	31
0	4096	4000	2000	10000	0.85	1.525	28
0.25	64	16000	1000	10000	0.5	1.65	47
0.25	128	16000	1000	10000	0.5	1.6	45
0.25	256	16000	1000	10000	0.5	1.6	45
0.25	512	8000	1000	10000	0.5	1.525	42
0.25	1024	8000	1000	10000	0.75	1.5	31
0.25	2048	4000	1000	10000	0.75	1.5	31
0.25	4096	4000	2000	10000	0.85	1.525	28

We focus on moments of the spin glass order parameter q where

$$q = \frac{1}{L} \sum_i S_i^{(1)} S_i^{(2)}, \tag{19}$$

in which (1) and (2) refer to two independent copies of the system with the same interactions. Of particular interest are the spin glass susceptibility,

$$\chi_{SG} = L\langle q^2 \rangle, \tag{20}$$

and the Binder ratio,

$$g = \frac{1}{2}\left(3 - \frac{[\langle q^4 \rangle]_{\text{av}}}{[\langle q^2 \rangle]_{\text{av}}^2}\right). \tag{21}$$

TABLE II. Simulation parameters for the diluted models. The parameters are the same as in Table I.

σ	L	N_{samp}	N_{equil}	N_{meas}	T_{min}	T_{max}	N_T
0	256	8000	400	8000	1.85	2.5	27
0	512	8000	800	16000	1.85	2.5	27
0	1024	8000	2000	40000	1.85	2.5	27
0	2048	4000	2000	40000	1.85	2.5	27
0	4096	4000	2000	40000	1.9	2.5	25
0	8192	2000	4000	80000	1.9	2.5	25
0	16384	2000	4000	80000	2.0	2.5	14
0.25	256	8000	800	16000	1.85	2.5	27
0.25	512	8000	800	16000	1.85	2.5	27
0.25	1024	8000	1200	24000	1.85	2.5	27
0.25	2048	4000	2000	40000	1.85	2.5	27
0.25	4096	4000	2000	40000	1.9	2.5	25
0.25	8192	2000	4000	80000	1.9	2.5	25
0.25	16384	2000	4000	80000	2.0	2.5	14
0.375	256	32000	1200	24000	1.863	4.0	24
0.375	512	26327	1200	24000	1.863	4.0	26
0.375	1024	16000	1200	24000	1.913	4.0	24
0.375	2048	15998	2000	40000	1.95	4.0	24
0.375	4096	8000	4000	80000	1.962	4.0	28
0.375	8192	7999	4000	80000	1.975	4.0	34
0.375	16384	4000	4000	80000	2.0	2.51	18

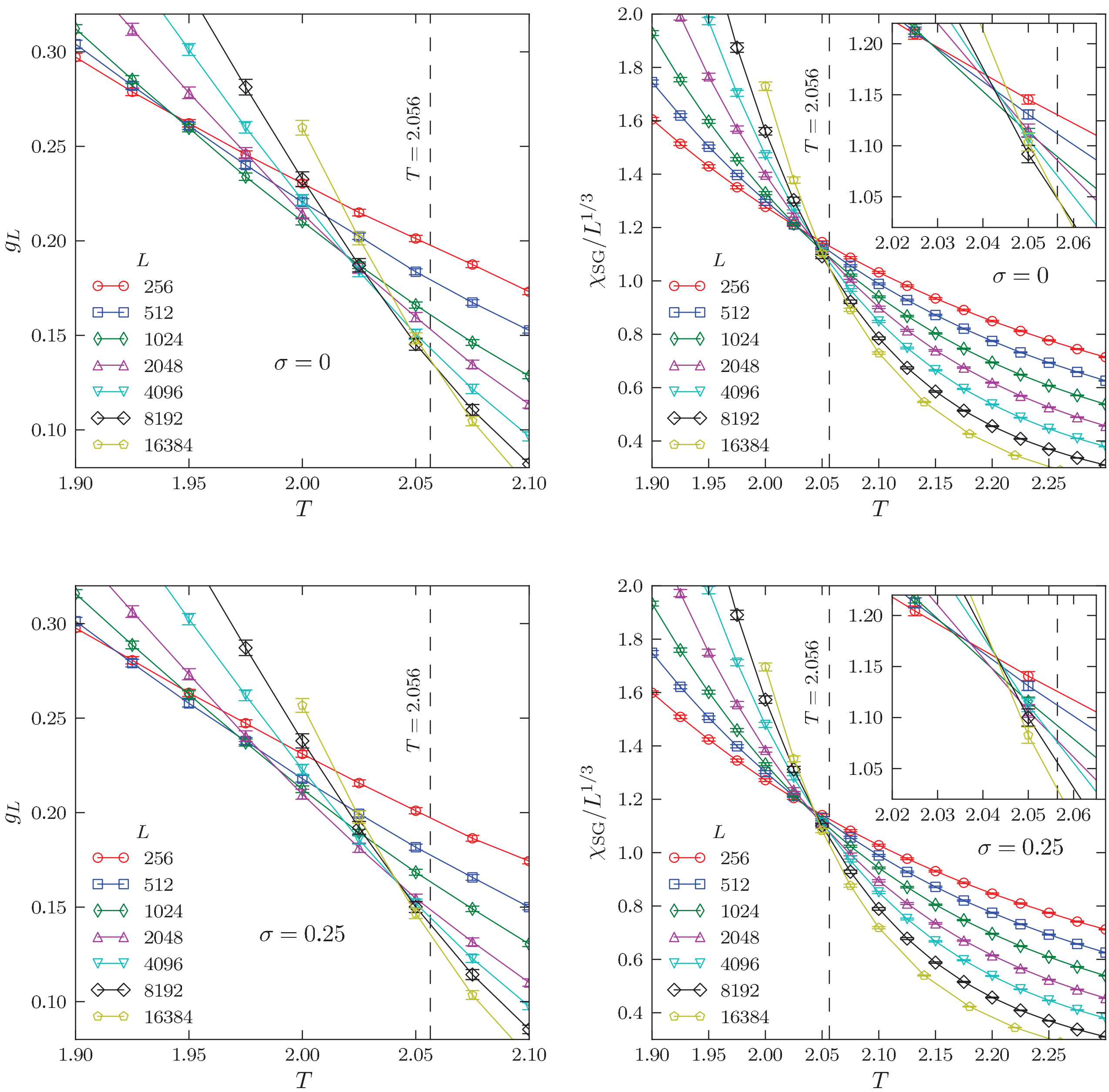

FIG. 3. (Color online) Data for the the scaled spin glass susceptibility and the Binder ratio for (top) the Viana-Bray model, i.e., the diluted model with $\sigma = 0$, and (bottom) the diluted model with $\sigma = 0.25$. The dashed vertical line shows the transition temperature for the Viana-Bray model obtained from Eq. (12) of the text.

Since the Binder ratio is dimensionless, its FSS behavior is simple. We are always in the regime of mean-field critical exponents ($0 \leqslant \sigma < 2/3$), so it has the form [19]

$$g = \widetilde{g}[(T - T_c)\, L^{1/3}]. \tag{22}$$

The spin glass susceptibility is not dimensionless, but since we are in the mean-field regime, its FSS form is also known exactly. It has the form [19]

$$\chi_{SG} = L^{1/3}\, \widetilde{\chi}[(T - T_c)\, L^{1/3}]. \tag{23}$$

One can therefore determine the transition temperature from where the data for g or $\chi_{SG}/L^{1/3}$ for different sizes intersect. However, we shall see that the data do not all intersect at a single temperature, showing that there are corrections to the FSS form in Eqs. (22) and (23). Consider Eq. (23). According to standard finite-size scaling, the spin glass susceptibility normally varies near the critical point according to [11]

$$\begin{aligned}\chi_{SG}(t,L) = {} & L^{a}[f(L^{b}t) + L^{-\omega}g(L^{y}t) + \cdots] \\ & + c_0 + c_1 t + \cdots,\end{aligned} \tag{24}$$

where $t = T - T_c$, $a = 2 - \eta$ ($=2\sigma - 1$ here), and $b = 1/\nu$. The $L^{-\omega}$ term is the leading *singular* correction to scaling, and c_0 is the leading *analytic* correction to scaling. However, in the mean-field limit, $\sigma < 2/3$, the exponents a and b are independent of σ [20–22] and take the value at σ_c for all

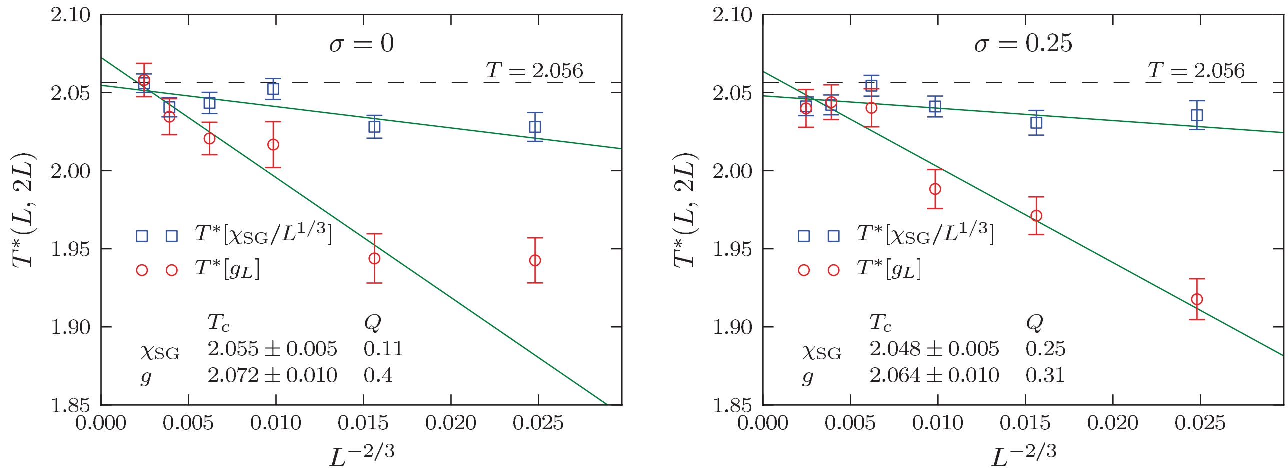

FIG. 4. (Color online) Results for the intersection temperatures for (left) the Viana-Bray model and (right) the diluted model with $\sigma = 0.25$.

$1/2 < \sigma < \sigma_c$, i.e., $a = b = 1/3$. Furthermore, although the $L^{2\sigma-1}$ term is replaced as the *largest* term by an $L^{1/3}$ term (due to the presence of a "dangerous irrelevant variable," cf. Refs. [20–22]), we expect [11] this term not to disappear but rather become a *correction to scaling*. Hence, we replace Eq. (24) by

$$\chi_{SG}(t,L) = L^{1/3}[f(L^{1/3}t) + L^{-\omega}g(L^{1/3}t) + \cdots] + d_0 L^{2\sigma-1} hg(L^{1/3}t) + c_0 + c_1 t \cdots. \tag{25}$$

The correction exponent ω can be obtained in the mean-field regime from the work of Kotliar *et al.* [4] and is given by $\omega = 2 - 3\sigma$. Hence, in the nonextensive regime, $\sigma < 1/2$, the dominant correction to scaling is the constant c_0.

Adding a constant to the right-hand side of Eq. (23), it is straightforward to show that the intersection temperature of the data for $\chi_{SG}/L^{1/3}$ for sizes L and $2L$ is given by

$$T^*(L,2L) = T_c + \frac{A}{L^{2/3}} + \cdots, \tag{26}$$

where A is a constant and the omitted terms are higher order in $1/L$. We expect that the intersection temperatures for the data for g have the same form. We shall use Eq. (26) to determine T_c for the models studied.

IV. RESULTS

We first present our results for the undiluted model. Data for the the scaled spin glass susceptibility and the Binder ratio are shown in Fig. 1. The top panels show the SK model, $\sigma = 0$, and the bottom panels show the undiluted model with $\sigma = 0.25$. One sees large corrections to scaling for the Binder ratio (the left-hand plots) but much smaller corrections for the scaled spin glass susceptibility (the right-hand plots). The inset enlarges the region of the intersections for the latter data.

Figure 2 shows values for the intersection temperature. These were determined by interpolation using cubic splines, and error bars were computed by a jackknife analysis. For *both* values of σ the data extrapolates to a value of 1, the exact value for the SK model (with very small errors). The quality of the fit, as represented by the goodness of fit parameter Q [23], is satisfactory except for the Binder ratio data for the SK model. We do not have a good explanation for this, except perhaps that multiple corrections to scaling are significant for the range of sizes studied. In any case we note that the result $T_c = 1$ for the SK model is rigorously correct. The result that $T_c = 1$ also for $\sigma = 0.25$, at the midpoint of the nonextensive region, provides strong evidence for the claim of Mori [15] that all models in the nonextensive region are identical to the SK model. While it would be useful to check this also in the space glass phase below T_c, such simulations would be difficult because relaxation times increase dramatically at low T, and so the range of sizes that could be studied would be much more limited than in the data presented here.

The corresponding results for the diluted model for $\sigma = 0$ and 0.25 are shown in Figs. 3 and 4. We also performed simulations for $\sigma = 0.375$ and show the resulting intersection temperatures in Fig. 5. For $\sigma = 0$, the Viana-Bray [16] model, the transition temperature is given by Eq. (12), which, for $z = 6$ taken here, gives the result in Eq. (13). In Fig. 3 we again see that corrections to scaling are larger for the Binder ratio than for the scaled spin glass susceptibility. The intersection

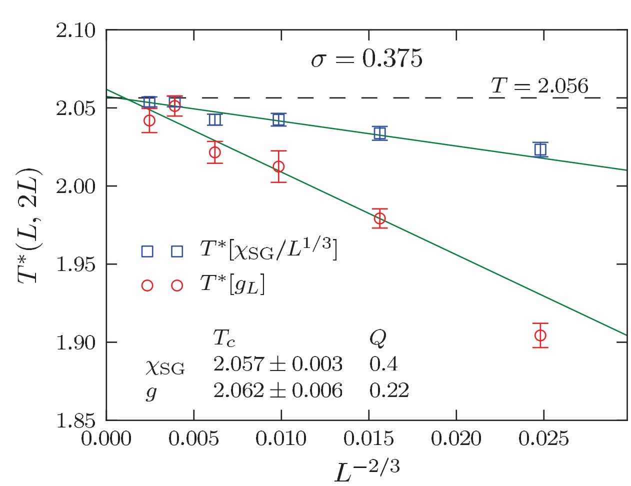

FIG. 5. (Color online) Results for the intersection temperatures for the diluted model with $\sigma = 0.375$.

temperatures *all* extrapolate to the exact value for $\sigma = 0$ within statistical uncertainty.[2]

V. SUMMARY AND CONCLUSIONS

We have performed Monte Carlo simulations to investigate the transition temperature of one-dimensional Ising spin glasses, both undiluted and diluted, for several values of σ in the nonextensive regime $0 \leqslant \sigma < 1/2$. For the undiluted model we studied two values of σ, $\sigma = 0$ and $\sigma = 0.25$. For $\sigma = 0.25$, which lies in the middle of the nonextensive regime, we find that the transition temperature agrees to high precision with the exact solution of the SK model. As a check, we also simulated the $\sigma = 0$ case, obtaining results consistent with the exact SK model result, though there seem to be multiple corrections to FSS for some of the data.

For the diluted model we studied three values of σ: $\sigma = 0$, which corresponds to the Viana-Bray model, $\sigma = 0.25$, which lies in the middle of the nonextensive regime, and $\sigma = 0.375$. In all cases we found the transition temperature to be consistent with the exact solution of the Viana-Bray model; all results are within $\sim$1.5 standard deviations.

To conclude, our results provide confirmation of the proposal [15] that the behavior of (undiluted) spin glasses everywhere in the nonextensive regime is identical to that of the SK model. We have also proposed that an analogous result applies to *diluted* spin glass models and provided numerical evidence for this too.

ACKNOWLEDGMENTS

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APPENDIX: SPHERICAL APPROXIMATION FOR THE FERROMAGNET

For the infinite-range *ferromagnet*, the interaction J_{ij} is equal to $1/(N-1)$ for $i \neq j$ and 0 for $i = j$. This fixes $T_c = 1$. The Fourier transform of this interaction is given by

$$J(k) = \delta_{k,0} - \frac{1}{N}, \tag{A1}$$

so only the $k = 0$ mode contributes to the transition.

For a power-law decay of the interactions in the nonextensive regime ($0 < \sigma < 1$), on dimensional grounds there is a singular piece which diverges like $k^{\sigma-1}$ for $k \to 0$. Furthermore the interactions have to be multiplied by a number of order $N^{\sigma-1}$ in order to satisfy the condition $T_c^{MF} = \sum_j J_{ij} = 1$. Hence, roughly speaking, we have

$$J(k) \propto (k\,N)^{\sigma-1}, \tag{A2}$$

where we note that $k \equiv k_n = 2\pi n/L$ ($n = 0,1,2,\ldots$). [For $n = 0, J(0)$ does not actually diverge but will be comparable to $J(k_1)$.] Hence other long wavelength modes, in addition to $k = 0$, are now significant. However, we shall now see that there are not enough of them to change the value of T_c from that of T_c^{MF} (=1).

We will do this by considering the "spherical approximation" [24], in which we reexpress the problem as a Gaussian one, with "soft" spins ϕ_i which take values from $-\infty$ to ∞, and a Hamiltonian

$$\mathcal{H}_{\text{Gauss}} = \frac{1}{2}\sum_{i,j}(\mu\delta_{ij} - J_{ij})\phi_i\phi_j, \tag{A3}$$

where μ is a Lagrange multiplier whose value is chosen to enforce the length constraint

$$\langle \phi_i^2 \rangle = 1. \tag{A4}$$

It turns out the the spherical approximation is exact for an m-component model in the limit $m \to \infty$ [25]. Fourier transforming Eq. (A4) and doing the Gaussian integrals gives

$$\frac{1}{T} = \frac{1}{L}\sum_k \frac{1}{\mu - J(k)}. \tag{A5}$$

The transition occurs when the denominator vanishes at $k = 0$, i.e., when $\mu = J(0)$, and so

$$\frac{1}{T_c^{\text{spher}}} = \frac{1}{L}\sum_k \frac{1}{J(0) - J(k)}. \tag{A6}$$

It is interesting to compare this with the mean-field result, $T_c = \sum_j J_{ij} = J(0)$. Since $J_{ii} = (1/L)\sum_k J(k) = 0$, we can rewrite the mean-field transition temperature as

$$T_c^{MF} = \frac{1}{L}\sum_k [\,J(0) - J(k)\,]. \tag{A7}$$

Thus, whereas in mean-field theory T_c is equal to the average of $J(0) - J(k)$, in the spherical approximation $1/T_c$ is equal to the average of the *inverse* of this.

For the infinite-range model, where $J(k)$ is given by Eq. (A1) and only the $k = 0$ mode contributes, the spherical result agrees with the mean-field result (consistent with the mean-field result being exact for this model).

We now estimate T_c from the spherical approximation, Eq. (A6), for the power-law model, where $J(k)$ varies like Eq. (A2). Because we normalize the interactions to $J(0) = 1$, we can include an extra factor of $J(0)$ and expand in powers of $J(k)/J(0)$, i.e.,

$$\frac{1}{T_c^{\text{spher}}} = \frac{1}{L}\sum_k \frac{1}{1 - J(k)/J(0)} \tag{A8a}$$

$$= \frac{1}{L}\sum_k \left[1 + \frac{J(k)}{J(0)} + \left(\frac{J(k)}{J(0)}\right)^2 + \cdots\right] \tag{A8b}$$

$$= 1 + \frac{1}{L}\sum_k \left[\left(\frac{J(k)}{J(0)}\right)^2 + \cdots\right] \tag{A8c}$$

$$= 1 + \frac{\sum_j J_{ij}^2}{\left(\sum_j J_{ij}\right)^2} + \cdots, \tag{A8d}$$

where in Eq. (A8c) we used $\sum_k J(k) = 0$. In Eq. (A8d) we have $\sum_j J_{ij} \propto L^{1-\sigma}$ while $\sum_j J_{ij}^2 = L^{1-2\sigma}$ ($0 \leqslant \sigma < 1/2$), $\sum_j J_{ij}^2 = \text{const}$ ($1/2 < \sigma < 1$). Hence the leading

[2] All the results are within one standard deviation, except for the data for g for $\sigma = 0$ and $\chi_{SG}/L^{1/3}$ for $\sigma = 0.25$, but even these are within $\sim$1.5 standard deviations, which we also consider acceptable.

correction term in Eq. (A8d) vanishes everywhere in the nonextensive regime.

To conclude, in this Appendix we have given a suggestive argument as to why T_c for the ferromagnet is given exactly by the mean-field value everywhere in the nonextensive regime. It is therefore also plausible that other properties are also identical to those of mean-field theory (i.e., the infinite-range model).

[1] S. Boettcher, Phys. Rev. Lett. **95**, 197205 (2005).
[2] F. Dyson, Commun. Math. Phys. **12**, 212 (1969).
[3] F. Dyson, Commun. Math. Phys. **21**, 269 (1971).
[4] G. Kotliar, P. W. Anderson, and D. L. Stein, Phys. Rev. B **27**, 602 (1983).
[5] H. G. Katzgraber and A. P. Young, Phys. Rev. B **72**, 184416 (2005).
[6] H. G. Katzgraber, D. Larson, and A. P. Young, Phys. Rev. Lett. **102**, 177205 (2009).
[7] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Phys. Rev. Lett. **101**, 107203 (2008).
[8] M. A. Moore, Phys. Rev. B **82**, 014417 (2010).
[9] L. Leuzzi, G. Parisi, F. Ricci-Tersenghi, and J. J. Ruiz-Lorenzo, Phys. Rev. Lett. **103**, 267201 (2009).
[10] A. J. Bray, M. A. Moore, and A. P. Young, Phys. Rev. Lett. **56**, 2641 (1986).
[11] D. Larson, H. G. Katzgraber, M. A. Moore, and A. P. Young, Phys. Rev. B **81**, 064415 (2010).
[12] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
[13] S. A. Cannas, A. C. N. de Magalhães, and F. A. Tamarit, Phys. Rev. B **61**, 11521 (2000).
[14] A. Campa, A. Giansanti, and D. Moroni, Phys. Rev. E **62**, 303 (2000).
[15] T. Mori, Phys. Rev. E **84**, 031128 (2011).
[16] L. Viana and A. J. Bray, J. Phys. C **18**, 3037 (1985).
[17] H. G. Katzgraber and A. P. Young, Phys. Rev. B **67**, 134410 (2003).
[18] K. Hukushima and K. Nemoto, J. Phys. Soc. Jpn. **65**, 1604 (1996).
[19] For a discussion of how finite-size scaling is modified in the region of mean-field exponents, see, for example, Refs. [20–22,26,27].
[20] K. Binder, M. Nauenberg, V. Privman, and A. P. Young, Phys. Rev. B **31**, 1498 (1985).
[21] E. Luijten, K. Binder, and H. W. J. Blöte, Eur. Phys. J. B **9**, 289 (1999).
[22] J. L. Jones and A. P. Young, Phys. Rev. B **71**, 174438 (2005).
[23] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in C++*, 3rd ed. (Cambridge University Press, Cambridge, 2007).
[24] T. H. Berlin and M. Kac, Phys. Rev. **86**, 821 (1952).
[25] H. E. Stanley, Phys. Rev. **176**, 718 (1968).
[26] E. Brézin, J. Phys. (Paris) **43**, 15 (1982).
[27] E. Brézin and J. Zinn-Justin, Nucl. Phys. B **257**, 867 (1985).